Fuzzy Strongly α-Continuous Maps on generalized Topological Spaces

Sujeet Chaturvedi
J.K. Maitra
Nazir Ahmad Ahengar
Department of Mathematics and Computer Sciences R.D.V.V Jabalpur 482001 (M.P) India

Abstract: In this paper we have studied g-fuzzy strongly α-continuous maps and have investigated equivalent conditions for a map to be g-fuzzy strongly α-continuous map. Also we have established some significant properties of g-fuzzy strongly α-continuous map.

IndexTerms - **g-fuzzy topology, g-fuzzy semi-open, g-fuzzy α-open, g-fuzzy continuous, g-fuzzy semi-continuous, g-fuzzy α -continuous.**

1. INTRODUCTION

Csaszar introduced the notions of generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. Moreover, he studied the simplest separation Y. Beceren has introduced the concept of strongly α-continuous maps in 2000. Bin Shahana has defined the concept of fuzzy α-continuous maps. Fuzzy strongly α-continuous map is the stronger form of fuzzy α-continuous map. In the present paper, we further carry out our investigations on generalized open fuzzy sets and generalized fuzzy topology. In this paper we have studied g-fuzzy strongly α-continuous maps and investigated necessary and sufficient conditions for a map to be g-fuzzy strongly α-continuous. Further we have established some important properties of g-fuzzy strongly α-continuous maps.

2. Preliminaries.

Definition 2.1: Let X be a crisp set and let τ_g be a collection of fuzzy sets on X. Then τ_g is said to be a generalized fuzzy (g-fuzzy) topological space on X if it satisfies following conditions:

i) The fuzzy sets 0 and 1 are in τ_g where $0,1\colon X \to I$ are defined as $0(x) = 0$ and $1(x) = 1 \quad \forall\, x \in X$

ii) If $\{\{\lambda_j\}, j \in J$ is any family of fuzzy sets on X where $\lambda_j \in \tau_g$ then $\cup_{j\in J} \lambda_j \in \tau_g \ \forall\, j \in J$.

The pair (X, τ_g) is called a generalized fuzzy topological space. We usually denote fuzzy topological space (X, τ_g) by X_g only. The members of the collection τ_g are called g-fuzzy open sets. In X_g A fuzzy set $\lambda\colon X_g \to I$ is called a fuzzy closed set in X_g provided its complement λ^c is a fuzzy open set in X_g. In a generalized fuzzy topological space X_g, the closure of a fuzzy set denoted by $Cl(\lambda)$ is defined to be the intersection of all fuzzy closed sets in X_g containing λ . The interior of a fuzzy set λ is denoted by $Int(\lambda)$ and is defined to be the union of all fuzzy open sets in X_g contained in λ. Clearly $Cl(\lambda)$ is a fuzzy closed set in X_g and $Int(\lambda)$ is a fuzzy open set in X_g. We note that $Cl(\lambda^c) = Int(\lambda)^c$ and $In(\lambda^c) = Cl(\lambda)^c$. If $\lambda_j\colon X_g \to I,\ j \in J$ is any arbitrary collection of fuzzy sets in X_g then $Int(\cup_{j\in J} \lambda_j) \geq \cup_{j\in J} Int(\lambda_j)$ and $Cl(\cup_{j\in J} \lambda_j) \geq \cup_{j\in J} Cl(\lambda_j)$.

Definition 2.2: Let (X, τ_g) be a generalized fuzzy topological space. A fuzzy set λ in X is called:

i) g-fuzzy semi-open if $\lambda \leq Cl(Int(\lambda))$.

ii) g-fuzzy pre-open if, $\lambda \leq Int(Cl(\lambda))$.

iii) g- fuzzy α-open if $\lambda \leq Int(Cl(Int(\lambda)))$.

iv) g-fuzzy semipro-open if $\lambda \leq Cl(Int(Cl(\lambda)))$.

Remark 2.1: Every fuzzy open set is g-fuzzy α-open, every g-fuzzy α-open set is g-fuzzy semi open (resp. fuzzy pre-open) and every g-fuzzy semi open (resp. Fuzzy pre-open) set is g-fuzzy semi pre-open. But the converses may not be true. The complement of a g-fuzzy semi-open (resp. fuzzy pre-open, fuzzy α-open, fuzzy semi pre-open) set is called g-fuzzy semi-closed (resp. fuzzy pre-closed, fuzzy α-closed, fuzzy semi pre-closed).

Definition 2.3: Let (X, τ_g) be g-fuzzy topological space. A fuzzy set $X_r : X \to 1$ where $x \in X$ and $0 < r \leq 1$ is said to be a g-fuzzy point in a g-fuzzy topological space X if.

$$\chi_r(y) = \begin{cases} r, & \text{if } y = x \\ 0, & \text{if } y \neq x, \qquad y \in X \end{cases}$$

The g-fuzzy point X_r belongs to a g-fuzzy set 1 in X if $r \leq \lambda(x)$ i.e. X_r is a subset of 1.

Definition 2.4: Let (X, τ_g) be a generalized fuzzy topological space. A sub-collection B of X is called a base for τ_g if each member of τ_g can be expressed as the union of members of B.

Definition 2.5: Let X and Y be generalized fuzzy topological spaces. If λ is a fuzzy set in X and μ is a fuzzy set in Y then $\lambda \times \mu$ is a g-fuzzy set in X × Y defined as $(\lambda \times \mu)(x, y) = \min\{\lambda(x)\mu(y), \forall\, x \in X,\ y \in Y$.. If λ is a g-fuzzy open set in X and μ is a g-fuzzy open set in Y then $\lambda \times \mu$ is called a g- fuzzy basic open set in X × Y. A g-fuzzy open set in X × Y is an arbitrary union of g-fuzzy basic open sets in X × Y. The collection τ_g of all g-fuzzy open sets in X × Y is a g-fuzzy topology on X × Y called g-fuzzy product topology on X × Y and X × Y together with τ_g is called the generalized fuzzy product topological space of X and Y.

3. Fuzzy Strongly α-Continuous Maps

Definition 3.1 Let X and Y be generalized fuzzy topological spaces and $f: X \to Y$ be a map. Then f is said to be:

i) g-fuzzy continuous if $f^{-1}(\lambda)$ is g-fuzzy open in X for each g-fuzzy open set λ in Y.

ii) g- fuzzy α-continuous if $f^{-1}(\lambda)$ is g-fuzzy α-open in X for each g-fuzzy open set λ in Y.

iii) g-fuzzy semi-continuous if $f^{-1}(\lambda)$ is g-fuzzy semi-open in X for each g-fuzzy open set λ in Y.

iv) g-fuzzy strongly α-continuous if $f^{-1}(\lambda)$ is g-fuzzy α-open in X for each g-fuzzy semi-open set λ in Y.

Example.3.1: Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and μ and λ be g-fuzzy sets in X, Y defined as, $\mu = \{(x_1, 0.5), (x_2, 0.6)\}$ and $\lambda = \{(y_1, 0.7), (y_2, 0.8)\}$. Let $\tau_g = \{0, \mu, 1\}$ and $\tau_g{}' = \{0, \lambda, 1\}$ be generalized fuzzy topologies on sets X and Y respectively. The map $f: X \to Y$ defined as $f(x_i) = y_i, i = 1,2$ f is g-fuzzy strongly α-continuous map. Since each g-fuzzy open set is g-fuzzy semi-open, it follows that every g-fuzzy strongly α-continuous map is g-fuzzy α-continuous. However a g-fuzzy α-continuous map may not be g-fuzzy strongly α-continuous. We have the following example.

Example 3.2: Let $X = \{x_1, x_2\}$ and μ, λ and ν be fuzzy sets in X defined as $\mu = \{(x_1, 0.2), (x_2, 0.3)\}$, $\lambda = \{(y_1, 0.5), (y_2, 0.6)\}$ and $\nu = \{(y_1, 0.7), (y_2, 0.7)\}$. Let $Y = \{y_1, y_2\}$ and γ be g-fuzzy set in Y defined as $\gamma = \{(y_1, 0.6), (y_2, 0.7)\}$. Consider the generalized fuzzy topologies $\tau_g = \{0, \mu, \lambda, \nu, 1\}$ and $\tau_g{}' = \{0, \gamma, 1\}$ on sets X and Y respectively. The map $f: X \to Y$ defined as $f(x_i) = y_i, i = 1,2$ is g-fuzzy α-continuous map. The fuzzy set η in Y defined as $\eta = \{(y_1, 0.7), (y_2, 0.8)\}$ is fuzzy semi-open set in Y but $f^{-1}(\eta)$ is not g-fuzzy α-open set in X. Hence the map $f: X \to Y$ is g-fuzzy α-continuous but not g-fuzzy strongly α-continuous. The concepts of g-fuzzy continuity and g-fuzzy strongly α-continuity are independent of each other. The map $f: X \to Y$ mentioned in Example 3.1 is g-fuzzy strongly α-continuous but not fuzzy continuous. In the following example we see that a g-fuzzy continuous map $f: X \to Y$ may not be g-fuzzy strongly α-continuous.

Example 3.3 Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Let μ, and λ be fuzzy sets in X and ν be fuzzy set in Y defined as $\mu = \{(x_1, 0.3), (x_2, 0.4)\}$, $\nu = \{(y_1, 0.5), (y_2, 0.6)\}$ and $\tau_g = \{0, \mu, \lambda, 1\}$ and $\tau_g{}' = \{0, \nu, 1\}$ be generalized fuzzy topologies on sets X and Y respectively. The map $f : X \to Y$ defined as $f(x_i) = y_i, i = 1,2$ is g-fuzzy continuous. The fuzzy set η in Y defined as $\eta = \{(y_1, 0.6), (y_2, 0.7)\}$ is fuzzy semi-open set in Y but $f^{-1}(\eta)$ is not g-fuzzy α-open set in X. Hence the map $f: X \to Y$ is g-fuzzy continuous but not g-fuzzy strongly α-continuous. In the following result we have obtained several equivalent conditions for a map $f: X \to Y$, where X and Y are generalized fuzzy topological spaces to be g-fuzzy strongly α-continuous.

Theorem 3.1 Let X and Y be generalized fuzzy topological spaces and $f: X \to Y$ be a map. Then following conditions are equivalent:

i) f is g- fuzzy strongly α-continuous.

ii) For each g-fuzzy point x_β in X and each g-fuzzy semi-open set λ in Y containing $f(x_\beta)$, there exists a g- fuzzy α-open set μ in X containing x_β such that hat $f(\mu \leq \lambda)$.

iii) For each g-fuzzy semi-closed set λ in Y, $f^{-1}(\lambda)$ is g-fuzzy closed set in X.

iv) For each g-fuzzy set μ in X, $f(\alpha Cl(\mu) \leq sCl(f(\mu))$.

v) For each g-fuzzy set λ in Y, $\alpha Cl(f^{-1}(\lambda)) \leq f^{-1}(sCl(\lambda))$.

vi) For each g-fuzzy set λ in Y, $f^{-1}(sInt(\lambda)) \leq \alpha Int(f^{-1}(\lambda))$.

vii) For each g-fuzzy set λ in Y, $Cl(Int(Cl(f^{-1}(\lambda)))) \leq f^{-1}(sCl(\lambda))$.

viii) For each g-fuzzy set μ in X, $f(Cl\left(Int(Cl(\mu))\right)) \leq sCl(f(\mu))$.

Proof (i) ⟹ (ii): Let $f: X \to Y$ be a g-fuzzy strongly α-continuous map. Let x where $x_\beta \in X$ and $0 < \beta \leq 1$ be a g-fuzzy point in X and λ be a g-fuzzy semi-open set in Y containing the g-fuzzy point $f(x_\beta)$. Since $f(x_\beta)\left(f(x)\right) = \beta \leq \lambda(f(x))$, we have $\beta \leq \lambda\left(f(x)\right)(x)$ i.e.$\mu = f^{-1}(\lambda)$ contains the g-fuzzy point x_β. Since f is g-fuzzy strongly α-continuous, μ is g-fuzzy α-open set in X containing the g-fuzzy point x_β and $f(\mu) \leq \lambda$.

(ii) ⟹ (i): Let λ be a g-fuzzy semi-open set in Y. For $x \in X$ and $0 < \beta \leq 1$. Let x_β be a g-fuzzy point in $f^{-1}(\lambda)$. Then λ contains $f(x_\beta)$ and so by given condition (ii) there exists a g-fuzzy α-open set μ in X containing the g-fuzzy point x_β and $f(\mu) \leq \lambda$. This implies $\mu \leq Int(Cl\left(Int(\mu)\right)) \leq Int(Cl\left(Int((f^{-1}(\lambda)))\right))$ and therefore $Int(Cl\left(Int((f^{-1}(\lambda)))\right))$ contains the g-fuzzy point x_β. Thus each g-fuzzy point of $f^{-1}(\lambda)$ is also a g-fuzzy point of $Int(Cl\left(Int((f^{-1}(\lambda)))\right))$. This shows that in $f^{-1}(\lambda) \leq Int(Cl\left(Int((f^{-1}(\lambda)))\right))$ i.e., $f^{-1}(\lambda)$. is a g-fuzzy α-open set in X. Hence $f : X \to Y$ is g-fuzzy strongly α-continuous map.

(i) ⟹ (iii): Let λ be a g-fuzzy semi-closed set in Y. Then $\lambda^c=1- \lambda$ is g-fuzzy semi-open set inY. Since $f : X \to Y$ is g- fuzzy strongly α-continuous, $f^{-1}(\lambda^c) = f^{-1}(1- \lambda)$ is g-fuzzy α-open set in X. This implies $f^{-1}(\lambda) = f^{-1}(1- \lambda) = 1- f^{-1}(\lambda^c)$ is g-fuzzy α-closed set in X.

(iii) ⟹ (i): Let λ be a fuzzy semi-open set in Y. Then $\lambda^c=1- \lambda$ is fuzzy semi-closed set in Y. Therefore by given condition (iii), $f^{-1}(\lambda^c) = f^{-1}(1- \lambda)$ is g-fuzzy α-closed set in X. Hence $f^{-1}(\lambda) = f^{-1}(1- \lambda^C) =1-) = 1 - f^{-1}(\lambda^c)$ is g-fuzzy α-open set in X. Thus $f : X \to Y$ is g-fuzzy strongly α-continuous map

(iii) ⟹ (iv): Let μ be a g-fuzzy set in X. Since $\mu \leq f^{-1}(f(\mu))$ we have $\mu \leq f^{-1}(sCl(f(\mu)))$. Now $sCl(f(\mu))$ is a g-fuzzy semi-closed set in y. Hence by given condition (iii), $f^{-1}(sCl(f(\mu)))$ is g-fuzzy α-closed set in X containing μ. Since $\alpha Cl(\mu)$ is the smallest g-fuzzy α-closed set containing μ, it follows that $\alpha Cl(\mu) \leq f^{-1}(sCl(f(\mu))$ This implies $f(\alpha Cl(\mu) \leq (sCl(f(\mu))$.

(iv) ⟹ (iii): Let λ be a g-fuzzy semi-closed set in Y. Then by given condition (iv), we have $f^{-1}(\alpha Cl(f^{-1} (\lambda))) \leq sCl(f(f^{-1} (\lambda)))=\lambda$ This implies $f^{-1}(\alpha Cl(f^{-1} (\lambda))) \leq f^{-1} (\lambda)$ Since $f^{-1} (\lambda) \leq \alpha Cl(f^{-1} (\lambda))$, we deduce that $f^{-1} (\lambda) = \alpha Cl(f^{-1} (\lambda))$. Now $\alpha Cl(f^{-1} (\lambda))$ is g-fuzzy α-closed set in X, it follows that $f^{-1} (\lambda)$ is g-fuzzy α-closed set in X.

(iv) ⟹ (v): Let be a g-fuzzy set in Y. Then by given condition (iv), $f (\alpha Cl(f^{-1} (\lambda))) \leq sCl(f (f^{-1} (\lambda))) \leq sCl(\lambda)$ This implies $\alpha Cl(f^{-1} (\lambda)) \leq f^{-1} (sCl(\lambda))$.

(v) ⟹ (iv): Let μ be a g-fuzzy set in X. Then by given condition (v), $\alpha Cl(f^{-1} (f(\mu))) \leq f^{-1} (sCl(f (\mu)))$. This implies, $\alpha Cl(\mu) \leq f^{-1}(sCl(f(\mu))$, and hence $f(\alpha Cl(\mu) \leq (sCl(f(\mu))$.

(i) ⟹ (vi): Let λ be a g- fuzzy set in Y. Since $sInt(\lambda)$ is a g-fuzzy semi-open set in Y, by given condition (i), $f^{-1}(sInt(\lambda))$ is g-fuzzy α-open set in X. Hence we have $f^{-1}(sInt(\lambda)) = \alpha Int(f^{-1}(sInt(\lambda)))$. Since, $\alpha Int(f^{-1}(sInt(\lambda))) \leq \alpha Int(f^{-1} (\lambda))$.we find that $f^{-1}(sInt(\lambda)) \leq \alpha Int(f^{-1} (\lambda))$.

(vi) ⟹ (i): Let λ be a g-fuzzy semi-open set in Y. Then we have $sInt(\lambda)=\lambda$. Therefore by given condition (vi) $f^{-1}(\lambda),= f^{-1}(sInt(\lambda)) \leq \alpha Int(f^{-1}(\lambda))$, i.e.) $f^{-1}(\lambda) \leq \alpha Int(f^{-1}(\lambda))$. Since $\alpha Int(f^{-1}(\lambda)) \leq f^{-1}(\lambda)$, we find that $f^{-1}(\lambda)= \alpha Int(f^{-1}(\lambda))$. Hence) $f^{-1}(\lambda)$ is a g-fuzzy α-open set in X. Thus $f : X \rightarrow Y$ is a g-fuzzy strongly α-continuous map.

Theorem 3.2: Let X and Y be generalized fuzzy topological spaces and $f : X \rightarrow Y$ be a bijective map. Then f is g-fuzzy strongly α-continuous iff for each g-fuzzy set μ in X, $sInt(f(\mu)) \leq f(\alpha Int(\mu))$.

Proof: Let $f : X \rightarrow Y$ be a bijective map. Suppose f is g-fuzzy strongly α-continuous. If is a g-fuzzy set in X then $f(\mu)$is a g-fuzzy set in Y. Since f is g-fuzzy strongly α-continuous, from Theorem 3.6 we have, $f^{-1}(sInt(f(\mu))\leq \alpha Int(f^{-1}(\mu))$. Since is one-one, $\alpha Int(f^{-1}(\mu)) = \alpha Int(\mu)$. This shows that, $f^{-1}(sInt(f(\mu))\leq \alpha Int(\mu))$. Further since f is onto we have, , $sInt(f(\mu))=f (f^{-1}(sInt(f(\mu)))) \leq f(\alpha Int(\mu))$.Thus $sInt(f (f^{-1}(\lambda)))\leq f(\alpha Int(\mu))$.

Conversely let λ be a g-fuzzy semi-open set in Y. Then $sInt(f(\lambda))=\lambda$. Now $f^{-1}(\lambda)$ is a g-fuzzy set in X, from hypothesis, $f(\alpha Int(f^{-1}(\lambda)) \geq sInt(f (f^{-1}(\lambda)))$. Since f is onto, $sInt(f (f^{-1}(\lambda))) = sInt(\lambda) =\lambda$. Therefore $f(\alpha Int(f^{-1}(\lambda))) \geq \lambda$. Further since f is one-one, $\alpha Int(f^{-1}(\lambda)) = f^{-1}(f(\alpha Int(f^{-1}(\lambda))) \geq f^{-1}(\lambda)$. As $\alpha Int(f^{-1}(\lambda))\leq f^{-1}(\lambda)$, we deduce that $f^{-1}(\lambda) = \alpha Int(f^{-1}(\lambda))$. Thus $f^{-1}(\lambda)$ is a fuzzy α-open set in X. Hence $f : X \rightarrow Y$ is g-fuzzy strongly α-continuous map.

References

1. Azad, K.K., On fuzzy semi continuity, fuzzy almost continuity and fuzzy weak continuity, J. Math. Anal. Appl. 82 14-32, (1981).
2. Beceren, Y., On strongly α-continuous functions, Far East J. Math. Sci. (FJMS), Special Volume, Part I-12, 51-58, (2000).
3. Bin Shahana, A.S., On fuzzy strong semi-continuity and fuzzy precontinuity, Fuzzy Sets and Systems 44 303-308, (1991).
4. Bin Shahana, A.S., Mappings in fuzzy topological spaces,Fuzzy Sets and Systems 61, 209-213, (1994).
5. Chang, C.L., Fuzzy topological spaces, J.Math. Anal. Appl.24, 182-190 (1968).
6. Csaszar, A., Generalized open sets in generalized topologies, Acta MathematicaHungaria 96, 351-357, (2002).
7. Csaszar, A., Generalized open sets in generalized topologies, Acta Mathematica Hungaria 106, 53-66, (2005).
8. Shrivastava, M., Maitra, J.K., and Shukla, M., A note on fuzzy α-continuous maps, Vikram Mathematical Journal 29, (2008).
9. Zadeh, L.A., Fuzzy sets, Inform. and Control 8, 338-353, (1965).