Theorem of Limits of Triple Sequences

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Abstract : In this paper triple sequences and triple series method for finding a Cauchy, The Sandwich theorem triple sequences for proof the theorem. This method finds a triple sequence and triple series in the feasible proof. In this method the number of allocation *n, m, l* is satisfied for all theorems for example double sequence and double series method. This method does not require the theorem. It is easy to understand and this theorem is very efficient for those who are dealing with triple sequence and triple series. It can easily adapt an existing theorem.

***IndexTerms* -** Theorem of Limits, The Sandwich Theorem.

THEOREM OF LIMITS

In this section, we prove some results which enable us to evaluate the triple and iterated limits of a triple sequence.

Theorem

If $s(n,m,l)$ can be written as $s(n,m,l) = a_n a_m a_l$ such that the limits

$$\lim_{n\to\infty} a_n = l_1, \lim_{m\to\infty} a_m = l_2, \lim_{l\to\infty} a_l = l_3, \lim_{n,m\to\infty} a_{n,m} = l_1\, l_2,$$

$$\lim_{m,l\to\infty} a_{m,l} = l_2\, l_3, \lim_{l,n\to\infty} a_{l,n} = l_3\, l_1.$$

Then

$$\lim_{n\to\infty}\left\{\lim_{m,l\to\infty} s(n,m,l)\right\} = \lim_{n\to\infty}\left\{\lim_{m,l\to\infty} s(n,m,l)\right\}$$

$$= \lim_{n\to\infty}\left\{\lim_{m,l\to\infty} s(n,m,l)\right\}$$

$$= l_1 l_2 l_3.$$

Proof:

Use hypothesis, we have

$$\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,m,l)\right]\right\} = \lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty}(a_n a_m a_l)\right]\right\}$$

$$= \lim_{n\to\infty} a_n, \lim_{m\to\infty} a_m, \lim_{l\to\infty} a_l$$

$$= l_1 l_2 l_3.$$

$$\lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} s(n,m,l)\right]\right\} = \lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty}(a_n a_m a_l)\right]\right\}$$

$$= \lim_{n\to\infty} a_n, \lim_{m\to\infty} a_m, \lim_{l\to\infty} a_l$$

$$= l_2 l_3 l_1.$$

$$\lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty} s(n,m,l)\right]\right\} = \lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty}(a_n a_m a_l)\right]\right\}$$

$$= \lim_{n\to\infty} a_n, \lim_{m\to\infty} a_m, \lim_{l\to\infty} a_l$$

$$= l_3 l_1 l_2$$

Hence

$$\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,m,l)\right]\right\} = \lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty}(a_n a_m a_l)\right]\right\}$$

$$= l_1 l_2 l_3.$$

Next to show that $\lim_{n,m,l\to\infty} s(n,m,l) = l_1 l_2 l_3.$

Let $\varepsilon > 0$ be given, since $[a_n]$is bounded (being convergent), there exists $k \in N$ such that $|a_n| \le k, \forall\, n \in N$ again let $\varepsilon > 0$ be given, since $[a_m]$ is bounded (being convergent), there exists $j \in N$ such that $|a_m| \le j, \forall\, m \in N$

And since $a_n \to l_1, a_m \to l_2, a_l \to l_3$ There exists a natural number $N = N(\varepsilon)$

Such that $|a_n - l_1| < \varepsilon/_{3b}, |a_m - l_2| < \varepsilon/_{3b}, |a_l - l_3| < \varepsilon/_{3b}, \forall\ n, m, l \ge N$

Where b = max $\{k, j, |l_1|, |l_3|\}$. Hence it follows that $n, m, l \ge N$.

$$\begin{aligned}|s(n,m,l) - l_1 l_2 l_3| &\le |s(n,m,l) - a_n, a_m, l_3| + |a_n, a_m, l_3 - l_1, a_m, l_3| + |l_1, a_m, l_3 - l_1 l_2 l_3| \\ &\le |a_n, a_m a_l \ - a_n, a_m, l_3| + |a_n, a_m, l_3 - l_1, a_m, l_3| + |l_1, a_m, l_3 - l_1 l_2 l_3| \\ &= \ |a_n a_m||a_l - l_3| + |a_m l_3||a_n - l_1| + |l_1 l_3||a_m - l_2| \\ &= \ |a_n||a_m||a_l - l_3| + |a_m||l_3||a_n - l_1| + |l_1||l_3||a_m - l_2| \\ &< KJ\ \varepsilon/_{3b} + J\ \ |l_1|\,\varepsilon/_{3b} + |l_1||l_3|\,\varepsilon/_{3b} \\ &\le\ b\,\varepsilon/_{3b} + b\,\varepsilon/_{3b} + b\,\varepsilon/_{3b} \\ &\le 3b\ \ \varepsilon/_{3b} < \varepsilon.\end{aligned}$$

There it follows that $\lim\limits_{n,m,l\to\infty} s(n,m,l) = l_1 l_2 l_3.$

Example

Consider the triple sequence $s(n,m,l) = \frac{1}{n,m,l}$ $n, m, l \in N$ we claim that

$$\begin{aligned}\lim_{n,m,l\to\infty} s(n,m,l) &= \lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,m,l)\right]\right\} \\ &= \lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} s(n,m,l)\right]\right\} \\ &= \lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty}\ s(n,m,l)\right]\right\} \\ &= 0.\end{aligned}$$

Indeed, write $s(n,m,l) = a_n, a_m a_l \ = \left(\frac{1}{n}\right)\left(\frac{1}{m}\right)\left(\frac{1}{l}\right), \forall\ \ n, m, l \in N.$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \ = \ \frac{1}{\infty} = 0$$

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{1}{m} \ = \ \frac{1}{\infty} = 0$$

$$\lim_{l\to\infty} a_l = \lim_{l\to\infty} \frac{1}{l} \ = \ \frac{1}{\infty} = 0$$

It follows form 4.1 that

$$\begin{aligned}\lim_{n,m,l\to\infty}\left(\frac{1}{nml}\right) &= \lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty}\frac{1}{nml}\right]\right\} \\ &= \lim_{n\to\infty}\frac{1}{n}\lim_{m\to\infty}\frac{1}{m}\lim_{l\to\infty}\frac{1}{l} \\ &= 0.\end{aligned}$$

Theorem

If $s(n,m,l)$ can be written as $s(n,m,l)= a_n + a_m + a_l$ such that the limits

$$\lim_{n\to\infty} a_n = l_1, \lim_{m\to\infty} a_m = l_2, \lim_{l\to\infty} a_l = l_3,$$

Then $\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,m,l)\right]\right\} = \lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} s(n,m,l)\right]\right\}$

$$= \lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty} s(n,m,l)\right]\right\}$$

$$= \lim_{n,m,l\to\infty} s(n,m,l)$$

$$= l_1 + l_2 + l_3.$$

Proof

Use hypothesis, we have $\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,m,l)\right]\right\} = \lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty}(a_n + a_m + a_l)\right]\right\}$

$$= \lim_{n\to\infty} a_n + \lim_{m\to\infty} a_m + \lim_{l\to\infty} a_l$$

$$= l_1 + l_2 + l_3$$

$$\lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} s(n,m,l)\right]\right\} = \lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty}(a_n + a_m + a_l)\right]\right\}$$

$$= \lim_{n\to\infty} a_n + \lim_{m\to\infty} a_m + \lim_{l\to\infty} a_l$$

$$= l_1 + l_2 + l_3$$

$$\lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty} s(n,m,l)\right]\right\} = \lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty}(a_n + a_m + a_l)\right]\right\}$$

$$= \lim_{n\to\infty} a_n + \lim_{m\to\infty} a_m + \lim_{l\to\infty} a_l$$

$$= l_1 + l_2 + l_3$$

Next, to show that $\lim_{n,m,l\to\infty} s(n,m,l) = l_1 + l_2 + l_3$ let $\varepsilon > 0$ be given,

Use hypothesis there exists a natural number $N = N(\varepsilon)$ such that

$$|a_n - l_1| < {}^{\varepsilon}/_{3}, |a_m - l_2| < {}^{\varepsilon}/_{3}, |a_l - l_3| < {}^{\varepsilon}/_{3}, \quad \forall \; n,m,l \geq N$$

Hence, we have

$$\forall\, n,m,l \geq N \quad |s(n,m,l) - (l_1 + l_2 + l_3)| = |a_n + a_m + a_l - (-l_1 - l_2 - l_3)|$$

$$\leq |a_n - l_1| + |a_m - l_2| + |a_l - l_3|$$

$$< {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3}$$

$$\leq {}^{3\varepsilon}/_{3}$$

$$< \varepsilon.$$

Therefore, it follows that $\mathit{lim}_{n,m,l\to\infty} s(n,m,l) = l_1 + l_2 + l_3$.

Example

Consider the triple sequence $s(n,m,l) = \frac{1}{n} + \frac{1}{m} + \frac{1}{l}\, n,m,l \geq N$ we showed in example 1.8 that

$$\mathit{lim}_{n,m,l\to\infty} s(n,m,l) = \mathit{lim}_{n,m,l\to\infty}\left(\frac{1}{n} + \frac{1}{m} + \frac{1}{l}\right)$$

$$= \mathit{lim}_{n\to\infty}\frac{1}{n} + \mathit{lim}_{m\to\infty}\frac{1}{m} + \mathit{lim}_{l\to\infty}\frac{1}{l}$$

$$= \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{\infty}$$

$$= 0.$$

Theorem (The Sandwich Theorem)

Statement

Suppose that $x(n,m,l), s(n,m,l)$ *and* $y(n,m,l)$ are triple sequence of real numbers such that

$$x(n,m,l) \leq s(n,m,l) \leq y(n,m,l) \forall n,m,l \leq N \text{ and that}$$

$\lim_{n,m,l\to\infty} x(n,m,l) = \lim_{n,m,l\to\infty} y(n,m,l)$ then $s(n,m,l)$ is convergent and

$$\lim_{n,m,l\to\infty} x(n,m,l) = \lim_{n,m,l\to\infty} s(n,m,l) = \lim_{n,m,l\to\infty} y(n,m,l)$$

Proof

Let a = $\lim_{n,m,l\to\infty} x(n,m,l) = \lim_{n,m,l\to\infty} y(n,m,l)$, then given $\varepsilon > 0$, there exists a real number N such that

$$n,m,l \geq N \quad \Rightarrow |x(n,m,l) - a| < \varepsilon \ \ and \ \ |x(n,m,l) - a| < \varepsilon$$

Since the hypothesis implies that

$$x(n,m,l) - a \leq s(n,m,l) - a \leq y(n,m,l) - a \forall n,m,l \leq N$$

It follows that $-\varepsilon < s(n,m,l) - a < \varepsilon \forall n,m,l \leq N$ since $\varepsilon > 0$ was arbitrary, this implies that $\lim_{n,m,l\to\infty} s(n,m,l) = 0$.

CONCLUSION:

In this paper, a new triple sequence and triple series were introduced. Cauchy sequence and monotone sequence proof the theorems. The double sequences and double series while comparing to triple sequences and triple series. It is easy to proof and we get the required proof of theorems.

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