Efficient Strong (Weak) Dominating (γ,eD)-Number of Graphs

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Abstract : ***In this paper, we introduce the new concept of efficient strong (weak) dominating (γ,eD)-number of graphs. Also, this number is found for some standard graphs, subdivision graphs.***

Keywords: Edge Detour, Edge detour domination, efficient strong (weak) dominating (γ,eD)- number

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1. Introduction: The concept of domination was introduced by Ore and Berge[8]. Let G be a finite, undirected connected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to atleast one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number γ(G) of G. We consider connected graphs with atleast two vertices. A set of vertices is independent if no two vertices are adjacent. A regular spanning subgraph of degree 1 is called 1-factor (1F). A subdivision of an edge e = uv of a graph G is the replacement of the edge e by a path {u, v, w}. If every edge of G is subdivided exactly once, then the resulting graph is called the subdivision graph S(G). For basic definitions and terminologies, we refer Harary[1].

For vertices u and v in a connected graph G, the detour distance D(u,v) is the length of longest u-v path in G. A u-v path of length D(u,v) is called a u-v detour. A subset S of V is called a detour set if every vertex in G lies on a detour joining a pair of vertices of S. The detour number dn(G) of G is the minimum order of a detour set and any detour set of order dn(G) is called a detour basis of G. These concepts were studied by Chartrand[3].

A subset S of V(G) is called an edge detour set of G if every edge in G lie on a detour joining a pair of vertices of S. The edge detour number $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order dn_1 is an edge detour basis. A graph G is called an edge detour graph if it has an edge detour set. Edge detour graph were introduced and studied by Santhkumaran and Athisayanathan[11]. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ $(|N_w[v] \cap S| = 1)$, where $N_s(v) = \{u \in V(G): uv \in E(G), deg(u) \geq deg(v)\}$, $(N_w(v) = \{u \in V(G): uv \in E(G), deg(v) \geq deg(u)\})$. The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient number and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is a strong(weak) efficient domination graph if and only if there exists a strong(weak) efficient dominating set of G. Strong(weak) efficient dominating graphs were introduced and studied by N.Meena, A.Subramanian, V.Swaminathan[7]. An edge detour dominating set is a subset S of V(G) which is both dominating and an edge detour set of G. An edge detour dominating set is said to be a minimal edge detour dominating set of G if no proper subset of S is an edge detour dominating set of G. An edge detour dominating S is said to be minimum edge detour dominating set of G if there exist no edge detour dominating set S' such that |S'|<|S|. The smallest cardinality of an edge detour dominating set of G is called the edge detour domination number of G. It is denoted by $\gamma_{eD}(G)$. Any edge detour dominating set S of G of minimum cardinality $\gamma_{eD}(G)$ is called a γ_{eD}- set of G. are Edge detour dominating graphs were studied by Mahalakshmi.A, Palani.K and Somasundaram.S[5].

The following results are from [4].

Theorem 1.1: The domination numbers of some standard graph are given as follows.

1. $\gamma(P_p) = \left\lceil \frac{p}{3} \right\rceil$, $p \geq 3$
2. $\gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil$, $p \geq 3$
3. $\gamma(K_p) = \gamma(W_p) = \gamma(K_{1,n}) = 1$.
4. $\gamma(K_{m,n}) = 2$ if m, $n \geq 2$.

The following are from Sampathkumar.E and Pushpa Latha .L,[10].

Definition 1.2: A subset S of V(G) is called a strong dominating set of G if for every $v \in V-S$ there exists $u \in S$ such that u and v are adjacent and $deg(u) \geq deg(v)$.

The following results are from [5].

Remark 1.3:

1. $\gamma_{eD}(G) \geq dn_1(G)$ and $\gamma_{eD}(G) \geq \gamma(G)$.
2. If the set of all pendant vertices of a graph G forms an edge detour dominating set S of G, then S is the unique minimum edge detour dominating set of G.

Theorem 1.4: $\gamma_{eD}(K_{1,n}) = n$.

Theorem 1.5: $\gamma_{eD}(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & \text{if } n \geq 5; \\ 2 & \text{if } n = 2,3 \text{ or } 4. \end{cases}$

Theorem 1.6: For n > 5, $\gamma_{eD}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

The following are from Mahalakshmi.A, Palani.K and Somasundaram.S[6]

Definition 1.7: Let G be a connected graph. An efficient dominating (γ,eD)-set of G is an edge detour dominating set of G such that for every $v \in V(G)$, $|N[v] \cap S_i|$ = 1. The minimum cardinality of among all efficient dominating (γ,eD) is the efficient dominating (γ,eD)-number of G and is denoted by $e\gamma_{eD}(G)$. An efficient dominating (γ,eD)-set of minimum cardinality $e\gamma_{eD}(G)$ is called a $e\gamma_{eD}$-set of G. A graph G is said to be an efficient dominating (γ, eD)-graph if it has an efficient dominating (γ,eD)-set.

2. Efficient Strong (Weak) Dominating (γ,eD)-Number of Graph

Definition 2.1: Let G = (V, E) be a simple graph. A subset S of V(G) is called an efficient strong (weak) dominating (γ,eD)-set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1 (|N_w[v] \cap S| = 1)$, where $N_s(v) = \{u \in V(G): uv \in E(G), \deg(u) \geq \deg(v)\}$($N_w(v) = \{u \in V(G): uv \in E(G), \deg(v) \geq \deg(u)\}$. The minimum cardinality of an efficient strong (weak) dominating (γ,eD)-set of G is called the efficient strong (weak) dominating (γ,eD)-number of G and it denoted by $es\gamma_{eD}(G)$ ($ew\gamma_{eD}(G)$). A graph G is an efficient strong (weak) dominating (γ,eD)-graph if and only if there exists an efficient strong (weak) dominating (γ,eD)-set of G.

Example 2.2:

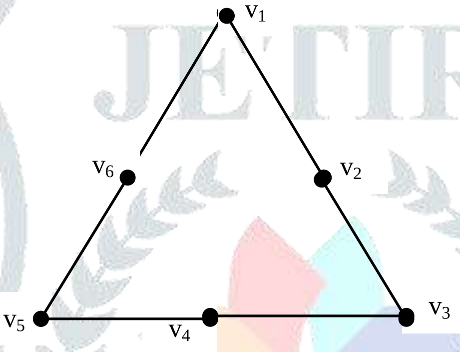
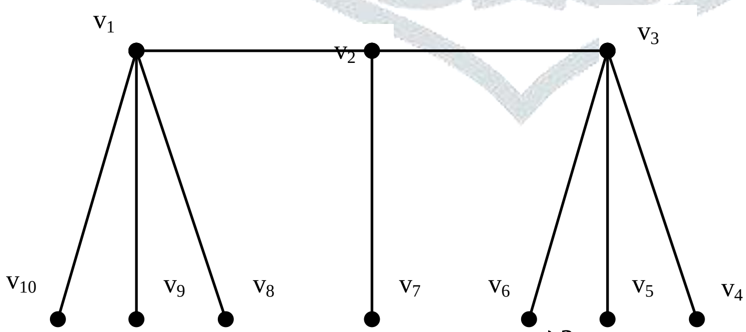


Figure 2.1

Here, S = {v_3, v_6} is an efficient strong dominating (γ,eD)-set of G. Hence, $es\gamma_{eD}(G) = 2$.
In this graph, $\gamma_s(G) = \gamma_{eD}(G) = e\gamma_{eD}(G) = es\gamma_{eD}(G) = ew\gamma_{eD}(G) = 2$.

Remark 2.3: All (γ,eD)-graph need not be an efficient strong (weak) dominating (γ,eD)-graph.



2.2

Here, S = {v_4, v_5, v_6, v_7, v_8, v_9, v_{10}} is a (γ,eD)-set. Therefore, $\gamma_{eD}(G) = 7$. But, it is not an efficient strong (weak) dominating (γ,eD)-set. Since, $|N_s[v] \cap S| \neq 1 (|N_w[v] \cap S| \neq 1)$.

Remark 2.4:

1. Not all graph admit efficient strong (weak) dominating (γ,eD)-set.
2. If $es\gamma_{eD}(ew\gamma_{eD})$ -set exists then (γ,eD)-set and $es\gamma_{eD}(ew\gamma_{eD})$-set are equal.
3. If a regular graph G is an efficient dominating (γ,eD)-graph, then G is obviously an efficient strong (weak) dominating (γ,eD)-graph.

Therefore, $es\gamma_{eD}(G) = ew\gamma_{eD}(G) = e\gamma_{eD}(G)$. For, $N_s[v] = N_w[v] = N[v]$ for every v ∈ G, where G is regular.

Lemma 2.5: For cycle C_n, $n \equiv 1 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$, there is no efficient dominating (γ,eD)-set.
Proof: Let $C_n = \{v_1, v_2, v_3, \ldots, v_1\}$.
Case 1: $n \equiv 1$ (mod 3)
The graph C_n has more than one (γ,eD)-set. In each set, there exists elements u and v such that, d(u,v) = 1 or d(u,v) = 2.
Sub Case 1a: d(u,v) = 1
Let $u,v \in S_i$ where S_i is one of the (γ,eD)-sets of C_n.
Therefore, $|N[u] \cap S_i| \neq 1$ and $|N[v] \cap S_i| \neq 1$. Therefore, C_n has no efficient dominating (γ,eD)-set.
Sub Case 1b: d(u,v) = 2
Let u, $v \in S_i$ where S_i is one of the (γ,eD)-sets of C_n. Let $w \in C_n$ be the vertex which lies between u and v. Then, $|N[w] \cap S_i| \neq 1$. Therefore, C_n has no efficient dominating (γ,eD)-set.
Case 2: $n \equiv 2$ (mod3)
Sub Case 2a: n = 5
Here the cycle is C_5 and every (γ,eD)-set of C_5 contains a pair of adjacent vertices.
Therefore, C_5 has no efficient dominating (γ,eD)-set.
Sub Case 2b: n > 5
Here, C_n has more than one (γ,eD)-set. In each set, there exists elements u and v such that d(u,v) = 2. Let $u,v \in S_i$ where S_i is one of the (γ,eD)-sets of C_n. Let $w \in C_n$ be the vertex which lies between u and v. Then, $|N[w] \cap S_i| \neq 1$. Therefore, C_n has no efficient dominating (γ,eD)-set.
Lemma 2.6: For path P_n, $n \equiv 0 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$, there are no efficient dominating (γ,eD)-graphs.
Proof: Let $P_n = \{v_1, v_2, v_3, \ldots, v_n\}$.
Case i: $n \equiv 0$ (mod 3) Then, n = 3k, k > 0
In this case, P_n has more than one (γ,eD)-set. In each set any one of the vertices $v_2, v_5, v_8, \ldots, v_{n-1}$ does not satisfy the condition $|N[v] \cap S| = 1$.
Therefore, P_n, $n \equiv 0$ (mod 3) has no efficient dominating (γ,eD)-set.
Case ii: $n \equiv 2$ (mod 3)
Then, n = 3k + 2, k > 0
In this case also, P_n has more than one (γ,eD)-set. In each set any one of the vertices $v_2, v_4, v_7, \ldots, v_{n-1}$ does not satisfy the condition $|N[v] \cap S| = 1$.
Therefore, P_n, $n \equiv 2$ (mod 3) has no efficient dominating (γ,eD)-set.
Theorem 2.7: Every efficient strong dominating (γ,eD)-set of a graph G is independent.
Proof: Let S be an efficient strong dominating (γ,eD)-set of a graph G. Suppose $u, v \in S$ such that u and v are adjacent. Then, either $|N_s[u] \cap S| \neq 1$ or $|N_s[v] \cap S| \neq 1$. Therefore, S is not an efficient strong dominating (γ,eD)-set of G, which is a contradiction. Hence, S is independent.
Theorem 2.8: Every efficient weak dominating (γ,eD)-set is independent.
Proof: The proof is similar to the proof of Theorem 2.7.
Theorem 2.9: For cycle C_{3n}, $es\gamma_{eD}(C_{3n}) = ew\gamma_{eD}(C_{3n}) = n$ for all n > 1.
Proof: Let $G = C_{3n}$, n>1 and $V(G) = \{v_1, v_2, \ldots, v_{3n}\}$. The (γ,eD)-set of C_n are $S_1 = \{v_1, v_4, v_7, \ldots, v_{3n-2}\}$; $S_2 = \{v_2, v_5, v_8, \ldots, v_{3n-1}\}$; $S_3 = \{v_3, v_6, v_9, \ldots, v_{3n}\}$. And it can be easily verified that $|N_s[v] \cap S_i| = 1$ and $|N_w[v] \cap S_i| = 1$ for all $v \in C_{3n}$ and i = 1, 2, 3. Hence, S_i, i=1,2,3 are efficient strong and weak dominating (γ,eD)-sets of C_{3n}. Therefore, the cycle C_{3n} is an efficient strong and weak dominating (γ,eD)-graph and $es\gamma_{eD}(C_{3n}) = ew\gamma_{eD}(C_{3n}) = \gamma_{eD}(C_{3n})$. Hence, by Theorem 1.7, $es\gamma_{eD}(C_{3n}) = ew\gamma_{eD}(C_{3n}) = \left\lceil \frac{3n}{3} \right\rceil = n.$

Theorem 2.10: For cycle C_n, $n \equiv 1 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$, there is no efficient strong (weak) dominating (γ,eD)-set.
Proof: Let $C_n = \{v_1, v_2, v_3, \ldots, v_1\}$.
Case 1: $n \equiv 1$ (mod 3)
The graph C_n has more than one efficient dominating (γ,eD)-set. In each set, there exists elements u and v such that, $d(u,v) \leq 2$.
Sub Case 1a: d(u,v) = 1
Let $u,v \in S_i$ where S_i is one of the efficient dominating (γ,eD)-sets of C_n.
Therefore, $|N_s[u] \cap S_i| \neq 1$ ($|N_w[u] \cap S_i| \neq 1$) and $|N_s[v] \cap S_i| \neq 1$ ($|N_w[v] \cap S_i| \neq 1$). Hence, S_i's are not efficient strong (weak) dominating (γ,eD)-sets of C_n.
Sub Case 1b: d(u,v) = 2
Let $u,v \in S_i$ where S_i is one of the efficient dominating (γ,eD)-sets of C_n. Let $w \in C_n$ be the vertex which lies between u and v. Then, $|N_s[w] \cap S_i| \neq 1$ ($|N_w[w] \cap S_i| \neq 1$). Therefore, S_i's are not efficient strong (weak) dominating (γ,eD)-sets of C_n. Since every efficient strong(weak) dominating (γ,eD)-set is also a (γ,eD)-set, it is enough to prove that no (γ,eD)-set of C_n, $n \equiv 1 (\text{mod } 3)$, is an efficient strong(weak) dominating (γ,eD)-set of C_n. Since S_i is arbitrary, by subcases 1a and 1b, C_n, $n \equiv 1$ (mod 3) has no efficient strong(weak) dominating (γ,eD)-set.
Case 2: $n \equiv 2$ (mod 3)
In this case, the graph C_n has more than one (γ,eD)-set and in each set there exists u, v such that d(u, v) = 2. Let $u, v \in S_i$ where S_i is one of the (γ,eD)-sets of C_n. Let $w \in C_n$ be the vertex which lies between u and v. Then, $|N_s[w] \cap S_i| \neq 1$ ($|N_w[w] \cap S_i| \neq 1$). Therefore, S_i's are not efficient strong (weak) dominating (γ,eD)-sets of C_n. Therefore, C_n has no efficient strong (weak) dominating (γ,eD)-set.
Theorem 2.11: The star graph $K_{1,n}$ is not an efficient strong and weak dominating (γ,eD)-graph for all n≥2.
Proof: Let $G = K_{1,n}$; n≥2.
Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ and v be the central vertex. By Theorem 1.5, $S = \{v_1, v_2, \ldots, v_n\}$ is the minimum (γ,eD)-set of $K_{1,n}$ and $\gamma_{eD}(K_{1,n}) = n$.

(i) $N_s[v] = \{v\}$ and $v \notin S$ and so $|N_s[v]\cap S| \neq 1$. Therefore, S is not an efficient strong dominating (γ,eD)-set of $K_{1,n}$. Since every efficient strong dominating (γ,eD)-set is also a (γ,eD)-set, there exists no efficient strong dominating (γ,eD)-set for $K_{1,n}$. Therefore, star graph is not an efficient strong dominating (γ,eD)-graph.
(ii) $N_w[v] = \{v,v_1,v_2,...,v_n\}$ and so $|N_w[v]\cap S| = n \geq 2$. Therefore, S is not an efficient weak dominating (γ,eD)-set of $K_{1,n}$. Since every efficient weak dominating (γ,eD)-set is also a (γ,eD)-set. Therefore, star graph is not an efficient weak dominating (γ,eD) - graphs. Hence, star graph is not an efficient strong and weak dominating (γ,eD) - graph.

Theorem 2.12: The path P_n, $n \equiv 2 \pmod 3$ has efficient strong dominating (γ,eD)-set and $es\gamma_{eD}(P_n) = \gamma_{eD}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

Proof: Let $P_n = (v_1, v_2, v_3, \ldots, v_n)$. Then, $S=\{v_1, v_3, v_6, v_9,\ldots,v_{n-5}, v_{n-2},v_n\}$ is the unique minimum efficient strong dominating (γ,eD)-set of P_n which is also a minimum (γ,eD)-set. Therefore, by Theorem 1.6, $es\gamma_{eD}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

Theorem 2.13: The paths P_n, $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 3$ have no efficient strong dominating (γ,eD)-set.
Proof: Let $P_n = (v_1, v_2, v_3, \ldots, v_n)$.
Case (i): $n \equiv 0 \pmod 3$.
Let S be a (γ,eD)-set of P_n. Then, $|N_s[v_2]\cap S| \neq 1$ or $|N_s[v_{n-1}]\cap S| \neq 1$. Therefore, S is not an efficient strong dominating (γ,eD)-set of P_n. Since every efficient strong dominating (γ,eD)-set is also a (γ,eD)-set, it is enough to prove that no (γ,eD)-set of P_n, $n \equiv 0 \pmod 3$, is an efficient strong dominating (γ,eD)-set of P_n. Since S is arbitrary, P_n, $n \equiv 0 \pmod 3$ has no efficient strong dominating (γ,eD)-set.
Case (ii): $n \equiv 1 \pmod 3$.
In this case, $S = \{v_1, v_4, v_7, ..., v_n\}$ is the unique (γ,eD)-set of P_n and $|N_s[v_2]\cap S| = |N_s[v_{n-1}]\cap S| = 0$. Therefore, S is not an efficient strong dominating (γ,eD)-set. Therefore, proceeding in case (i) P_n, $n \equiv 1 \pmod 3$ has no efficient strong dominating (γ,eD)-set.
Theorem 2.14: The path P_n, $n \equiv 1 \pmod 3$ has an efficient weak dominating (γ,eD)-set and

$$ew\gamma_s(P_n) = \gamma_{eD}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2.$$

Proof: Let $P_n = (v_1, v_2, v_3, \ldots, v_n)$. Then, $S = \{v_1, v_4, v_7, ..., v_n\}$ is the unique (γ,eD)−set of P_n, which is also the minimum efficient weak dominating (γ,eD)-set of P_n. Therefore, by Theorem 1.6,

$$ew\gamma_{eD}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2.$$

Theorem 2.15: The paths P_n, $n \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$ have no efficient weak dominating (γ,eD)-set.
Proof: Let $P_n = (v_1, v_2, v_3, \ldots, v_n)$.
Case (i): $n \equiv 0 \pmod 3$
Let S be a (γ,eD)-set of P_n. Then, $|N_w[v_2]\cap S| \neq 1$ or $|N_w[v_{n-1}]\cap S| \neq 1$. Therefore, S is not an efficient weak dominating (γ,eD)-set of P_n. Since every efficient weak dominating (γ,eD)-set is also a (γ,eD)-set, it is enough to prove that no (γ,eD)-set of P_n, $n \equiv 0 \pmod 3$, is an efficient weak dominating (γ,eD)-set of P_n. Since S is arbitrary, P_n, $n \equiv 0 \pmod 3$ has no efficient weak dominating (γ,eD)-set.
Case (ii): $n \equiv 2 \pmod 3$
In this case also, P_n has more than one (γ,eD)-set. In each set the vertices either v_2 or v_{n-1} does not satisfy the condition $|N_w[v_i]\cap S| = 1$. Therefore, proceeding as in case (i), P_n, $n \equiv 2 \pmod 3$ has no efficient weak dominating (γ,eD)-set.
Theorem 2.16: The complete graph K_p, $p>2$ is an not efficient strong (weak) dominating (γ,eD)-graph.
Proof: Let $G = K_p$ be a graph and S be a (γ,eD)-set of G. Suppose $u,v \in S$ such that u and v are adjacent. (Since, in K_p any two vertices are adjacent). Therefore, $|N_s[v]\cap S| \neq 1 (|N_w[v]\cap S| \neq 1)$. So that, S is not an efficient strong (weak) dominating (γ,eD)-set of G. Hence, it is not an efficient strong (weak) dominating (γ,eD)-graphs.
Theorem 2.17: Complete bipartite graphs $K_{m,n}$ are not an efficient strong (weak) dominating (γ,eD)-graphs.
Proof:
Case (i): m = n = 1
Then, $K_{m,n} \cong K_2$. Therefore, by Theorem 2.16, $K_{m,n}$ is not an efficient strong(weak) dominating (γ,eD)-graph.
Case (ii): $n \geq 2$, $m = 1$
We get a star graph. By Theorem 2.11, $K_{m,n}$ is not an efficient strong(weak) dominating (γ,eD)-graph.
Case (iii): $m, n \geq 2$
Let $V(K_{m,n}) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m\}$. Let S be a (γ,eD)-set of $K_{m,n}$. Then, the vertices of V – S does not satisfy the condition $|N_s[v]\cap S| = 1$ and $|N_w[v]\cap S| = 1$ for all $v \in V - S$. Therefore, S is not an efficient strong(weak) dominating (γ,eD)-set of $K_{m,n}$. Since every efficient strong(weak) dominating (γ,eD)-set is also a (γ,eD)-set, it is enough to prove that no (γ,eD)-set of $K_{m,n}$, is an efficient strong(weak) dominating (γ,eD)-set of $K_{m,n}$. Since S is arbitrary, $K_{m,n}$ has no efficient strong(weak) dominating (γ,eD)-set. Hence, complete bipartite graphs $K_{m,n}$ are not efficient strong(weak) dominating (γ,eD)-graphs.
Theorem 2.18: The wheel graph $W_{1,p}$ $p \geq 4$ has no efficient strong and weak dominating (γ,eD)-graphs.
Proof: Let $V(W_{1,p}) = \{v,v_1,v_2, ...,v_p\}$ and v be the central vertex.
(i) $N_s[v] = \{v\}$ and so $|N_s[v]\cap S| = 0$. Therefore, S is not an efficient strong dominating (γ,eD)-set of $W_{1,p}$. Also, there exists no efficient strong dominating (γ,eD)-set for $W_{1,p}$. Therefore, wheel graphs are not an efficient strong dominating (γ,eD)-graphs.
(ii) $N_w[v] = \{v,v_1,v_2,...,v_p\}$ and so $|N_w[v]\cap S| = p - 2 \geq 2$. Therefore, S is not an efficient weak dominating (γ,eD)-set of $W_{1,p}$. Also, there exists no efficient weak dominating (γ,eD)-set for $W_{1,p}$. Therefore, wheel graphs are not an efficient weak dominating (γ,eD)-graphs.
Hence, wheel graphs are not an efficient strong and weak dominating (γ,eD)-graphs.
Remark 2.19: If S is an efficient strong dominating (γ,eD)-set of a connected graph G, then V-S is a dominating set of G.
Proof: Since, every efficient strong dominating (γ,eD)-set is independent and G is connected every vertex in S is adjacent to at least one vertex in V-S. Therefore, V-S is a dominating set of G.
Theorem 2.20: $K_{n,n} - 1F$ is an efficient strong dominating (γ, eD)-graph and

$es\gamma_{eD}(K_{n,n} - 1F) = 2$ for all $n \geq 3$.

Proof: Let $G = K_{n,n} - 1F$ and $V(G) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since we remove 1F from $K_{n,n}$, degree of each vertex is reduced to n-1. Each v_i is not adjacent to one u_j for all $i,j \geq 3$. Therefore, $\{v_i, u_j\}$ is an efficient strong dominating (γ, eD)-set. Hence, $es\gamma_{eD}(K_{n,n} - 1F) = 2$ for all $n \geq 3$.

Theorem 2.21: $[K_n]$ is an efficient strong dominating (γ, eD)-set,

$$es\gamma_{eD}[K_n] = p - \Delta([K_n]) = \frac{n^2 - 3n + 4}{2} \text{ where } p = |V([K_n])|.$$

Proof: Let $n \geq 3$. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. Let $G = [K_n]$. $V([K_n]) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\binom{n}{2}}\}$. By the definition of $[K_n]$, each u_i is adjacent to exactly two vertices of K_n. Therefore, $|V(K_n)| = p - n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{n^2+n}{2}$. $\Delta(G) = \deg(v_i)$, for any i = 1 to n. Each v_i is adjacent to the remaining $(n - 1)v_i$'s and $(n - 1)u_j$'s. Therefore, $\Delta(G) = (n - 1) + (n - 1) = 2n - 2$.

Total number of u_j's which are not adjacent to $v_i = \left(\frac{n^2 - n}{2}\right) - (n-1) = \frac{n^2 - 3n + 2}{2}$. These $\frac{n^2 - 3n + 2}{2}$. u_j's together with v_i form an efficient strong dominating (γ, eD)-set S of G. Therefore, G is an efficient strong dominating (γ, eD)-set, $|S| = 1 + \frac{n^2 - 3n + 2}{2}. = \frac{n^2 - 3n + 4}{2}$. Therefore, $es\gamma_{eD}[G] \leq \frac{n^2 - 3n + 4}{2}$.

Let T be any efficient strong dominating (γ, eD)-set S of G. Since T is independent, T can contain at most one v_i, $1 \leq i \leq n$. Since for $n \geq 3$, no u_j can strongly dominate any v_i, T contains at least one v_i, $(1 \leq i \leq n)$. Therefore T contains exactly one v_i. Any u_j can dominate only two v_i's and all u_j's are independent. Therefore, T contains all u_j's not adjacent with $v_i \in T$.

And $|T| \geq 1 + \left(\frac{n^2 - n}{2}\right) - (n-1) = \frac{n^2 - 3n + 4}{2}$. Therefore, $es\gamma_{eD}[G] \geq \frac{n^2 - 3n + 4}{2}$.

Hence, $es\gamma_{eD}[G] = \frac{n^2 - 3n + 4}{2}$.

Example 2.22: Consider the following graph $G = [K_4]$.

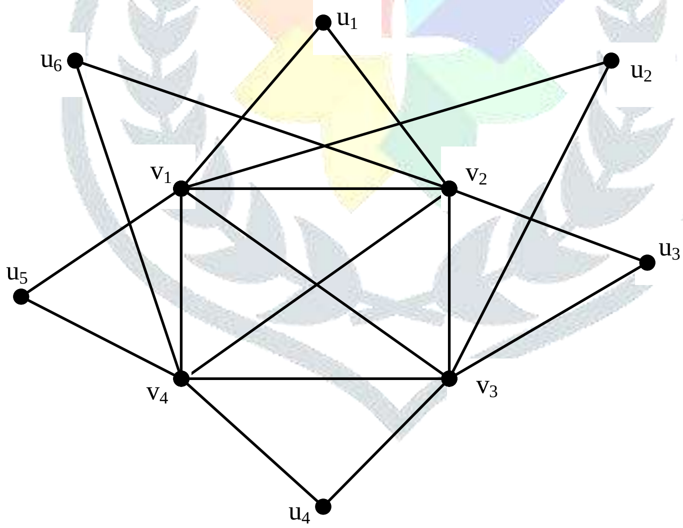


Figure 2.3

Here, $\{v_1, u_3, u_4, u_6\}$, $\{v_2, u_2, u_4, u_5\}$, $\{v_3, u_1, u_5, u_6\}$, $\{v_4, u_1, u_2, u_3\}$ are the efficient strong dominating (γ, eD)-set of G and p = 10, $\Delta(G) = 6$.

Hence, $es\gamma_{eD}[G] = \frac{n^2 - 3n + 4}{2} = 4 = p - \Delta(G)$.

3. Efficient Strong (Weak) Dominating (γ,eD)-Number of Subdivision Graphs

Theorem 3.1: $S(P_n)$ is an efficient strong dominating (γ,eD)-graph if $n \equiv 0 \pmod 3$ it is an efficient weak dominating (γ,eD)-graph if $n \equiv 1 \pmod 3$ and it is neither an efficient strong dominating (γ,eD)-graph nor an efficient weak dominating (γ,eD)-graph if $n \equiv 2 \pmod 3$.

Proof:

Case (i): $n \equiv 0 \pmod 3$

Here, $S(P_n)$ is some P_m with $m \equiv 2 \pmod 3$. Therefore, by Theorem 2.2, $S(P_n)$ where $n \equiv 0 \pmod 3$ is an efficient strong dominating (γ,eD)-graph.

Case (ii): $n \equiv 1 \pmod 3$

Here, $S(P_n)$ is some P_m with m≡1(mod 3). Therefore, by Theorem 2.14, $S(P_n)$ where n≡1(mod 3) is an efficient weak dominating (γ,eD)-graph.

Case (iii): n≡2 (mod 3)

Here, $S(P_n)$ is some P_m with m≡0(mod 3). Therefore, by Theorem 2.13(i) and 2.15, $S(P_n)$ where n≡2(mod 3) is neither an efficient strong dominating (γ,eD)-graph nor an efficient weak dominating (γ,eD)-graph.

Theorem 3.2: When n ≡ 0(mod 3), $S(C_n)$ is an efficient strong and weak dominating (γ,eD)-graph and $es\gamma_{eD}(S(C_n)) = ew\gamma_{eD}(S(C_n)) = 2n$ and it is neither an efficient strong dominating (γ,eD)-graph nor an efficient weak dominating (γ,eD)-graph if n ≡ 1 (mod 3) and n≡2 (mod 3).

Proof:

Case (i): n≡0 (mod 3)

Here, the graph $S(C_{3n})$ is also a cycle $C_{2(3n)}$. Therefore by Theorem 2.9, $S(C_{3n})$ is both an efficient strong and weak dominating (γ,eD)-graph and also $es\gamma_{eD}(S(C_{2(3n)})) = ew\gamma_{eD}(S(C_{2(3n)}))=2n$.

Case (ii): n≡1 (mod 3)

Here, the graph $S(C_n)$, n≡1(mod 3) is some C_m with m ≡ 2(mod 3). Therefore, by case (ii) of Theorem 2.13, $S(C_n)$, n≡1(mod 3) has no efficient strong (weak) dominating (γ,eD)-set.

Case (iii): n ≡ 2(mod 3)

Here, the graph $S(C_n)$, n≡2(mod 3) is some C_m with m≡1 (mod 3). Therefore, by case (i) of Theorem 2.13, $S(C_n)$, n≡2(mod 3) has no efficient strong (weak) dominating (γ,eD)-set.

Theorem 3.3: The subdivision graph $S(W_{1,p})$, $p \geq 3$ has no efficient strong and weak dominating (γ,eD)-set.

Proof: Let v be the central vertex of the graph $W_{1,p}$. Let $\{u_1, u_2, \ldots , u_p\}$ be the vertices which subdivide the edges of the outer cycle of the graph $W_{1,p}$. Then, $W = \{v, u_1, u_2, \ldots , u_p\}$ is the minimum edge detour dominating set of the graph $S(W_{1,p})$. Therefore, the edge detour domination number of $S(W_{1,p})$ is $\gamma_{eD}(S(W_{1,p})) = p + 1$.

Let $\{v_1, v_2, \ldots, v_p\}$ be the vertices of the outer cycle of the graph $S(W_{1,p})$ which are the original vertices of $W_{1,p}$ then $N_s[v_i] = \{v_i\}$ and $N_s([v_i]\cap S) = 0$ for all i=1,2,...,p. Therefore, $S(W_{1,p})$ has no efficient strong dominating (γ,eD)-set.

Also, $N_w([v_i]\cap S) = 2$ for all i=1,2,...,p. Therefore, $S(W_{1,p})$ has no efficient weak dominating (γ,eD)-set. Hence, $S(W_{1,p})$ has no efficient strong and weak dominating (γ,eD)-set.

Theorem 3.4: The subdivision of the star graph $S(K_{1,n})$, $n \geq 2$ has efficient strong and weak dominating (γ,eD)-set and $es\gamma_{eD}(S(K_{1,n})) = ew\gamma_{eD}(S(K_{1,n})) = n+1$.

Proof: Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ where v is the central vertex of the star and $\{u_1, u_2, \ldots, u_n\}$be the vertices which subdivide the n edges of the star graph. Then, $V((S(K_{1,n})) = \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $S=\{v, v_1, v_2, \ldots,v_n\}$ is the minimum edge detour dominating set of $S(K_{1,n})$. Therefore, $\gamma_{eD}(S(K_{1,n}) = n+1$.

(i)Here, $N_s[v] = \{v\}$ and so, $N_s([v]\cap S) = 1$. Then, for each v_i, $N_s[v_i] = \{ u_i, v_i\}$ and $N_s([v_i]\cap S)=1$ for all i=1,2,...,n. Also, for each u_i, $N_s[u_i] = \{ u_i, v\}$ and $N_s([u_i]\cap S)=1$ for all i=1,2,...,n. Therefore, for every vertex v of $S(K_{1,n})$ satisfy the condition $N_s([v]\cap S) = 1$. Hence, S is the minimum efficient strong dominating (γ,eD)-set and $es\gamma_{eD}(S(K_{1,n})) = n+1$.

(ii)Now, $N_w[v] = \{v, u_1, u_2, \ldots, u_n\}$ and so, $N_w([v]\cap S) = 1$. Then, for each v_i, $N_w[v_i]=\{v_i\}$ and $N_w([v_i]\cap S)=1$ for all i=1,2,...,n. Also, for each u_i, $N_w[u_i] = \{u_i, v_i\}$ and $N_w([u_i]\cap S)=1$ for all i=1,2,...,n. Therefore, for every vertex v of $S(K_{1,n})$ satisfy the condition $N_w([v]\cap S) = 1$. Hence, S is the minimum efficient weak dominating (γ,eD)-set and $ew\gamma_{eD}(S(K_{1,n})) = n+1$.

Hence, The subdivision of the star graph $S(K_{1,n})$, $n \geq 2$ has efficient strong and weak dominating (γ,eD)-set and $es\gamma_{eD}(S(K_{1,n})) = ew\gamma_{eD}(S(K_{1,n})) = n+1$.

Theorem 3.5: The subdivision graph $S(K_n)$, $n \geq 3$ has no efficient strong and weak dominating (γ,eD)-set.

Proof: Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let $\{u_1, u_2, \ldots,u_{\binom{n}{2}}\}$ be the vertices which subdivide the edges of K_n. Then, $V(S(K_n)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots,u_{\binom{n}{2}}\}$ and $S = \{v_1, v_2, \ldots, v_n\}$ is the minimum edge detour dominating set of K_n and $\gamma_{eD}(S(K_n) = n$.

(i) Here, $N_s([u_i]\cap S)=2$ for every i=1,2,..., $\binom{n}{2}$. Also, there exists no efficient strong dominating (γ,eD)-set for $S(K_n)$. Therefore, $S(K_n)$ has no efficient strong dominating (γ,eD)-set.

(ii) Here, $N_w[u_i] = \{u_i\}$ and $N_w([u_i]\cap S)=0$ for every i=1,2,..., $\binom{n}{2}$. Also, there exists no efficient weak dominating (γ,eD)-set for $S(K_n)$. Therefore, $S(K_n)$ has no efficient weak dominating (γ,eD)-set.

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