APPLICATIONS OF MOVING PLANE METHOD IN PROVING SYMMETRY OF SOLUTIONS OF BIHARMONIC EQUATIONS

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Abstract: The aim of this paper is to study the symmetry properties of solutions of bi-harmonic differential equations of the type

$$\Delta^2 \boldsymbol{u} + \boldsymbol{\alpha} \boldsymbol{u} = \mathbf{0}$$

and

$$\Delta^2 \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}) = \mathbf{0} \qquad \boldsymbol{in} \qquad \boldsymbol{\Omega} \subset \boldsymbol{R}^2$$

We employ the method of moving planes, which is based on the Maximum principles in bounded domains to obtain the result of symmetry of solutions of the bi-harmonic problems.

Index Terms **– Moving Plane Method, Symmetry, Bi-harmonic equations, Maximum Principles.**

I. INTRODUCTION

In recent years, a lot of interest has been shown in the study of symmetry properties of solutions of nonlinear elliptic equations, reflecting the symmetry of the domain. Linear elliptic equations arise in several models describing various phenomena in the applied sciences, Maximum principles have been some of the most useful properties used to solve a wide range of problems in the study of partial differential equations over the years. Starting from the basic fact from calculus that if a function f(x) satisfies $f'' > 0$ on an interval [a , b], then it can only achieve its maximum on the boundary of that interval. For partial differential equations, the same idea allows to draw very useful conclusions from the properties of the solutions and the domain of a given problem. We will look over some results such as the Hopf Maximum Principle and its generalization, approximations and uniqueness of solution for elliptic operators.

It is well known that a classical tool to study this question is the moving plane method which goes back to Alexandrov and Serrin [8] and was successfully used by Gidas-Ni-Nirenberg in the famous paper [2] to prove the radial symmetry of positive solutions to (1.1) when B is a ball and f has some monotonicity in the radial variable. Since the last four decades or so, " the method of moving planes" has been numerous applications in studying non linear partial differential equations See [1 , 6 , 7, 10 ,11, 12 , 13]. It can be used to prove symmetry of solutions. It is an important goal in mathematical analysis to establish symmetry properties of solutions of differential equations both from a theoretical point of view and for the applications.

To prove the symmetry J. Serrin introduced the method of moving plane in the differential equations, which has been previously used by A. D. Alexandrov in differential geometry. After some years the same method was employed by Gidas, Ni and Nirenberg to obtain the symmetry results and monotonicity for positive solutions of nonlinear elliptic equations. Moving plane method has been improved and simplified by Berestycki and Nirenberg in [14] with the aid of maximum principle in small omain. After that many other results followed with different operators, different boundary conditions, different geometries. In his paper [5] D. B. Dhaigude proved the Maximum principles for fourth order semilinear elliptic equations; He also stated result by Dunninger which we are now going to use. [4] Author D P Patil studied elliptic boundary value problems.

In this paper, we will denote an open bounded domain in R^N with C^1 boundary. We will say that is strictly convex if for all $x;\ y \in \Omega$ and for all $t \in (0,1)$, $(1-t)x + ty \in \Omega$ Remark that some symmetry results for solutions of elliptic partial .

In section (2) we state the theorem and the preliminary results and statements of main theorems. In section (3) we state and prove some useful lemmas required to prove the theorem 2.4. We prove the theorem for equation

$$\Delta^2 u + \alpha\, u = 0.$$

In section (4) we state and prove some useful lemmas required to prove the theorem 2.5, we prove the theorem for equation

$$\Delta^2 u + f(u) = 0.$$

2. Preliminaries and Main Result:

Before proceeding to the statement of our main result we shall set forth some preliminaries and hypotheses.

Theorem 2.1 [9] Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a non constant solution of

$$\Delta^2 u + \alpha\, u = 0 \qquad \Omega \subset R^n,\ \alpha \in R \qquad 2.1.$$

$$\boldsymbol{\Delta u = 0} \qquad \boldsymbol{\partial\,\Omega} \qquad \mathbf{2.2}$$

then u satisfies the maximum principle.

Theorem 2.2 [Dunninger (3)] The non constant solution u of

$$\Delta^2 u + C\, u = 0 \qquad C > 0 \quad in \quad \Omega \subset R^n$$.

$$\boldsymbol{\Delta\, u = 0 \qquad \partial\, \Omega}$$

Satisfies the inequality

$$|u(x)| \le |u(x_0)| \qquad\qquad x \in \bar{\Omega}$$

for some point x_0 on the boundary $\partial\Omega$ of Ω.

Theorem 2.3:[Serrin [7]] Let u = $u(x_1 ,\ x_2 , x_3 , \ldots\ldots , x_n)$ be a non-constant solution of

$$\Delta^2 u + \alpha\, u = 0 \qquad 2.3.$$

where α is positive constant and $f(u)$ is positive non decreasing, differentiable function; and if $\Delta\, u = 0$ on $\partial\, \Omega$, then u attains its maximum on $\boldsymbol{\partial\, \Omega}$.

Theorem 2.4 Let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be a non constant solution of

$$\Delta^2 u + \alpha\, u = 0 \qquad |\alpha| < 1 ,\ \alpha \in R \qquad in\, \Omega \subset R^n \qquad 2.4.$$

$$\boldsymbol{\Delta\, u = 0 \quad on \quad \partial\, \Omega} \qquad \mathbf{2.5}$$

$$u(x) \to 0 \qquad as \qquad |x| \to \infty \qquad 2.6$$

Define $U(r) = \sup\{(u) : |x| \ge R\}$

$\Phi\, (r) = 1 \quad B(r) = \{\, x \in R^n : |x| < r_0\}$

Assume that there exists a positive function w on $|x| \ge R_0$ for some $R_0 > 0$ satisfying

$$\Delta^2 u + \Phi(|x|)w \le 0 \qquad in \quad |x| > R_0 \qquad 2.7$$

$$\Delta\, u = 0 \qquad on\ |x| > R_0 \qquad 2.8$$

$$\lim_{|x|\to\infty} \frac{U(|x|)}{w(x)} = 0 \qquad 2.9$$

Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r \le 0\ for\ R_0 \ge 0$

We shall prove this theorem in section (3).

Theorem 2.5 Let

$$\Delta^2 u + f(\, u) = 0 \qquad in\, \Omega \subset R^n \qquad 2.10$$

$$\boldsymbol{\Delta\, u = 0 \quad on \quad \partial\, \Omega} \qquad \mathbf{2.11}$$

$$u(x) \to 0 \qquad as \qquad |x| \to \infty \qquad 2.12$$

Assume that f(u) is positive non decreasing and differentiable function.

Let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$

Define $U(r) = \sup\{(u) : |x| \ge R\}$

Then u must be radially symmetric about some point $x_0 \in \Omega$ and $u_r \le 0\ for\ r > 0$

We shall prove this theorem in section (4).

Before proceeding to the main theorems we shall set forth some preliminaries and hypothesis.

3. Lemmas and Proof of theorem 2.4

Let $\lambda > 0$ be a real number.

Define $T_\lambda = \{\, x :\ x = (x_1 , x_2 , x_3 , \ldots , x_n) : x_1 = \lambda\, \}$ which is the plane perpendicular to x_1 axis . We will move this plane continuously normal to itself to new position till it begins to intersect the region Ω . After that point the plane advances in Ω along X_1 - axis and cut off cap $\sum_\lambda$; which is the portion of Ω , and lies in the same side of the plane T_λ as the original plane T. Let $x^\lambda = (\, 2\lambda - x_1 , x_2 , x_3 , \ldots , x_n)$ be the reflection of the point $x = (x_1 , x_2 , x_3 , \ldots , x_n)$ about the plane T_λ.

Define $V_\lambda\, (x) = u(x) - \left(u(x)^\lambda\right)$.

We have $\left|x^\lambda\right| \ge |x|$ and $u(x) = u\ (x_1 , x_2 , x_3 , \ldots , x_n)$, $u(x^\lambda) = u(\, 2\lambda - x_1 , x_2 , x_3 , \ldots , x_n)$

By simple calculations we can obtain

$$\Delta^2 u\left(x^\lambda\right) = \Delta^2 u(x)$$

Lemma 3.1 Let $\boldsymbol{\lambda} > 0$ then $\boldsymbol{V_\lambda}$ satisfies $\Delta^2\, \boldsymbol{V_\lambda} + \boldsymbol{C_\lambda(x) V_\lambda} = \mathbf{0}$ in $\sum_\lambda$ where $\boldsymbol{C_\lambda(x) = \alpha}$

Proof: We have $\left|x^\lambda\right| \ge |x|$ for $x \in \sum_\lambda$

Also

$$\Delta^2 u(x) + \alpha\, u(x) = 0 \qquad 3.1$$

$u\left(x^\lambda\right)$ satisfies the same equation that u(x) does

$$\Delta^2 u\, (x^\lambda) + \alpha\, u\, (x^\lambda) = 0 \qquad 3.2$$

Subtracting

$$\begin{aligned} 0 &= [\Delta^2 u(x) + \alpha\, u(x)] - [\, \Delta^2 u\, (x^\lambda) + \alpha\, u\, (x^\lambda)] \\ &= [\Delta^2 u(x) - \Delta^2 u\, (x^\lambda)] - [\, \alpha\, u(x) - \alpha\, u\, (x^\lambda)] \\ &= \Delta^2 (u(x) - u\, (x^\lambda)) - \alpha\, (u(x) - u\, (x^\lambda)) \end{aligned}$$

$$0 = \Delta^2 \boldsymbol{V_\lambda} + \boldsymbol{C_\lambda(x)V_\lambda(x)}$$

where $\boldsymbol{C_\lambda(x) = \alpha}$ bounded. □

Define $\Lambda = \{ \lambda \in (0, \infty) : \boldsymbol{V_\lambda(x)} > 0 \quad in \ \sum_\lambda \}$

Lemma 3.2 Let $\boldsymbol{\lambda} > 0$ **.** If $\boldsymbol{V_\lambda} > 0$ on $\sum_\lambda \cap \overline{B_0}$ then $\lambda \in \Lambda$.

Proof: Let $\boldsymbol{\lambda} > 0$

$\therefore \lambda \in (0, \infty)$ and $\boldsymbol{V_\lambda} > 0$ on $\sum_\lambda \cap \overline{B_0}$.

By Lemma 3.1

$$\Delta^2 \boldsymbol{V_\lambda(x)} + \boldsymbol{C_\lambda(x)V_\lambda(x) = 0} \quad \boldsymbol{in} \ \ \textstyle\sum_\lambda \backslash \overline{B_0}$$
$$\boldsymbol{V_\lambda(x)} > 0 \quad on \ \ \partial(\textstyle\sum_\lambda \backslash \overline{B_0})$$

Since U(r) is non-increasing we have

$$0 \le u(x^\lambda) + t\left(u(x) - u(x^\lambda)\right) \le U(r) \quad \text{for } 0 \le t \le 1.$$

We have $C_\lambda(x) = \int_0^1 \alpha \, dt = \alpha = 1 \le \Phi(|x|)$

From $\Delta^2 w + \Phi(|x|)w \ge 0$ in $|x| \ge R_0$

$\Delta w = 0$ on $|x| = R_0$

The positive function w satisfies the equation

$\Delta^2 w + \Phi(|x|)w \le 0$ in $\sum_\lambda \backslash \overline{B_0}$

$\Delta w = 0$ on $\sum_\lambda \backslash \overline{B_0}$

Hence by maximum principle we have

$\boldsymbol{V_\lambda(x)} > 0$ **in** $\sum_\lambda \backslash \overline{B_0}$.

$\therefore \lambda \in \Lambda$. □

Lemma 3.3 Let $\lambda \in \Lambda$ then $\frac{\partial u}{\partial x_1} < 0$ on T_λ.

Proof: By lemma [3.1] we have

$$\Delta^2 \boldsymbol{V_\lambda(x)} + \boldsymbol{\alpha V_\lambda(x) = 0} \quad \boldsymbol{in} \ \ \textstyle\sum_\lambda$$
$$\boldsymbol{V_\lambda(x)} > 0 \quad on \ \ \partial(\textstyle\sum_\lambda)$$

But $\boldsymbol{V_\lambda(x) = 0}$ $\boldsymbol{on}$ T_λ

We have $\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ.

By Hopf boundary lemma,

$\frac{\partial u}{\partial x_1} = \frac{1}{2} \frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ. □

Proof of theorem 2.4

Let u(x) is positive non constant solution of boundary value problem [2.4 , 2.5] and $\lim_{|x| \to \infty} u(x) = 0$ then there exist $R_1 > R_0$ such that

$$\max\{ u(x) : |x| > R_1 \} < \min\{ u(x) : |x| \le R_0 \}$$

where R_0 is the constant taken in the theorem. We prove the theorem in following steps.

Step 1: To prove $[\boldsymbol{R_1}, \infty) \subset \boldsymbol{\Lambda}$ **.**

Let $\lambda \in [\boldsymbol{R_1}, \infty)$ **.**

$\therefore \lambda \ge R_1$ and we have $\overline{B_0} \subset \sum_\lambda$.

$\boldsymbol{V_\lambda(x)} > 0$ **in** $\overline{B_0}$.

By lemma 3.2 $\lambda \in \Lambda$.

$\therefore [\boldsymbol{R_1}, \infty) \subset \boldsymbol{\Lambda}$ **.**

Step 2: Let $\boldsymbol{\lambda_0} \in \boldsymbol{\Lambda}$**.** To prove that there exist $\boldsymbol{\epsilon} > 0$ such that $(\lambda_0 - \epsilon, \lambda_0] \subset \Lambda$ **.**

We shall prove this by method of contradiction. If possible suppose that there exist an increasing sequence, $\{\lambda_i\}$, i = 1 , 2 , 3 , …. Such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$. By converse of the lemma [3.2] we have a sequence $\{x_i\}$, i = 1 , 2 , 3 , …. Such that $x_i \in \sum_\lambda \cap \overline{B_0}$ and $V_{\lambda_i}(x_i) \le 0$. A subsequence which we call again $\{X_i\}$, converges to some point $x_0 \in \overline{\sum_{\lambda_0}} \cap \overline{B_0}$.

Then $V_{\lambda 0} \le 0$.

Since $V_{\lambda 0} > 0$ in $\sum_{\lambda 0}$, we must have $x_0 \in T_{\lambda 0}$.

By mean value theorem, we observe that there exists a point y_i satisfying $\frac{\partial u}{\partial x_1}(y_i) \ge 0$, on the straight segment joining x_i to ${x_i}^{\lambda i}$ for i = 1 , 2 , 3 , ...

Since $y_i \to x_0$ as $i \to \infty$, we have $\frac{\partial u}{\partial x_1}(x_0) \ge 0$.

On the other hand since $x_0 \in T_{\lambda 0}$ we have $\frac{\partial u}{\partial x_1}(x_0) < 0$.

By lemma [3.3] this is a contradiction and step 2 is established.

Step 3: To prove either statement (A) or statement (B) holds.
(A) $V_\lambda(x) > 0$ for $\lambda_1 > 0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$.
(B) $V_\lambda(x) > 0$ in $\sum_0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$.
Define $\lambda_1 = \inf\{\lambda > 0 : (\lambda, \infty) \subset \Lambda\}$ then either $\lambda_1 > 0$ or $\lambda_1 = 0$.
Case 1 $\lambda_1 > 0$
We have $V_{\lambda_1}(x) = u(x) - \left(u(x)^{\lambda_1}\right)$
From the continuity of the function u, we have

$$V_{\lambda_1}(x) > 0 \quad in \quad \textstyle\sum_{\lambda_1} .$$

Hence by strong maximum principle we have that either $V_{\lambda_1}(x) > 0$ in $\sum_{\lambda_1}$ or $V_{\lambda_1}(x) = 0$ in $\sum_{\lambda_1}$
Suppose that $V_{\lambda_1}(x) > 0$ in $\sum_{\lambda_1}$, then $\lambda_1 \in \Lambda$.
From step 2 there exist $\epsilon > 0$ such that $(\lambda_1 - \epsilon, \lambda_1] \subset \Lambda$.
This contradicts to the definition of λ_1.

$$\therefore \quad V_{\lambda_1}(x) = 0 \quad in \quad \textstyle\sum_{\lambda_1}$$

$$\therefore u(x) = u\left(x^{\lambda 1}\right) \quad in \quad \textstyle\sum_{\lambda_1}$$

Since $[\lambda_1, \infty) \subset \Lambda$ we have $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $> \lambda_1$. By lemma [3.3]
Thus we get statement (A).
Case 2 $\lambda_1 = 0$
Since u is continuous and $\lim_{|x|\to\infty} u(x) = 0$ we have $u(x) \geq u(x^0)$ in $\sum_0$
By lemma [3.3] $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > 0$.
Thus statement (B) occurs.
If statement (B) occurs in step 3 we can repeat the previous steps 1, 2, and 3 for the opposite X_1 direction about some plane
$x_1 = \lambda_1 < 0$ or u(x) < u(x^0) in $\sum_0$
Therefore
u(x) < u(x^0) in $\sum_0$
Therefore, u must be radially symmetric in X_1 direction about some plane and strictly decreasing away from the plane.
Since we can place X_1axis along any direction, we can conclude that u is radially symmetric.

4. Proof of theorem 2.5

Before proving the theorem we shall prove some lemmas which are required in the proof of the theorem.
Lemma 4.1 Let $\lambda > 0$ then V_λ satisfies $\Delta^2 \boldsymbol{V_\lambda} + \boldsymbol{C_\lambda(x) V_\lambda} = \mathbf{0}$ in $\sum_\lambda$ where $\boldsymbol{C_\lambda(x)} = \int_0^1 \boldsymbol{f_u}\left(\boldsymbol{u(x)} + \boldsymbol{t}\left(\boldsymbol{u(x)} - \boldsymbol{u(x^\lambda)}\right)\right) \boldsymbol{dt}$
Proof : Let

$$\Delta^2 u + f(u) = 0$$

and $(x) = u\ (x_1, x_2, x_3, \ldots, x_n)$. Let $\lambda > 0$ be a real number.
Define $T_\lambda = \{x : x = (x_1, x_2, x_3, \ldots, x_n) : x_1 = \lambda\}$ which is the plane perpendicular to x_1 axis . We will move this plane continuously normal to itself to new position till it begins to intersect the region Ω . After that point the plane advances in Ω along X_1 - axis and cut off cap $\sum_\lambda$; which is the portion of Ω , and lies in the same side of the plane T_λ as the original plane T. Let $x^\lambda = (2\lambda - x_1, x_2, x_3, \ldots, x_n)$ be the reflection of the point $x = (x_1, x_2, x_3, \ldots, x_n)$ about the plane T_λ.
Define $V_\lambda(x) = u(x) - \left(u(x)^\lambda\right)$.
We have $|x^\lambda| \geq |x|$ and $u(x) = u\ (x_1, x_2, x_3, \ldots, x_n)$, $u(x^\lambda) = u(2\lambda - x_1, x_2, x_3, \ldots, x_n)$
By simple calculations we can obtain

$$\Delta^2 u\left(x^\lambda\right) = \Delta^2 u(x)$$

$u\left(x^\lambda\right)$ satisfies the same equation that u(x) does

$$\Delta^2 u\ (x^\lambda) + f(u\ (x^\lambda)) = 0$$

On subtracting we obtain

$$0 = [\Delta^2 u(x) + f(u(x))] - [\Delta^2 u\ (x^\lambda) + f(u\ (x^\lambda))]$$

$$= [\Delta^2 u(x) - \Delta^2 u\ (x^\lambda)] - [f\left(u(x)\right) - f * \left(u\ (x^\lambda)\right)]$$

$$= \Delta^2(u(x) - u\ (x^\lambda)) - \frac{f\left(u(x)\right) - f\left(u\left(x^\lambda\right)\right)}{u(x) - u(x^\lambda)}[u(x) - u(x^\lambda)]$$

$$0 = \Delta^2 \boldsymbol{V_\lambda} + \boldsymbol{C_\lambda(x) V_\lambda(x)}$$

where $\boldsymbol{C_\lambda(x)} = \frac{f\,(u(x)) - f\left(u\,(x^\lambda)\right)}{u(x) - u(x^\lambda)} = \int_0^1 f_u\left(u(x^\lambda) + t[u(x) - u(x^\lambda)]\right)dt$
From $\Delta^2 w + w \geq 0$ in $|x| \geq R_0$

$\Delta\, w = 0$ on $|x| = R_0$

The positive non decreasing function u satisfies the equation

$\Delta^2\, \boldsymbol{w} + \boldsymbol{C_\lambda(x) w} \geq \boldsymbol{0}$ in $\sum_\lambda \backslash \overline{B_0}$

$\Delta\, w = 0$ on $\sum_\lambda \backslash \overline{B_0}$

Hence by maximum principle we have

$\boldsymbol{V_\lambda(x)} > 0$ **in** $\sum_\lambda \backslash \overline{B_0}$.

$\therefore \lambda \in \Lambda$. □

Lemma 4.2: Let > 0 **.** If $\boldsymbol{V_\lambda} > 0$ on $\sum_\lambda \cap \overline{B_0}$ then $\lambda \in \Lambda$.

Proof: Let $\boldsymbol{\lambda} > 0$ and $\boldsymbol{V_\lambda} < 0$ on $\sum_\lambda \cap \overline{B_0}$, then by lemma [4.1] and assumptions we have

$$\Delta^2\, \boldsymbol{V_\lambda(x)} + \boldsymbol{C_\lambda(x) V_\lambda(x)} = \boldsymbol{0} \quad \boldsymbol{in} \quad \sum_\lambda \backslash \overline{B_0}$$
$$\boldsymbol{V_\lambda(x)} > 0 \quad on \quad \partial(\sum_\lambda \backslash \overline{B_0})$$

Since U(r) is non-increasing we have

$$0 \leq u(x^\lambda) + t\left(u(x) - u(x^\lambda)\right) \leq U\,(\,|x|\,) \quad \text{for } 0 \leq t \leq 1.$$

We have $C_\lambda\,(\,x) = \int_0^1 f_u\left[u(x^\lambda) + t\left(u(x) - u(x^\lambda)\right)\right] dt \;\leq\; \int_0^1 f_u\, U(|x|) dt \;=\; \Phi\,(|x|)\; in\; \sum_\lambda$.

Where $\Phi\,(|x| = \sup\{\, f_u\,(\,r\,, s\,) : 0 \leq s \leq U(r)\}$

From $\Delta^2\, w + \Phi\,(\,|x|)w \geq 0$ in $|\,x| \geq R_0$

$\Delta\, w = 0$ on $|\,x| = R_0$

The positive non decreasing function w satisfies the equation

$\Delta^2\, w + \Phi\,(\,|x|)w \leq 0$ in $\sum_\lambda \backslash \overline{B_0}$

$\Delta\, w = 0$ on $\sum_\lambda \backslash \overline{B_0}$

Hence by maximum principle we have

$\boldsymbol{V_\lambda(x)} > 0$ **in** $\sum_\lambda$

$\therefore \lambda \in \Lambda$. □

Lemma 4.3: Let $\lambda \in \Lambda$ then $\frac{\partial u}{\partial\, x_1} < 0$ on T_λ.

Proof: Let $\lambda \in \Lambda$

$$\therefore \lambda \in (\,0\,, \infty)$$
$$\therefore \lambda > 0$$

By lemma [4.1] we have

$$\Delta^2\, \boldsymbol{V_\lambda(x)} + \boldsymbol{C_\lambda(x)\, V_\lambda(x)} \geq \boldsymbol{0} \quad \boldsymbol{in} \quad \sum_\lambda$$
$$\boldsymbol{V_\lambda(x)} > 0 \quad in \quad (\sum_\lambda)$$

But $\boldsymbol{V_\lambda(x) = 0}$ $\boldsymbol{on}$ T_λ

We have $\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ.

By Hopf boundary lemma,

$\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ. □

Proof of theorem 2.5:

Let u(x) is positive and and $\lim_{|x|\to\infty} u(x) = 0$ then there exist $R_1 > R_0$ such that

$$\max\{\, u(x) : |x| > R_1 \,\} < \min\{\, u(x) : |x| \leq R_0 \,\}$$

where R_0 is the constant taken in the theorem. We prove the theorem in following steps.

Step 1: To prove $[\boldsymbol{R_1}\,, \infty\,) \subset \boldsymbol{\Lambda}$ **.** We shall prove this step by method of contradiction.

Let $\lambda \notin [\boldsymbol{R_0}\,, \infty\,)$ **.**

$\therefore \lambda < R_0$ and we note that $\overline{B_0} \subset \sum_\lambda$.

Since $u(x) \leq u\,(x^\lambda)$ we have

$\boldsymbol{V_\lambda(x)} \leq 0$ **in** $\overline{B_0}$.

Which is a contradiction.

$\therefore \lambda \in \Lambda$.

$\therefore [\boldsymbol{R_1}\,, \infty\,) \subset \boldsymbol{\Lambda}$ **.**

Step 2: To prove if $\boldsymbol{\lambda_0} \in \boldsymbol{\Lambda}$**,** then there exist $\boldsymbol{\epsilon} > 0$ such that $(\lambda_0 - \epsilon\,,\, \lambda_0\,] \subset \Lambda$ **.**

Assume to the contrary that there exist an increasing sequence, $\{\lambda_i\}$, i = 1 , 2 , 3 , …. Such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$. By converse of the lemma [4.2] we have a sequence $\{x_i\}$, i = 1 , 2 , 3 , …. Such that $x_i \in \sum_\lambda \cap \overline{B_0}$ and $V_{\lambda_i}(x_i\,) \geq 0$. A subsequence which we call again $\{X_i\}$, converges to some point $x_0 \in \overline{\sum_{\lambda_0}} \cap \overline{B_0}$.

Then $V_{\lambda 0} \geq 0$.

Since $V_{\lambda 0} > 0$ in $\sum_{\lambda 0}$, we must have $x_0 \in T_{\lambda 0}$.

By mean value theorem, we observe that there exists a point y_i satisfying $\frac{\partial u}{\partial x_1}(y_i) \geq 0$, on the straight segment joining x_i to ${x_i}^{\lambda i}$ for i = 1 , 2 , 3 , ... Since $y_i \rightarrow x_0$ as $i \rightarrow \infty$, we have $\frac{\partial u}{\partial x_1}(x_0) \geq 0$.
On the other hand since $x_0 \in T_{\lambda 0}$ we have $\frac{\partial u}{\partial x_1}(x_0) < 0$.
By lemma [4.3] this is a contradiction and step 2 is established.
Step 3: To prove either statement (A) or statement (B) holds.
(A) $u(x) = u(x^{\lambda 1})$ for $\lambda_1 > 0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$.
(B) $u(x) > u(x^{\lambda 1})$ in $\sum_0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$.
Define $\lambda_1 = \inf\{\lambda > 0 : (\lambda, \infty) \subset \Lambda\}$ then either $\lambda_1 > 0$ or $\lambda_1 = 0$.
Case 1 $\lambda_1 > 0$
We have $V_{\lambda_1}(x) = u(x) - \left(u(x)^{\lambda_1}\right)$
From the continuity of the function u, we have

$$V_{\lambda_1}(x) > 0 \quad in \quad \sum_{\lambda_1}.$$

By lemma [4.1] we have $\Delta^2 \boldsymbol{V_\lambda(x)} + \boldsymbol{C_\lambda(x) V_\lambda(x)} \geq \boldsymbol{0} \quad \boldsymbol{in} \quad \sum_\lambda$.

Hence by strong maximum principle we have that either $V_{\lambda_1}(x) > 0 \quad in \quad \sum_{\lambda_1}$ or $V_{\lambda_1}(x) = 0 \quad in \quad \sum_{\lambda_1}$
Suppose that $V_{\lambda_1}(x) > 0 \quad in \quad \sum_{\lambda_1}$, then $\lambda_1 \in \Lambda$.
From step 2 there exist $\epsilon > 0$ such that $(\lambda_1 - \epsilon, \lambda_1] \subset \Lambda$.
This contradicts to the definition of λ_1.

$$\therefore \quad V_{\lambda_1}(x) = 0 \quad in \quad \sum_{\lambda_1}$$
$$\therefore u(x) = u\left(x^{\lambda 1}\right) \quad in \quad \sum_{\lambda_1}$$

Since $[\lambda_1, \infty) \subset \Lambda$ we have $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $> \lambda_1$. By lemma [4.3]
Thus we get statement (A).
Case 2 $\lambda_1 = 0$
Since u is continuous and $\lim_{|x| \to \infty} u(x) = 0$ we have $u(x) \geq u(x^0) \quad in \quad \sum_0$
By lemma [4.3] $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > 0$.
Thus statement (B) occurs.
If statement (B) occurs in step 3 we can repeat the previous steps 1, 2, and 3 for the opposite X_1 direction about some plane
$x_1 = \lambda_1 < 0$ or u(x) < u(x^0) in $\sum_0$
Therefore
u(x) < u(x^0) in $\sum_0$
Therefore, u must be radially symmetric in X_1 direction about some plane and strictly decreasing away from the plane.
Since we can place X_1axis along any direction, we can conclude that u is radially symmetric about origin.

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