Bernoulli Wavelet Operational Matrix of Integration and its Application in Solving Differential Equations

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Abstract: The aim of this paper is to propose an efficient Bernoulli wavelet numerical method for solving differential equations. First Bernoulli operational matrix of integration is derived using Bernoulli wavelet and then applied to solve the differential equation of one and higher order. Some illustrative examples are given to demonstrate the efficiency and validity of the proposed method.

Keywords: Differential equation, Bernoulli Wavelet, Operational matrix of integration.
Subject Classification AMS: 65T60, 47N70, 34A08, 26A33.

1. Introduction

In recent years, wavelets with their characteristic properties have found their place in a wide range of engineering disciplines. Wavelets are used in numerical analysis, signal analysis, system analysis, optimal control and solution of many differential and integral equations. The main characteristic of wavelets is its ability to convert the given differential equations, fractional order differential equations and integral equations to a system of nonlinear and linear algebraic equations, which are then solved by existing numerical methods. Many researchers started using various wavelets for analyzing problems of greater computational complexity and show that wavelets to be an powerful tools to explore a new direction in solving differential equations of first and higher order.

In general it is not always possible to obtain exact solution of an arbitrary differential equation. This necessitates either discretization of differential equation leading to numerical solutions, or their qualitative study which is concerned with deduction of important properties of the solutions without actually solving them. . Many authors are used to different types of wavelet and approximating functions for solving initial and boundary value problems. Hsiao and Wang [1] used Haar wavelet for solution of nonlinear stiff systems, Chen and Hsiao [2] time-varying functional differential equations and lumped and distributed-parameter systems, U. Lepik [3] solving differential equation and Evolution equation using Haar wavelets, S. A. Youefi [4] used Legendre wavelet method for solving differential equation of Lane-Emden type, H. Kaur et. al. [5] give Haar wavelet approximate solutions for the generalized Lane Emden equations, R. K. Pandey et. al. [6] used Legendre operational matrix of differentiation for solution of Lane- Emden equation. N. Berwal et. al. [7] solving differential equation through Haar operational matrix.

There are two different approaches for solving differential equations. One approach is based on converting differential equation into integral equations through integration, and approximating various signals involved in the equation by truncated orthogonal series, and using the operational matrix of integration, to eliminate the integral operations [8]. Another one is based on using operational matrix of derivatives in order to reduce the problem into solving a system of linear or nonlinear algebraic equations. There are some papers in the literature about using the operational matrix of derivatives to solve differential equations.

The outline of this paper is as follows: in second section, we discussed how to construct Bernoulli wavelet. In third section, we discuss the function approximation and operational matrix of integration for Bernoulli wavelet. In section fourth, we apply Bernoulli wavelet technique and find out an approximate solution of the three examples.

2. Bernoulli Wavelet

Bernoulli wavelets [9] $\psi_{n,m}(t)=\psi\left(k,\hat{n},m,t\right)$ have four arguments; $\hat{n}=n-1,\ \ n=1,2,3,...,2^{k-1},$ k can assume any positive integers, m is the order for Bernoulli polynomials and t is the normalized time. We define them on the interval $[0,1)$ as follows

$$\psi_{n,m}(t)=\begin{cases}2^{\frac{k-1}{2}}\tilde{\beta}_m\left(2^{k-1}t-\hat{n}\right), & \dfrac{\hat{n}}{2^{k-1}}\le t<\dfrac{\hat{n}+1}{2^{k-1}},\\ 0, & otherwise,\end{cases} \tag{1}$$

with

$$\tilde{\beta}_m(t)=\begin{cases}1, & m=0,\\ \dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\beta_m(t), & m>0,\end{cases} \tag{2}$$

Where $m=0,1,2,...,M-1$ and $n=1,2,...,2^{k-1}$. The coefficient $\dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for normality, the dilation parameter is $a=2^{-(k-1)}$ and translation parameter $b=\hat{n}2^{-(k-1)}$. Here, $\beta_m(t)$ are the well-known Bernoulli polynomials of order m which can be defined by [9]

$$\beta_m(t)=\sum_{i=0}^{m}\binom{m}{i}\alpha_{m-i}t^i,$$

where α_i, $i=0,1,...,m$ are Bernoulli numbers. These numbers are a sequence of rational numbers which arise in the series expansion of trigonometric functions [10] and can be defined by the identity

$$\frac{t}{e^t-1}=\sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!}. \tag{3}$$

The first few Bernulli numbers are

$$\alpha_0=1,\ \ \alpha_1=\frac{-1}{2},\ \ \alpha_2=\frac{1}{6},\ \ \alpha_3=\frac{-1}{30},...$$

with $\alpha_{2i+1}=0,\ i=1,2,3,...$

the first few Bernoulli polynomials are

$$\beta_0=1,\ \ \beta_1=t-\frac{1}{2},\ \ \beta_2=t^2-t+\frac{1}{6},\ \ \beta_3=t^3-\frac{3}{2}t^2+\frac{1}{2}t,.... \tag{4}$$

These polynomials satisfy the following formula [10]

$$\int_0^1\beta_n(t)\beta_m(t)dt=(-1)^{n-1}\frac{m!n!}{(m+n)!}\alpha_{n+m},\qquad m,\ n\geq 1.$$

According to [10], Bernoulli polynomial form a complete basis over the interval $[0,1]$.

3. Function approximation

Suppose that $\{\psi_{10}(t),\psi_{11}(t),...,\psi_{2^{k-1}M-1}(t)\}\subset L_2[0,1]$ is the set of Bernoulli wavelets and a function $f(t)$ defined over $L_2[0,1]$ can be approximated by Bernoulli wavelet series as follows

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}C_{nm}\psi_{nm}(t), \tag{5}$$

where

$$C_{nm}=\int_0^1 f(t)\psi_{nm}(t)dt$$

if the infinite series in equation (5) is truncated then the equation (5) can be written as

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{nm}\psi_{nm}(t)=C^T\Psi(t). \tag{6}$$

where T indicated transposition and, C and $\Psi(t)$ are $2^{k-1}M\times 1$ column vectors given by

$$C=\left[c_{10},c_{11},...,c_{1M-1},c_{20},...,c_{2M-1},...,c_{2^{k-1}0},...,c_{2^{k-1}M-1}\right]^T,$$

and

$$\Psi(t)=\left[\psi_{10}(t),\psi_{11}(t),...,\psi_{1M-1}(t),\psi_{20}(t),...,\psi_{2M-1}(t),...,\psi_{2^{k-1}0}(t),...,\psi_{2^{k-1}M-1}(t)\right]^T.$$

Since the truncated Bernoulli wavelet series can be an approximate solution of differential equation, one has an error function $E(t)$ for $f(t)$ as follows:

$$E(t)=\left|f(t)-C^T\Psi(t)\right|.$$

4 Bernoulli wavelet operational matrix of integration

We derive Bernoulli wavelet operational matrix of integration. To illustrate the working procedure, we choose $k=2$ and $M=3$ .We have:

$$\left.\begin{array}{l}\psi_{10}=\begin{cases}\sqrt{2}, & 0\le t<\frac{1}{2},\\ 0, & otherwise.\end{cases}\\ \psi_{11}=\begin{cases}\sqrt{6}\left(4t-1\right), & 0\le t<\frac{1}{2},\\ 0, & otherwise.\end{cases}\\ \psi_{12}=\begin{cases}\sqrt{10}\left(24t^2-12t+1\right), & 0\le t<\frac{1}{2},\\ 0, & otherwise.\end{cases}\end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l}\psi_{20}=\begin{cases}\sqrt{2}, & \frac{1}{2}\le t<1,\\ 0, & otherwise.\end{cases}\\ \psi_{21}=\begin{cases}\sqrt{6}\left(4t-3\right), & \frac{1}{2}\le t<1,\\ 0, & otherwise.\end{cases}\\ \psi_{22}=\begin{cases}\sqrt{10}\left(24t^2-36t+13\right), & \frac{1}{2}\le t<1,\\ 0, & otherwise.\end{cases}\end{array}\right\} \tag{8}$$

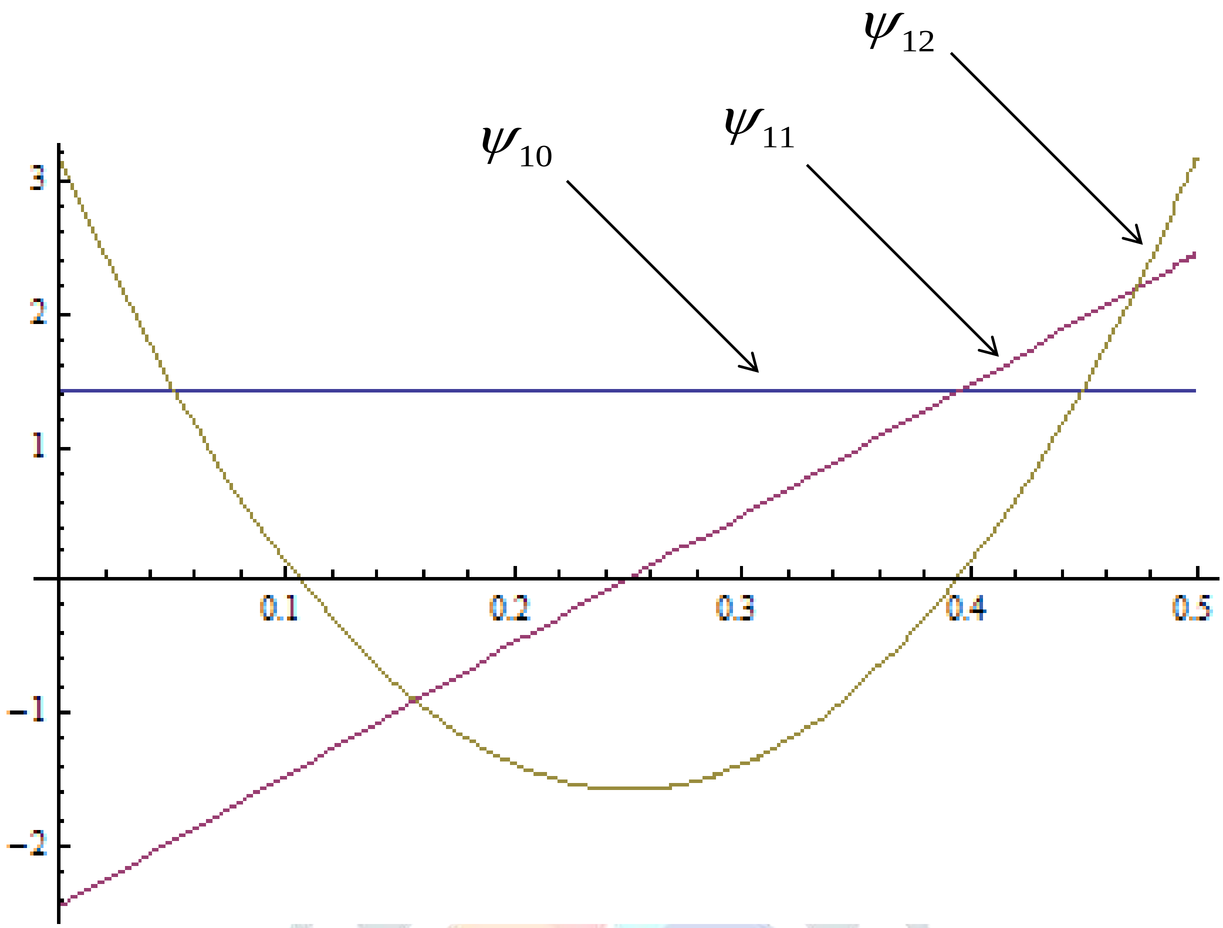

Figure 4.1: *Bernoulli wavelets for* $k = 2;\ M = 3$ *and t in* $[0,1/2]$

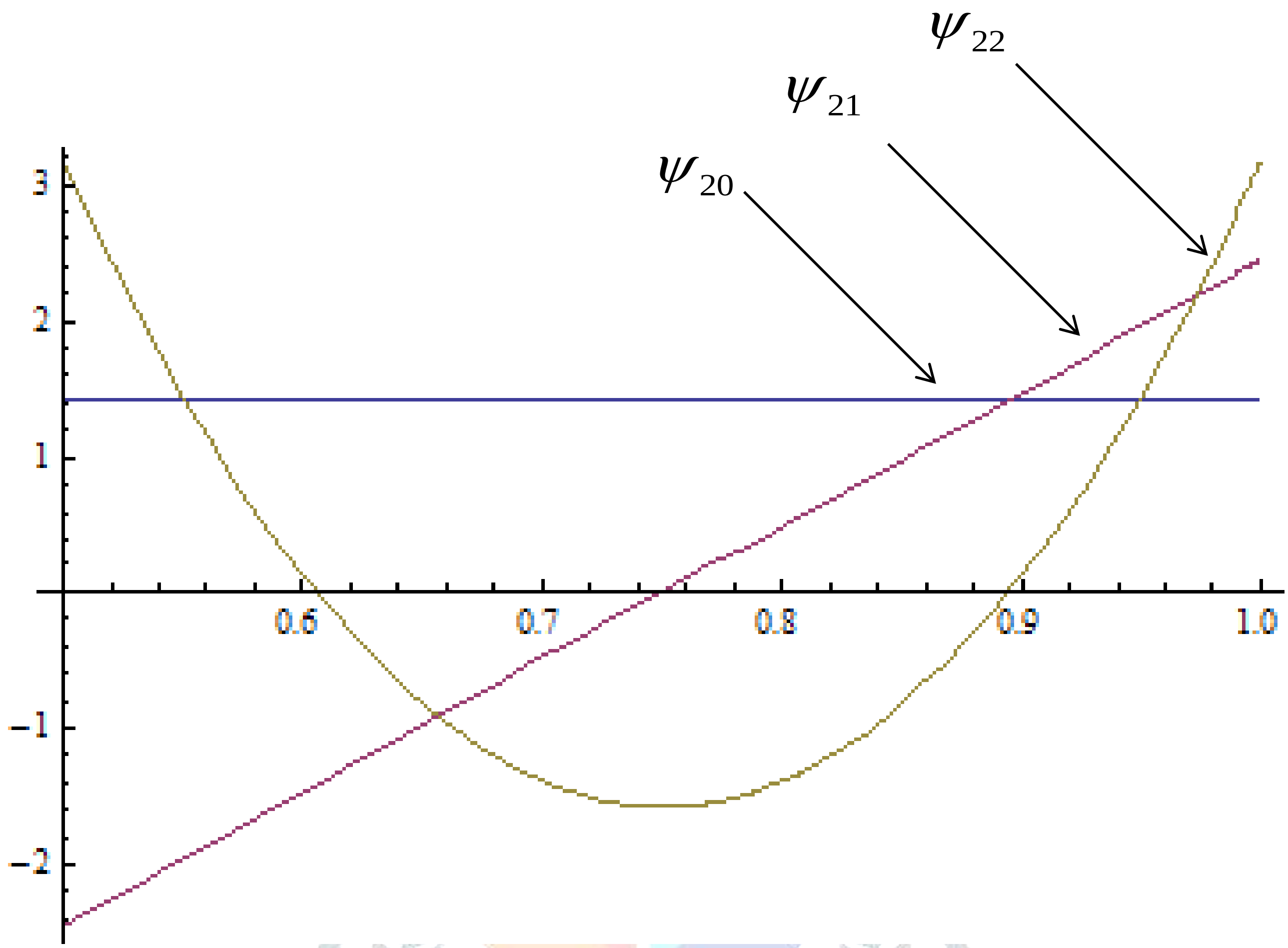

Figure 4.2: *Bernoulli wavelets for* $k = 2;\ M = 3$ *and t in* $[1/2, 1]$

Figure 5.4.1 and Figure 5.4.2 shown the plot of Bernoulli wavelets for $k = 2$ and $M = 3$.

By integrating the equations *(5.4.8), (5.4.9)* from 0 to t and using $C_{n,m} = \left\langle f(t), \psi_{n,m} \right\rangle$, obtained as

$$\int_0^t \psi_{10}(t)dt = \begin{cases} 2^{1/2}t, & 0 \le t < \dfrac{1}{2} \\ 0, & otherwise, \end{cases}$$

$$= \begin{bmatrix} \dfrac{1}{4} & \dfrac{1}{4\sqrt{3}} & 0 & \dfrac{3}{4} & \dfrac{1}{4\sqrt{3}} & 0 \end{bmatrix}^T \Psi_6(t).$$

$$\int_0^t \psi_{11}(t)dt = \begin{cases} 6^{1/2}\left(2t^2 - t\right), & 0 \le t < \dfrac{1}{2} \\ 0, & otherwise, \end{cases}$$

$$= \begin{bmatrix} \dfrac{-1}{4\sqrt{3}} & 0 & \dfrac{1}{4\sqrt{15}} & \dfrac{5}{4\sqrt{3}} & \dfrac{1}{2} & \dfrac{1}{4\sqrt{15}} \end{bmatrix}^T \Psi_6(t).$$

Similarly,

$$\int_0^t \psi_{12}(t)dt = \begin{bmatrix} 0 & \dfrac{-1}{4\sqrt{15}} & 0 & \sqrt{5} & \dfrac{29}{4\sqrt{15}} & \dfrac{1}{2} \end{bmatrix}^T \Psi_6(t)$$

$$\int_0^t \psi_{20}(t)dt = \left[\frac{1}{4} \quad \frac{1}{4\sqrt{3}} \quad 0 \quad \frac{3}{4} \quad \frac{1}{4\sqrt{3}} \quad 0\right]^T \Psi_6(t)$$

$$\int_0^t \psi_{21}(t)dt = \left[\frac{-7}{4\sqrt{3}} \quad \frac{-1}{2} \quad \frac{1}{4\sqrt{15}} \quad \frac{-13}{4\sqrt{3}} \quad 0 \quad \frac{1}{4\sqrt{15}}\right]^T \Psi_6(t)$$

$$\int_0^t \psi_{22}(t)dt = \left[2\sqrt{5} \quad \frac{-29}{4\sqrt{15}} \quad \frac{-1}{2} \quad 3\sqrt{5} \quad \frac{-1}{4\sqrt{15}} \quad 0\right]^T \Psi_6(t).$$

Thus,

$$\int_0^t \Psi(t)_{6\times 1}\, dt \approx P_{6\times 6}\Psi_6(t), \tag{9}$$

Where $\chi_6(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \psi_{20}(t), \psi_{21}(t), \psi_{22}(t)]^T$ and operational matrix of integration is

$$P_{6\times 6} = \frac{1}{2}\begin{bmatrix} 1/4 & 1/4\sqrt{3} & 0 & 3/4 & 1/4\sqrt{3} & 0 \\ -1/4\sqrt{3} & 0 & 1/4\sqrt{15} & 5/4\sqrt{3} & 1/2 & 1/4\sqrt{15} \\ 0 & -1/4\sqrt{15} & 0 & \sqrt{5} & 29/4\sqrt{15} & 1/2 \\ 1/4 & 1/4\sqrt{3} & 0 & 3/4 & 1/4\sqrt{3} & 0 \\ -7/4\sqrt{3} & -1/2 & 1/4\sqrt{15} & -13/4\sqrt{3} & 0 & 1/4\sqrt{15} \\ 2\sqrt{5} & 29/4\sqrt{15} & -1/2 & 3\sqrt{5} & -1/4\sqrt{15} & 0 \end{bmatrix}$$

5 Application of the Bernoulli wavelet method

In this section, operational matrix of integration of Bernoulli wavelet is used for finding the numerical solution of ordinary differential equations.

Assume that

$$y^n(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t), \tag{10}$$

where C^T is an unknown vector which should be determined, $\Psi(t)$ is the vector defined in *(5.3.7)* and $n \in \square$, denote the order of higher order derivate, which involves in given problem. Equation *(5.5.11)* is integrated *n* time with respect to *t*, in this way the solution $y(t)$ and its *nth* derivative are expressed in terms of Bernoulli wavelet functions and their integrals.

The expression of $y(t), y^{'}(t), ..., y^n(t)$ are substituting in the given differential equations. Thus we get a system of equation with $2^{k-1}M$ unknowns. Then we can obtain the unknown vector *C* by solving this system of equations. By inserting these values of *C*, we can easily find the corresponding expression of $y(t)$. All calculations have been done by mathematica-7.

Example 5.1: Let us consider the following differential equation [7]

$$y^{"}(t) + 2y^{'}(t) + 5y(t) = f(t) \tag{11}$$

with condition

$$y(0) = 0, \quad y^{'}(0) = 1. \tag{12}$$

where $f(t) = 3\exp(-t)\sin t$ and the exact solution of equation (11) by homotopy perturbation method is

$$y(t) = \exp(-t)\sin t.$$

First, approximate second order derivative as

$$y^{"}(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{13}$$

and integrating equation (13) with respect to t two time and using equation (12), we get

$$y^{'}(t) = 1 + C^T P\Psi(t) \tag{14}$$

We approximate 1 as $1 \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} l_{nm} \psi_{nm}(t) = L^T \Psi(t)$ and using in equation (14), we get

$$y'(t) = L^T \Psi(t) + C^T P \Psi(t) = \left(L^T + C^T P\right) \Psi(t) \tag{15}$$

and

$$y(t) = \left(L^T P + C^T P^2\right) \Psi(t) \tag{16}$$

Also we have,

$$\exp(-t)\sin t \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} f_{nm} \psi_{nm}(t) = F^T \Psi(t),$$

so

$$f(t) \approx 3F^T \Psi(t) \tag{17}$$

Substituting equation (13) and (15-17) in equation (11), we obtain

$$C^T \Psi(t) + 2\left(L^T + C^T P\right) \Psi(t) + 5\left(L^T P + C^T P^2\right) \Psi(t) = 3F^T \Psi(t)$$

$$C^T = \left(3F^T - 2L^T - 5L^T P\right)\left(I + 2P + 5P^2\right)^{-1} \tag{18}$$

By solving this system of linear equations, we can find the coefficient vector C for different value of K and M Putting the value of the vector C in equation (16) we get the required numerical solution of the Example 5.1 for Bernoulli wavelet $K = 2$ and $M = 3, 4$. It can be seen from Figure 5.1.1 that the obtained solutions by the given method approach is too close to the exact solution. In order to analyses the effectiveness of the given approach has good convergence in the certain applicable domain and this is showing in the Table 1.

t	**ES by HPM**	**Haar Solution for** $M = 8$	**BWM for** $k = 2, M = 3$	**BWM for** $k = 2, M = 4$	**Absolute Error** $k = 2, M = 4$
0.0625	0.0587	0.0576	0.0563	0.0585	1.46×10^{-4}
0.1875	0.1545	0.1539	0.1522	0.1538	6.56×10^{-4}
0.3125	0.2249	0.2245	0.2237	0.2255	6.65×10^{-4}
0.4375	0.2735	0.2735	0.2708	0.2736	1.27×10^{-4}
0.5625	0.3039	0.3042	0.2911	0.3037	1.14×10^{-4}
0.6875	0.3191	0.3192	0.2833	0.3186	4.59×10^{-4}
0.8125	0.3222	0.3225	0.2457	0.3226	4.74×10^{-4}
0.9375	0.3157	0.3160	0.1763	0.3157	8.10×10^{-5}

Table 5.1.1: Numerical results of $y(t)$ for $k = 2$ and $M = 2, 3$ and exact solution of example 5.1.

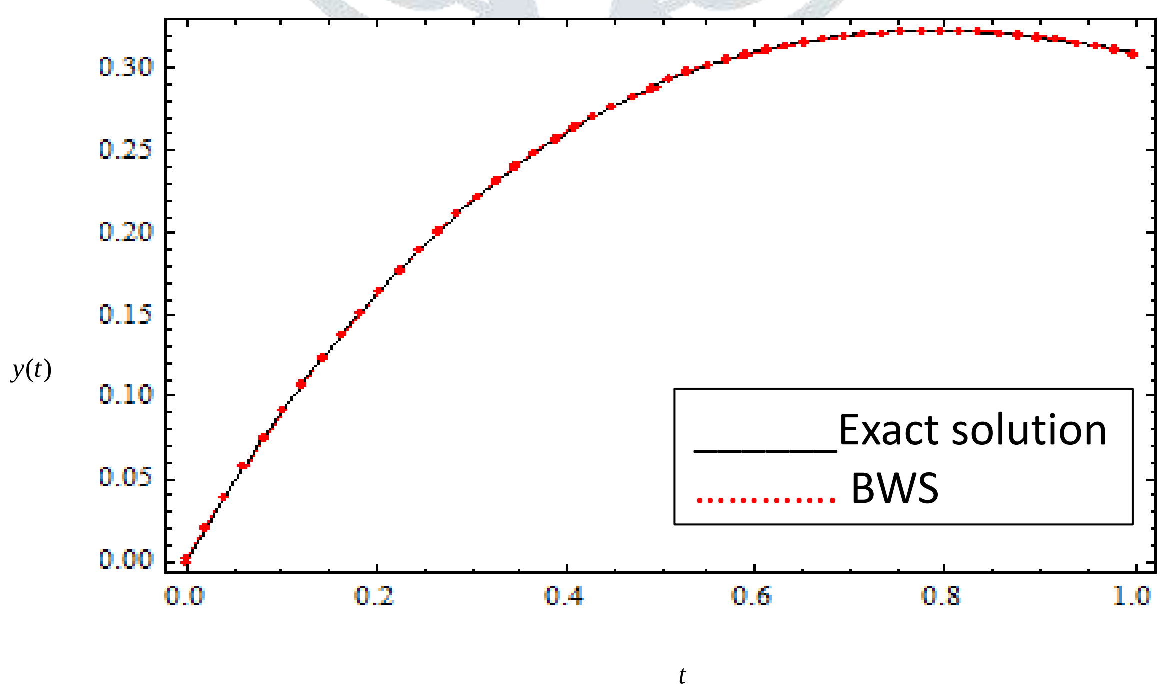

Figure 5.1.1: Coparision of $y(t)$ for $k=2$ and $M=3$ with exact solution of example 5.1.

Example 5.2: Consider the following eighth order differential equation [7]

$$y^{viii}(t) - y(t) = -8\exp(t) \qquad \text{where} \qquad 0 \le t \le 1, \tag{19}$$

with initial conditions

$$\begin{aligned} &y(0)=1, \quad y'(0)=0, \quad y''(0)=-1, \quad y'''(0)=-2, \\ &y^{iv}(0)=-3, \quad y^{v}(0)=-4, \quad y^{vi}(0)=-5, \quad y^{vii}(0)=-6, \end{aligned} \tag{20}$$

Exact solution of equation (19) is $y(t)=(1-t)\exp(t)$.

First, approximate $y^{viii}(t)$ as

$$y^{viii}(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{21}$$

Integrating equation (21) with respect to t and using initial conditions (20), we get

$$y(t) = 1 - 1\frac{t^2}{2!} - 2\frac{t^3}{3!} - 3\frac{t^4}{4!} - 4\frac{t^5}{5!} - 5\frac{t^6}{6!} - 6\frac{t^7}{7!} + C^T P^8 \Psi(t) \tag{22}$$

We also approximate $\exp(t)$ as

$$\exp(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} j_{nm}\psi_{nm}(t) = J^T\Psi(t) \tag{23}$$

Substututing equation (21), (22) and (23) in equation (19), we get

$$C^T\Psi(t) - \left(1 - 1\frac{t^2}{2!} - 2\frac{t^3}{3!} - 3\frac{t^4}{4!} - 4\frac{t^5}{5!} - 5\frac{t^6}{6!} - 6\frac{t^7}{7!} + C^T P^8 \Psi(t)\right) = -8J^T\Psi(t) \tag{24}$$

Let $1 - 1\frac{t^2}{2!} - 2\frac{t^3}{3!} - 3\frac{t^4}{4!} - 4\frac{t^5}{5!} - 5\frac{t^6}{6!} - 6\frac{t^7}{7!} \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} r_{nm}\psi_{nm}(t) = R^T\Psi(t)$, then equation (24) can be written as

$$C^T\left(I - P^8\right)\Psi(t) = \left(R^T - 8J^T\right)\Psi(t)$$

$$C^T = \left(R^T - 8J^T\right)\left(I - P^8\right)^{-1} \tag{25}$$

By manipulating system (25) of linear equations and obtain the coefficient vector C^T. Putting value of the vector C^T in equation (22) we acquired the approximate result of the Example 5.2 for Bernoulli wavelets ($k=2;\ \ M=3,4$) and shown in Figure 5.2 1. This has been shown that the solution obtained by the proposed method is very close to the exact solution. In order to analyses the effectiveness of the proposed approach, The results of example 5.2 are compared with Haar wavelet results. Which is showing in the Table 5.2.1.

t	ES	Haar Solution $M=8$	BWM for $k=2,\ M=3$	BWM for $k=2,\ M=4$	Absolute Error $k=2,\ M=4$
0.0625	0.9980	0.9970	0.9977	0.9981	1.6×10^{-4}
0.1875	0.9812	0.9790	0.9795	0.9811	1.0×10^{-3}
0.3125	0.9397	0.9386	0.9387	0.9387	9.9×10^{-4}
0.4375	0.8712	0.8693	0.8698	0.8709	2.6×10^{-4}
0.5625	0.7678	0.7661	0.7658	0.7681	3.2×10^{-4}
0.6875	0.6215	0.6193	0.6184	0.6235	2.0×10^{-3}
0.8125	0.4225	0.4200	0.4180	0.4205	1.9×10^{-3}
0.9375	0.1596	0.1565	0.1531	0.1590	5.3×10^{-4}

Table 5.2.1: Numerical results of $y(t)$ for $k=2$ and $M=2,3$ and exact solution of example 5.2

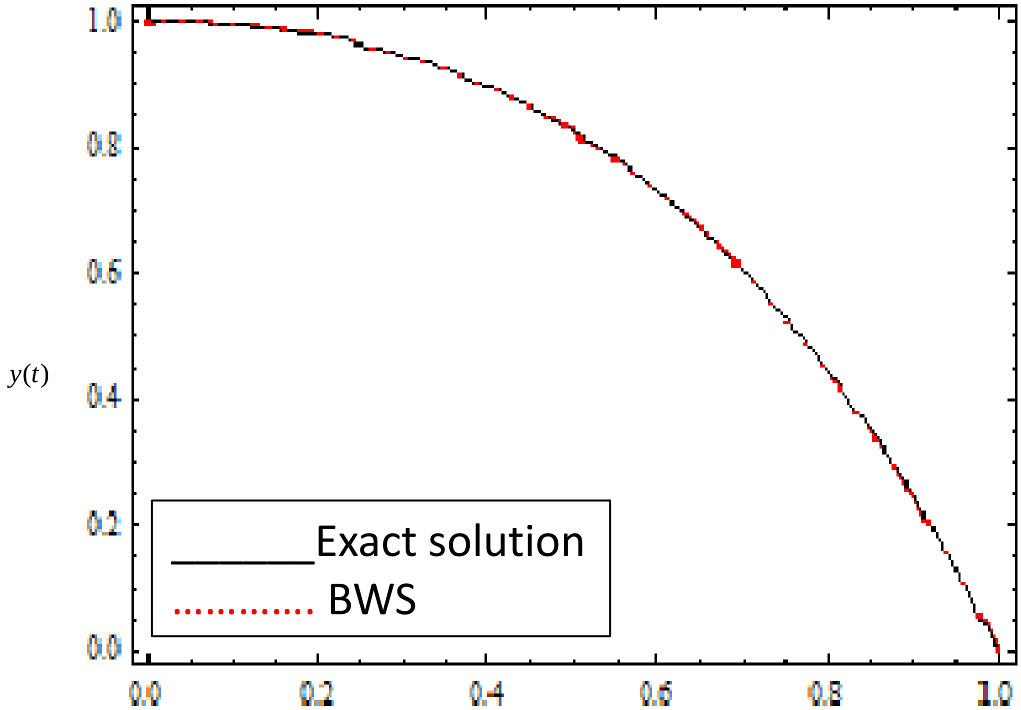

Figure 5.2.1: Coparision of $y(t)$ for $k=2$ and $M=3$ with exact solution of example 5.2.

Example 5.3: Consider the following linear Lane-Emden equation [12]

$$y^{''}(t)+\frac{1}{t}y^{'}(t)+y(t)=t^2-t^3-9t+4, \qquad \text{where} \qquad 0<t<1, \tag{26}$$

Subject to the boundary condition

$$y^{'}(0)=0, \quad y(1)=0. \tag{27}$$

The exact solution is $y(t)=t^2-t^3.$

First, approximate second order derivative as

$$y^{''}(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^T\Psi(t), \tag{28}$$

and integrating equation (28) with respect to t two time and using equation (27), we get

$$y^{'}(t)=C^T P\Psi(t), \tag{29}$$

and

$$y(t)=C^T P^2\Psi(t). \tag{30}$$

We approximate 1 as $1\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}l_{nm}\psi_{nm}(t)=L^T\Psi(t)$ and integrating it three time 0 to t, we get

$$t\approx L^T P\Psi(t), \quad t^2\approx 2L^T P^2\Psi(t) \quad \text{and} \quad t^3\approx 3!L^T P^3\Psi(t) \tag{31}$$

Substituting equations (28-31) in equation (26), we have

$$C^T\Psi(t)+\frac{1}{t}C^T P\Psi(t)+C^T P^2\Psi(t)=2!L^T P^2\Psi(t)-3!L^T P^3\Psi(t)-9L^T P\Psi(t)+4L^T\Psi(t)$$

$$C^T\left(I+\frac{1}{t}P+P^2\right)\Psi(t)=L^T\left(2!P^2-3!P^3-9P+4I\right)\Psi(t)$$

$$C^T=L^T\left(2!P^2-3!P^3-9P+4I\right)\left(I+\frac{1}{t}P+P^2\right)^{-1} \tag{32}$$

By solving system of linear equations (32), we obtain the coefficient vector C^T. Inserting the value of the vector C^T in equation (30) we find the acquired approximate result of the Example 5.3 for Bernoulli wavelets $(k=2; \quad M=3, 4)$ and shown in Figure 5.3 1. Figure 5.3.1 shows that the solution obtained by the proposed method is very close to the exact solution. The effectiveness of the proposed way has very good in the certain applicable domain and this is showing in the Table 5.3.1.

t	ES	BWM for $k=2,\ M=3$	BWM for $k=2,\ M=4$	Absolute Error $k=2,\ M=3$	Absolute Error $k=2,\ M=4$
0.0625	0.0036	0.0041	0.0037	4.3×10^{-4}	3.7×10^{-5}
0.1875	0.0285	0.0306	0.0283	2.0×10^{-3}	2.6×10^{-4}
0.3125	0.0671	0.0650	0.0670	2.0×10^{-3}	1.3×10^{-4}
0.4375	0.1076	0.1072	0.1075	4.3×10^{-4}	1.6×10^{-4}
0.5625	0.1384	0.1388	0.1384	4.3×10^{-4}	2.7×10^{-5}
0.6875	0.1477	0.1498	0.1475	2.0×10^{-3}	2.0×10^{-4}
0.8125	0.1237	0.1216	0.1234	2.0×10^{-3}	3.7×10^{-4}
0.9375	0.0549	0.0544	0.0548	4.3×10^{-4}	1.3×10^{-4}

Table 5.3.1: Numerical results of $y(t)$ for $k=2$ and $M=2,\ 3$ and exact solution of example 5.3.

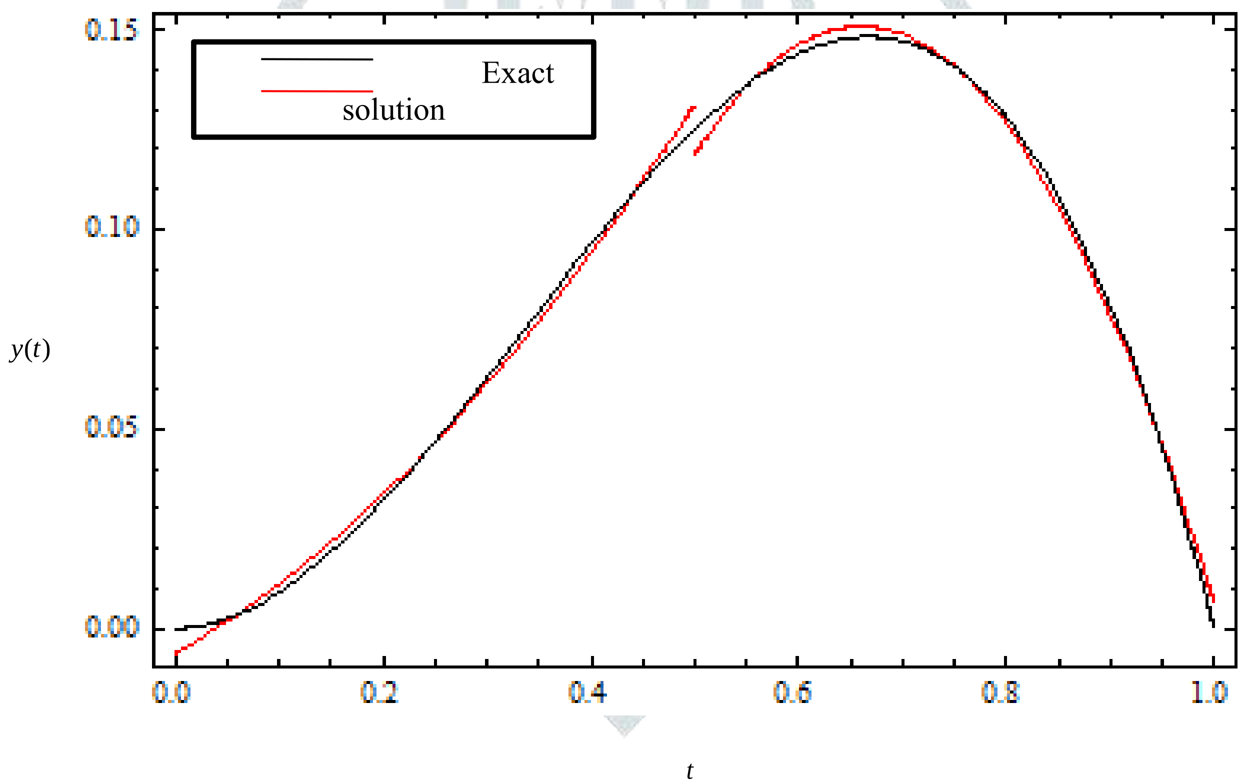

Figure 5.3.1: Coparision of $y(t)$ for $k=2$ and $M=3$ with exact solution of example 5.3.

6 Conclusions

The main goal of this paper is to develop an efficient and accurate method to solve linear differential equation. The Bernoulli wavelet operational matrix of integration is utilized to reduce the problem to the solution of linear algebraic equations. One of the most advantage of this method is that accuracy approximate solutions achieved using very small values of $k=2$ and $M=3,\ 4$. Illustrative examples are included to demonstrate the validity and applicability of the proposed method.

Acknowledgment

The authors are very thankful to respected Dr. Sag Ram Verma, Department of Mathematics and Statistics, Gurukula Kangri University, Haridwar, for encouragement and support.

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