Tg^n SPACES IN BITOPOLOGY

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1.1 INTRODUCTION

Levine introduced the notion of $T_{1/2}$-spaces which properly lie between T_1-spaces and T_0-spaces. Many authors studied properties of $T_{1/2}$-spaces: Dunham [9], Arenas et al. [4] etc. In this chapter, we introduce the notions called Tg^n -spaces, ${}_g$Tg^n -spaces and α Tg^n -spaces and obtain their properties and characterizations.

1.2 ABSTRACT

Bitopological spaces are equipped with two arbitrary topologies.

In this paper, Tg^n -spaces, ${}_g$Tg^n -spaces and α Tg^n -spaces are introduced in a bitopological space and their properties are investigated.

1.3 keywords

Tg^n -spaces, ${}_g$Tg^n -spaces , α Tg^n -spaces

1.4 PRELIMINARIES

Throughout this thesis (X, τ) (or X) represent topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition 1.4.1

A subset A of a space (X, τ) is called:

(i) semi-open set [11] if A $\subseteq$ cl(int(A));

(ii) preopen set [13] if A $\subseteq$ int(cl(A));

(iii) α -open set [14] if A $\subseteq$ int(cl(int(A)));

(iv) β-open set [1] (= semi-preopen [3]) if A $\subseteq$ cl(int(cl(A))).

The complements of the above mentioned open sets are called their respective closed sets.

The preclosure [15] (resp. semi-closure [6], α -closure [7], semi-pre-closure [2]) of a subset A of X, denoted by pcl(A) (resp. scl(A), α cl(A), spcl(A)), is defined to be the intersection of all preclosed (resp. semi-closed, α -closed, semi-preclosed) sets of (X, τ) containing A. It is known that pcl(A) (resp. scl(A), α cl(A), spcl(A)) is a preclosed (resp. semi-closed, α -closed, semi-preclosed) set. For any subset A of an arbitrarily chosen topological space, the semi-interior [6] (resp. α -interior [7], preinterior [15]) of A, denoted by sint(A) (resp. α int(A), pint(A)), is defined to be the union of all semi-open (resp. α -open, preopen) sets of (X, τ) contained in A.

Definition 1.4.2

A subset A of a space (X, τ) is called:

(i) a generalized closed (briefly g-closed) set [10] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of g-closed set is called g-open set;

(ii) a generalized semi-closed (briefly gs-closed) set [5] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of gs-closed set is called gs-open set;

(iii) an α -generalized closed (briefly α g-closed) set [12] if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of α g-closed set is called α g-open set;

(iv) a generalized semi-preclosed (briefly gsp-closed) set [15] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of gsp-closed set is called gsp-open set;

(v) a ĝ-closed set [17] (= ω-closed [16]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ). The complement of ĝ-closed set is called ĝ-open set;

(vi) a g^n -closed set [3] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is gs-open in (X, τ). The complement of g^n -closed set is called g^n -open set;

(vii) a g*-preclosed (briefly g*p-closed) set [18] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in (X, τ). The complement of g*p-closed set is called g*p-open set.

The collection of all g^n -closed (resp. ω-closed, α g-closed, gsp-closed, gs-closed, α-closed, g*p-closed) sets is denoted by G^n C(X) (resp. ωC(X), αGC(X), GSPC(X), GSC(X), αC(X), G^*PC(X)).

The collection of all g^n -open (resp. ω-open, α g-open, gsp-open, gs-open, α-open, g*p-open) sets is denoted by G^n O(X) (resp. ωO(X), αGO(X), GSPO(X), GSO(X), αO(X), G^*PO(X)).

We denote the power set of X by P(X).

Definition 1.4.3

A space (X, τ) is called:

(i) $T_{1/2}$-space [10] if every g-closed set is closed.

(ii) T_b-space [8] if every gs-closed set is closed.

(iii) αT_b-space [7] if every α g-closed set is closed.

(iv) T_ω-space [16] if every ω-closed set is closed.

(v) T_p*-space [18] if every g*p-closed set is closed.

(vi) $*_sT_p$-space [18] if every gsp-closed set is g*p-closed.

(vii) αT_d-space [7] if every α g-closed set is g-closed.

(viii) α-space [14] if every α-closed set is closed.

Definition 1.4.4 [45]

Let (X, τ) be a topological space and A $\subseteq$ X. We define the gs-closure of A (briefly gs-cl(A)) to be the intersection of all gs-closed sets containing A.

Remark 1.4.5 [4]

For a topological space X, the followings hold:

(i) Every closed set is g^n -closed but not conversely.

(ii) Every g^n -closed set is ω -closed but not conversely.

(iii) Every g^n -closed set is g-closed but not conversely.

(iv) Every g^n -closed set is α g-closed but not conversely.

(v) Every g^n -closed set is gs-closed but not conversely.

(vi) Every g^n -closed set is gsp-closed but not conversely.

Theorem 1.4.6 [4]

A set A is g^n -closed in X if and only if cl(A) – A contains no nonempty gs-closed set.

1.5 PROPERTIES OF Tg'''-SPACES

We introduce the following definition:

Definition 1.5.1

A space (X, τ) is called a Tg^n -space if every g^n -closed set in it is closed.

Example 1.5.2

Let X = {a, b, c} with τ = {ϕ, {b}, X}. Then G''' C(X) = {ϕ, {a, c}, X}. Thus (X, τ) is a Tg^n -space.

Example 1.5.3

Let X = {a, b, c} with τ = {ϕ, {a, c}, X}. Then G''' C(X) = {ϕ, {b}, {a, b},{b, c}, X}. Thus (X, τ) is not a Tg^n -space.

Proposition 1.5.4

Every $T_{1/2}$-space is Tg^n -space but not conversely.

Proof

Follows from Remark 1.4.5 (iii).

The converse of Proposition 1.3.4 need not be true as seen from the following example.

Example 1.5.5

Let X and τ be as in the Example 1.3.2, GC(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Thus (X, τ) is not a $T_{1/2}$-space.

Proposition 1.5.6

Every Tω-space is Tg^n -space but not conversely.

Proof

Follows from Remark 1.4.5 (ii).

The converse of Proposition 1.5.6 need not be true as seen from the following example.

Example 1.5.7

Let X = {a, b, c} with τ = {ϕ, {a}, {b, c}, X}. Then ωC(X) = P(X) and G^n C(X) = {ϕ, {a}, {b, c}, X}. Thus (X, τ) is Tg^n -space but not a Tω-space.

Proposition 1.5.8

Every α T_b-space is Tg^n -space but not conversely.

Proof

Follows from Remark 1.4.5 (iv).

The converse of Proposition 1.5.8 need not be true as seen from the following example.

Example 1.5.9

Let X and τ be as in the Example 1.3.2. Then αG C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Thus (X, τ) is not a α T_b-space.

Proposition 1.5.10

Every $*_sT_p$-space and T_p*-space is Tg^n -space but not conversely.

Proof

Follows from Remark 1.4.5 (vi) and Definition 1.4.3 (vi) and (v).

The converse of Proposition 1.5.10 need not be true as seen from the following example.

Example 1.5.11

Let X and τ be as in the Example 1.5.2. Then GSP C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X} and G^*P C(X) = {ϕ, {a}, {c}, {a, c}, {b, c}, X}. Thus (X, τ) is neither $*_sT_p$-space nor T_p*-space.

Proposition 1.5.12

Every T_b-space is Tg^n -space but not conversely.

Proof

Follows from Remark 1.4.5 (v).

The converse of Proposition 1.5.12 need not be true as seen from the following example.

Example 1.5.13

Let X and τ be as in the Example 1.5.2. Then GS C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Thus (X, τ) is not a T_b-space.

Remark 1.5.14

We conclude from the next two examples that Tg^n -spaces and α -spaces are independent.

Example 1.5.15

Let X and τ be as in the Example 1.3.2. Then α C(X) = {ϕ, {a}, {c}, {a, c}, X}. Thus (X, τ) is a Tg^n -space but not an α -space.

Example 1.5.16

Let X and τ be as in the Example 1.3.3. Then α C(X) = {ϕ, {b}, X}. Thus (X, τ) is an α -space but not a Tg^n -space.

Theorem 1.5.17

For a space (X, τ) the following properties are equivalent:

(i) (X, τ) is a Tg^n -space.

(ii) Every singleton subset of (X, τ) is either gs-closed or open.

Proof

(i) $\rightarrow$ (ii). Assume that for some $x \in X$, the set {x} is not a gs-closed in (X, τ). Then the only gs-open set containing $\{x\}^c$ is X and so $\{x\}^c$ is g^n -closed in (X, τ). By assumption $\{x\}^c$ is closed in (X, τ) or equivalently {x} is open.

(ii) $\rightarrow$ (i). Let A be a g^n -closed subset of (X, τ) and let $x \in cl(A)$. By assumption {x} is either gs-closed or open.

Case (a) Suppose that {x} is gs-closed. If $x \notin A$, then $cl(A)-A$ contains a nonempty gs-closed set {x}, which is a contradiction to Theorem 1.4.6. Therefore $x \in A$.

Case (b) Suppose that {x} is open. Since $x \in cl(A)$, $\{x\} \cap A \neq \phi$ and so $x \in A$. Thus in both case, $x \in A$ and therefore $cl(A) \subseteq A$ or equivalently A is a closed set of (X, τ).

Definition 1.5.18

A topological space (X, τ) is called generalized semi-R_0 (briefly gs-R_0) if and only if for each gs-open set G and $x \in G$ implies gs-cl({x}) $\subset$ G.

Definition 1.5.19

A topological space (X, τ) is called:

(i) generalized semi-T_0 (briefly gs-T_0) if and only if to each pair of distinct points x, y of X, there exists a gs-open set containing one but not the other.

(ii) generalized semi-T_1 (briefly gs-T_1) if and only if to each pair of distinct points x, y of X, there exists a pair of gs-open sets, one containing x but not y, and the other containing y but not x.

Theorem 1.5.20

For a topological space X, each of the following statement is equivalent:

(i) X is a gs-T_1.

(ii) Each one point set is gs-closed set in X.

Proof

(i) $\Rightarrow$ (ii) Let a space X be gs-T_1 and $x \in X$. Suppose gscl({x}) $\neq$ {x}. Then we can find an element $y \in$ gscl({x}) with $y \neq x$. Since X is gs-T_1, there exist gs-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Now $x \in V^c$ and V^c is gs-closed. Therefore gscl({x}) $\subseteq V^c$ which implies $y \in V^c$, contradiction. Hence gscl({x}) = {x} or {x} is gs-closed.

(ii) $\Rightarrow$ (i) Let $x, y \in X$ with $x \neq y$. Then {x} and {y} are gs-closed. Therefore $U = (\{x\})^c$ and $V = (\{y\})^c$ are gs-open and $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence X is gs-T_1

Theorem 1.5.21

For a space (X, τ) the following properties hold:

(i) If (X, τ) is gs-T_1, then it is Tg^n.

(ii) If (X, τ) is Tg^n, then it is gs-T_0 .

Proof

(i) The proof is obvious from Theorem 1.5.20.

(ii) Let x and y be two distinct elements of X. Since the space (X, τ) is Tg^n, we have that {x} is gs-closed or open. Suppose that {x} is open. Then the singleton {x} is a gs-open set such that $x \in \{x\}$and $y \notin \{x\}$. Also, if {x} is gs-closed, then $X \setminus \{x\}$ is gs-open such that $y \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. Thus, in the above two cases, there exists a gs-open set U of X such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Thus, the space (X, τ) is gs-T_0.

Theorem 1.5.22

For a gs-R_0 topological space (X, τ) the following properties are equivalent:

(i) (X, τ) is gs-T_0.

(ii) (X, τ) is T g^n.

(iii) (X, τ) is gs-T_1.

Proof

It suffices to prove only (i) ⇒ (iii). Let x ≠ y and since (X, τ) is gs-T_0, we may assume that x ∈ U ⊆ X \ {y} for some gs-open set U. Then x ∈ X \ gs-cl({y}) and X \ gs-cl({y}) is gs-open. Since (X, τ) is gs-R_0, we have gs-cl({x}) ⊆ X \ gs-cl({y}) ⊆ X \ {y} and hence y ∉ gs-cl({x}). There exists gs-open set V such that y ∈ V ⊆ X \ {x} and (X, τ) is gs-T_1.

1.6 $_g$T g^n-SPACES

Definition 1.6.1

A space (X, τ) is called a $_g$Tg^n -space if every g-closed set in it is g^n -closed.

Example 1.6.2

Let X and τ be as in the Example 1.5.3, is a $_g$Tg^n -space and the space (X, τ) in the Example 1.6.2, is not a $_g$Tg^n -space.

Proposition 1.6.3

Every $T_{1/2}$-space is $_g$Tg^n -space but not conversely.

Proof

Follows from Remark 1.5.5 (i).

The converse of Proposition 1.6.3 need not be true as seen from the following example.

Example 1.6.4

Let X and τ be as in the Example 1.5.3, is a $_{g}$Tg^n -space but not a $T_{1/2}$-space.

Remark 1.6.5

Tg^n -spaces and $_{g}$Tg^n -spaces are independent.

Example 1.6.6

The space (X, τ) in the Example 1.5.3, is a $_{g}$Tg^n -space but not a Tg^n -space and the space (X, τ) in the Example 1.5.2, is a Tg^n -space but not a $_{g}$Tg^n -space.

Theorem 1.6.7

If (X, τ) is a $_{g}$Tg^n -space, then every singleton subset of (X, τ) is either g-closed or g^n -open.

Proof

Assume that for some x ∈ X, the set {x} is not a g-closed in (X, τ). Then {x} is not a closed set, since every closed set is a g-closed set. So $\{x\}^c$ is not open and the only open set containing $\{x\}^c$ is X itself. Therefore $\{x\}^c$ is trivially a g-closed set and by assumption, $\{x\}^c$ is an g'''-closed set or equivalently {x} is g^n -open.

The converse of Theorem 1.6.7 need not be true as seen from the following example.

Example 1.6.8

Let X and τ be as in the Example 1.5.2. The sets {a} and {c} are g-closed in (X, τ) and the set {b} is g^n -open. But the space (X, τ) is not a $_{g}$Tg^n -space.

Theorem 1.6.9

A space (X, τ) is $T_{1/2}$ if and only if it is both Tg^n and $_{g}$Tg^n**.**

Proof

Necessity. Follows from Propositions 1.5.4 and 1.5.3.

Sufficiency. Assume that (X, τ) is both Tg^n and $_g$Tg^n. Let A be a g-closed set of (X, τ). Then A is g^n -closed, since (X, τ) is a $_g$Tg^n. Again since (X, τ) is a Tg^n, A is a closed set in (X, τ) and so (X, τ) is a $T_{1/2}$.

1.7 α Tg^n -SPACES

Definition 1.7.1

A space (X, τ) is called a α Tg^n -space if every α g-closed set in it is g^n -closed.

Example 1.7.2

Let X and τ be as in the Example 1.3.3, is a α Tg^n -space and the space (X, τ) in the Example 1.3.2, is not a α Tg^n -space.

Proposition 1.7.3

Every α T_b-space is α Tg^n -space but not conversely.

Proof

Follows from Remark 1.4.5 (i).

The converse of Proposition 1.5.3 need not be true as seen from the following example.

Example 1.7.4

Let X and τ be as in the Example 1.5.3, is a α Tg^n -space but not a α T_b-space.

Proposition 1.7.5

Every α Tg^n -space is a α T_d-space but not conversely.

Proof

Let (X, τ) be an α Tg^n -space and let A be an α g-closed set of (X, τ). Then A is a g^n -closed subset of (X, τ) and by Remark 1.4.5 (iii), A is g-closed. Therefore (X, τ) is an α T_d-space.

The converse of Proposition 1.7.5 need not be true as seen from the following example.

Example 1.7.6

Let X and τ be as in the Example 1.5.3, is a α T_d-space but not a α Tg^n -space.

Theorem 1.7.7

If (X, τ) is a α Tg^n -space, then every singleton subset of (X, τ) is either α g-closed or g^n -open.

Proof

Similar to Theorem 1.6.7.

The converse of Theorem 1.7.7 need not be true as seen from the following example.

Example 1.7.8

Let X and τ be as in the Example 1.5.2. The sets {a} and {c} are α g-closed in (X, τ) and the set {b} is g^n -open. But the space (X, τ) is not a α Tg^n -space.

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