A Study Of The Orthonormal Series Expansion of Generalized Functions

P.N. PATHAK[1], RAGHAVENDRA SINGH[2] and RAJENDRA KUMAR TRIPATHI[3]

[1]Deptt of Mathematics, AIHE, Axis Colleges Kanpur, [2]Deptt of Mathematics, IITM Bilhaur Kanpur , [3]Deptt. Of Applied Sciences and Humanities, Faculty of Engineering and Technology, Khwaja Moinuddin Chishti Language university Lucknow, U.P

In this paper we discuss Zemanian's approach to expand certain generalized functions into series of orthonormal functions which are the eigen values of a certain type self-adjoint differential operator for which we deal with Laguerre polynomials which are known to form a sequence of orthogonal polynomials and hence system of orthonormal functions ψ_n in the interval $(0,\infty)$ with respect to the weight function $x^{\alpha}.e^{-x}$ and construct a testing function space $A(L_n^{(\alpha)})$ as a subspace of $L_2(I)$ space and then expand the members of dual space $A'(L_n^{(\alpha)})$ of this space into orthonormal series of ψ_n.

1. Introduction:

The various methods for expansion of certain Schwartz distributions (generalized functions) with respect to different orthonormal systems and that for generalization of the integral transforms to the certain class of generalized functions by using these expansions have been developed by several authors. As a by-product of these expansions certain inversion formulae for the integral transforms of generalized functions are obtained therefore each of the methods of such expansions arises a whole new class of generalized integral transformations.

In 1950, Schwartz [11] has given the expansion of elements of S_n i.e. of tempered distributions with respect to orthonormal system of the Hermite polynomials.

Korevaar [6] in 1959, Widlund [14] in 1961, Giertz [4] in 1964 and Walter [13] in 1965 investigated the technique of orthogonal series expansions to generalized functions.

In 1966, Zemanian [16] has given new approach of expanding of certain Schwartz distributions into series of orthonormal functions which are eigen function of the self adjoint differential operator and he developed a straight forward technique to generalized in a distributions way a variety of integers transforms. Specially his work based on L_2-convergence theory of orthonormal series and in particular, he also identified few spaces of distributions corresponding to certain orthonormal system. e.g. space S_n' of tempered distributions in the case of Hermite system.

Further Zemanian`s method has been extended to all regular C^{∞}, self-adjoint ordinary differential operators by Judge [5] and in the same year, Guillemot-Teissiers [12] developed a technique of expanding of tempered distributions in orthonormal series of Laguerre polynomials.

In 1978, Panday and Pathak [7] discussed the same kind work as in Zemanian [18] by extending the L_1 convergence theory of orthogonal series in the distributional sense. Further in 1983, following Zemanian, Pathak [9] extended his work by applying L^p-convergence theory. In between certain summability methods also were employed by Ditzion [2], Pathak[8].

Recently in 1990 Duran [3] discussed expansion of distributions in the orthonormal series of the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$. Specially he extended the result of Guillenot-Tessiers and obtained some new and interesting properties applying these results. In this special case Duran`s space of distribution coincides with space S_n' of tempered distribution.

In our work we consider the orthonormal system $\{\psi_n\}$,dfined in (2.2), and then following Zemanian we study the expansions of members of $A'(L_n^{(\alpha)})$ with respect to this orthonormal system.

2. Preliminaries:

In this section we describe the space $L_2(I)$ Laguerre polynomial, transform and their important properties which are useful in our subsequent work.

2.1 The space $L_2(I)$

Let I denote the open interval , $a < x < b$, *on the real line. where* , $a = -\infty$ *and* b may be ∞. $L_2(I)$ is the space of (equivalence classes of) quadratically integrable functions on I with the customary inner product,

$$(f,g) = \int_a^b f(x)\overline{g(x)}\,dx \qquad f,g \in L_2(I) \qquad (2.1.1)$$

And norm $||f|| = [(f,f)]^{\frac{1}{2}}$. (2.1.2)

It is a sequentially complete space.

If k is a nonnegative integer and θ_k denotes a complex-valued smooth function on I such that $\theta_k(x) \neq 0$ on η shell denote a linear differentiation operator of the form,

$$\eta = \theta_0 D^{n_1}\theta_1 D^{n_2} \ldots \theta_v D^{n_v}, \qquad (2.1.3)$$

Where the n_k are nonnegative integer and $D^k = \frac{d^k}{dx^k}$. We also require that the θ^k *and* n^k be such that

$$\eta = \overline{\theta_v}(-D)^{n_v} \ldots (-D)^{n_2}\overline{\theta_1}(-D)^{n_1}\overline{\theta_0} \qquad (2.1.4)$$

Moreover, we shall assume that $\boldsymbol{\eta}$ possesses eigen values $\boldsymbol{\lambda_n}$ and normalized eigenfunctions $\boldsymbol{\psi_n}(\boldsymbol{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots)$ with the following properties. The $\boldsymbol{\psi_n}$ form a complete orthonormal system in $\boldsymbol{L_2(I)}$. the $\boldsymbol{\lambda_n}$ are real and have no finite point of accumulation. (By this we also mean that no particular value of $\boldsymbol{\lambda_n}$ is assumed for more that a finite number of n.) we shall always number the $\boldsymbol{\lambda_n}$ such that $|\boldsymbol{\lambda_0}| \leq |\boldsymbol{\lambda_1}| \leq |\boldsymbol{\lambda_2}| \leq \cdots$. this form of $\boldsymbol{\eta}$ will allow us to define a testing function subspace of $\boldsymbol{L_2(I)}$ on which $\boldsymbol{\eta}$ is self-adjoint.

2.2 Laguerre Polynomials

We will like to call the polynomial denoted by $L_n^{(\alpha)}(x)$ as the Laguerre polynomial and its special case for α=0 denoted as $L_n(x)$ as the simple Laguerre polynomial. Some authors call $L_n^{(\alpha)}(x)$ as the generalized Laguerre polynomial and $L_n(x)$ the Laguerre polynomial. We will use the term generalized Laguerre transform for the Laguerre transform of generalized function.

The one way of defining the Laguerre polynomial $L_n^{(\alpha)}(x)$ of order α and degree n in x is by means of the generating relation

$$e^t \; {}_0F_1(- \; ; \; 1+\alpha \; ; \; -xt) = \sum_{n=0}^{\infty} \frac{L_n{}^{(\alpha)}(x)t^n}{(1+\alpha)_n} \tag{2.2.1}$$

where ${}_0F_1$ is the generalized hypergeometric function with no numerator but one denominator parameter, n is a non negative integer and Re(α) >-1.

From (4.3.1) by expanding the function on the left we have

$$\sum_{n=0}^{\infty} \frac{L_n{}^{(\alpha)}(x)t^n}{(1+\alpha)_n} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m x^m t^m}{m! \; (1+\alpha)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m x^m t^{n+m}}{m! \; n! \, (1+\alpha)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m x^m t^n}{m! \; (n-m)! \, (1+\alpha)_m}$$

Now by comparing the coefficients of t^n on both sides we get

$$L_n{}^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-1)^m x^m (1+\alpha)_n}{m! \; (n-m)!(1+\alpha)_m} \tag{2.2.2}$$

which shows that $L_n^{(\alpha)}(x)$ form a simple set of polynomials i.e. a set of polynomials which has one polynomial of each degree.

Equation (4.2.2) can also be written as

$$L_n{}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \sum_{m=0}^{\infty} \frac{(-n)_m x^m}{(1+\alpha)_m m!} \qquad \text{Because } \frac{(-1)^m}{(n-m)!} = \frac{(-n)_m}{n!}$$

This gives

$$L_n{}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} , F, (-m; 1+\alpha; x) \tag{2.2.3}$$

It follows that Laguerre polynomial $L_n^{(\alpha)}(x)$ is a terminating confluent hypergeometric series.

It is also relevant to write below the further important and useful properties of $L_n^{(\alpha)}(x)$.

(i) From (2.2.2), we can obtained a very interesting formula

$$L_n{}^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} D_n[e^{-x}x^{n+\alpha}] \tag{2.2.4}$$

This is known as Rodrigues formula for $L_n{}^{(\alpha)}(x)$

(ii) For α >-1, we have

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) x^k dx = 0 \quad k = 0,1,2,\ldots,(n-1) \tag{2.2.5}$$

(iii) Product of two Laguerre Polynomials: For this Bailey's Formula [1] is

$$L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = \frac{\Gamma(n+\alpha+1)}{n!} \sum_{k=0}^{n} \frac{(xy)^k}{k!\Gamma(k+\alpha+1)} L_{n-k}^{(\alpha+2k)}(x+y) \tag{2.2.6}$$

(iv) We describe below some useful results known as bounds for the Laguerre Polynomials$L_n^{(\alpha)}(x)$. These bounds are discussed by Duran [3] and he used these results for Laguerre expansions of *tempered distributions* and generalized functions

(a) If α<0, x>0 and n∈N then

$$|L_n^{(\alpha)}(x)| \le e^{x/2} \frac{2^{-\alpha}}{2\pi} \frac{\Gamma(1/2)\Gamma(-\alpha/2)}{\Gamma(1-\alpha)/2} \tag{2.2.7}$$

Moreover if -2 < α <-1 then

$$\left|L_n^{(\alpha)}(x)\right| \le 2e^{t/2} \tag{2.2.8}$$

(b) For α > -2, x > 0 and n∈N we have

$$\left|L_n^{(\alpha)}(x)\right| \le 2\left(\frac{n+[\alpha]+2}{n}\right) e^{x/2} \tag{2.2.9}$$

In particular, for α =0

$$|L_n(x)| \le e^{-x/2}$$

2.3 System of Complete Orthonormal Functions ψ_n

We know that the sequence of polynomials $L_n^{(\alpha)}(x)$,n=0,1,2 and Re(α) >-1 forms an orthogonal set over the interval (0,∞) with respect to weight function $x^\alpha e^{-x}$, i.e.

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = 0 \quad if \quad m \ne n \tag{2.3.1}$$

We also have

$$\int_0^\infty x^\alpha e^{-x} (L_n^{(\alpha)}(x))^2 dx = \frac{\Gamma(1+\alpha+n)}{n!} \tag{2.3.2}$$

Moreover, from above we can show that the functions

$$\psi_n(x) = \left(\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right)^{1/2} x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x) \tag{2.3.3}$$

n=0, 1, 2,…, Re(α)> -1, form a system of complete orthonormal functions in the Hilbert space $L_2(I)$ where I= (0, ∞) and these functions $\psi_n(x)$ are eigen functions of the operator.

$$\eta = x^{-\alpha/2} e^{x/2} D x^{\alpha+1} e^{-x} D x^{-\alpha/2} e^{x/2} \tag{2.3.4}$$

corresponding eigen values $\lambda_n = -n$

In particular, when α=0 the polynomials $L_n^{(\alpha)}(x)$ reduce to $L_n(x)$ called simple Laguerre polynomial

$$L_n(x) = \sum_{m=0}^{n} \frac{(-1)^m n! x^m}{(m!)^2 (n-m)!} \tag{2.3.5}$$

Although the most of results for $L_n(x)$, which are used frequently, may be obtained by putting α=0 in the results already known for $L_n^{(\alpha)}(x)$ yet some of them, which are very important and useful in our subsequent work, be discussed as below:

$L_n(x)$ Satisfies the second order differential equation

$$xD^2 L_n(x) + (1-x) D L_n(x) + n L_n(x) = 0$$

which can also be written as

$$D[x\ e^{-x}\ D L_n(x)] + n e^{-x}\ L_n(x) = 0 \tag{2.3.6}$$

The polynomials $L_n(x)$, n=0,1,2,…form an orthogonal set with respect to the wt. function e^{-x} over the interval (0,∞) i.e.

$$\int_0^\infty e^{-x} L_n(x)\, L_m(x) = 0, \quad m \neq n \tag{2.3.7}$$

We also have

$$\int_0^\infty e^{-x} L_n{}^2(x)\, dx = 1, \tag{2.3.8}$$

From above it can be shown that the functions $\psi_n(x) = e^{-x/2}\, L_n(x)$ form a complete orthonormal set in the Hilbert space $L_2(I)$, where $I = (0, \infty)$.

Therefore in view of above last three equations we arrive at the conclusion that in the interval $I = (0, \infty)$ the functions

$$\psi_n(x) = e^{-x/2}\ L_n(x) \quad n=0,1,2,\ldots \tag{2.3.9}$$

are eigen functions of the differential operator

$$\eta = e^{x/2} Dx\ e^{-x} D\ e^{x/2} \tag{2.3.10}$$

corresponding eigen values $\lambda_n = -n$

Next, if α is a non-negative integer, say k, then polynomials $L_n^{(k)}(x)$ are called associated Laguerre polynomials which is explicit written as

$$L_n{}^{(k)}(x) = \sum_{m=0}^{n} \frac{(-1)^m (n+k)!}{(k-m)!(k+m)!m!} \tag{2.3.11}$$

We also write here a useful relation between associated Laguerre polynomials and simple Laguerre polynomials:

$$L_n{}^{(k)}(x) = \frac{d^k}{dx^k}\{L_{n+k}(x)\} \tag{2.3.12}$$

Also From expansion theorem for orthogonal polynomials, for any f(x) defined on (0,∞), we have

$$f(x) = \sum_{n=0}^{\infty} a_n\, L_n{}^{(\alpha)}(x) \tag{2.3.13}$$

at the point of continuity of f(x) and

$$\frac{1}{2}[f(x+0) + f(x-0)] = \sum_{n=0}^{\infty} a_n\, L_n{}^{(\alpha)}(x)$$

at the points of discontinuity of f(x),

where

$$a_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty x^\alpha e^{-x} f(x) L_n{}^{(\alpha)}(x)\, dx \tag{2.3.14}$$

for Re(α) > -1.

In the above view, for non negative integer n, we have

$$x^n = \sum_{k=0}^{n} \frac{(-1)^k n! (1+\alpha)_n L_k{}^{(\alpha)}(x)}{(n-k)!(1+\alpha)_k} \tag{2.3.15}$$

We now give expansions of Hermite polynomial $H_n(x)$ and Legendre polynomial $P_n(x)$ in the series of $L_n^{(\alpha)}(x)$

$$H_n(x) = 2^n(1+\alpha)n\sum_{m=0}^{n} {}_2F_2\left[\begin{matrix}-\frac{1}{2}(n-m), -\frac{1}{2}(-n-m-1); \\ -\frac{1}{2}(\alpha+n), -\frac{1}{2}(-\alpha+n-1);\end{matrix} \frac{-1}{4}\right] \times \frac{(-n)_m L_m^{(\alpha)}(x)}{(1+\alpha)_m} \tag{2.3.16}$$

And

$$P_n(x) = \frac{2^n\left(\frac{1}{2}\right)_n(1+\alpha)_n}{n!}\sum_{m=0}^{n} {}_2F_3\left[\begin{matrix}-\frac{1}{2}(n-m), -\frac{1}{2}(-n-m-1); \\ \frac{1}{2}-n', \frac{1}{2}(\alpha+n), -\frac{1}{2}(-\alpha+n-1);\end{matrix} \frac{-1}{4}\right] \times \frac{(-n)_m L_m^{(\alpha)}(x)}{(1+\alpha)_m} \tag{2.3.17}$$

3. Testing function Space A($L_n^{(\alpha)}$) and expansion of test functions

we now construct a subset of $L_2(I)$ which serves as a testing function space for the generalized functions . Let us denote this subset of $L_2(I)$ by A($L_n^{(\alpha)}$).

Let A($L_n^{(\alpha)}$)consists of all functions φ (x) that possess the following three properties:

(i) φ (x) is defined, real valued and smooth on *I=(0, ∞)*

(ii) For each k=0, 1, 2, ……

$$\gamma_k(\varphi) = \gamma_0(\eta^k\varphi) = \left[\int_0^\infty |\eta^k\varphi(x)|^2\right]^{1/2} < \infty \tag{3.1.1}$$

where operator η is defined by (4.4.2), and

(iii) for each n and k as above, we have

$$(\eta^k\varphi, \psi_n) = (\varphi, \eta^k\psi_n) \tag{3.1.2}$$

where $\psi_n(x) = \left(\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right)^{1/2} x^{\alpha/2}e^{-x/2}L_n{}^{(\alpha)}(x)$

It can easily be proved that A($L_n^{(\alpha)}$) is a vector space this vector space is made into a topological vector space by defining a topology generated by separating collection of semi norms γ_k, $k = 0,1,2,\ldots$ …defined by (3.1.1). This topology having a countable local base is metrizable through the metric defined by

$$d(\varphi,\psi) = \sum_{k=0}^{\infty} \frac{2^{-k}\gamma_k(\varphi-\psi)}{1+\gamma_k(\varphi-\psi)} \tag{3.1.3}$$

where $\varphi,\psi \in$ A($L_n^{(\alpha)}$). Following Rudin [10] and Zemanian [15], it is clear that d is complete hence A($L_n^{(\alpha)}$) is a Frechet space and therefore A($L_n^{(\alpha)}$)turns out to be a testing function space.

Further from (2.3.4) we have ψ_n be the eigen functions of the differential operator η corresponding eigen values λ_n = -n, so that it is evident that $\eta\psi_n = -n\psi_n$ and, in general,

$$\eta^k\psi_n = (-n)^k\psi_n \tag{3.1.4}$$

And then we have

$$[\gamma_k(\psi_n)]^2 = \int_0^\infty |\eta^k\psi_n|^2\,dx = n^{2k}\int_0^\infty (\psi_n)^2 dx = n^{2k} \tag{3.1.5}$$

also for $n \neq m$ we have

$$(\eta^k\psi_n,\psi_m) = (-n)^k(\psi_n,\psi_m) = 0 = (-m)^k(\psi_n,\psi_m) = (\psi_n,\eta^k\psi_m) \tag{3.1.6}$$

Now from (3.1.5) and (3.1.6) it follows that every ψ_n satisfies conditions (ii) and (iii) and also condition (i) by its definition and hence every ψ_n is a member of A($L_n^{(\alpha)}$). Also from (3.1.6) it can be shown that the operator η is a continuous mapping of A($L_n^{(\alpha)}$) into itself.

Now from condition (ii) it follows that for every $\phi\in$ A($L_n^{(\alpha)}$), $\eta^k\phi$ is in $L_2(I)$ and then by the Fourier Laguerre expansion of $\eta^k\phi$, we have

$$\eta^k\varphi = \sum_{n=0}^{\infty}(\eta^k\varphi,\psi_n)\psi_n = \sum_{n=0}^{\infty}(\varphi,\eta^k\psi_n)\psi_n$$

$$= \sum_{n=0}^{\infty} \lambda_n{}^k(\varphi,\psi_n)\psi_n \qquad where\ \lambda_n = -n$$

$$= \sum_{n=0}^{\infty} (\varphi,\psi_n)\lambda_n{}^k\psi_n = \sum_{n=0}^{\infty} (\varphi,\psi_n)\eta^k\psi_n$$

These series converge in $L_2(I)$, consequently for each k

$$\gamma_k[\varphi - \sum_{n=0}^{N}(\varphi,\psi_n)\,\psi_n] = \gamma_0[\eta^k\varphi - \sum_{n=0}^{N}(\varphi,\psi_n)\eta^k\,\psi_n] \to 0 \tag{3.1.7}$$

as N→ ∞

Therefore this shows that every $\phi\in$ A($L_n^{(\alpha)}$)can be represented as

$$\varphi = \sum_{n=0}^{\infty}(\varphi,\psi_n)\psi_n \tag{3.1.8}$$

and the series on right hand side converges in A($L_n^{(\alpha)}$). Also (3.1.8) defines inverse Laguerre transform of $\phi\in$ A($L_n^{(\alpha)}$).

4. Generalized function Space A′($L_n^{(\alpha)}$) and expansion of generalized functions

We denote the dual of A($L_n^{(\alpha)}$) by A′($L_n^{(\alpha)}$) which is a space of generalized functions on which Laguerre transform will be defined. A′($L_n^{(\alpha)}$) is also complete since A ($L_n^{(\alpha)}$) is a complete countable multinorm space.

We now discuss below some useful and important properties of the generalized function space A′($L_n^{(\alpha)}$).

We define a differential operator η′ on A′($L_n^{(\alpha)}$) where η′ is obtained by reversing the order in which differentiation and multiplication by functions $x^{\alpha/2}e^{-x/2}$ and $x^{\alpha+1}e^{-x}$ occur in $\eta = x^{\alpha/2}e^{-x}Dx^{\alpha+1}e^{-x}Dx^{\alpha/2}e^{x/2}$and replacing each D by –D. Then we see that η′=η. Thus the differential operator on A′($L_n^{(\alpha)}$) is defined by the relation

$$(\eta f,\varphi) = (f,\eta\varphi), \quad f \in A'(L_n{}^{(\alpha)}), \quad \varphi \in A(L_n{}^{(\alpha)}) \tag{4.1.1}$$

Since η is a continuous linear mapping of A ($L_n^{(\alpha)}$) into A ($L_n^{(\alpha)}$) therefore it is also a continuous linear mapping of A′($L_n^{(\alpha)}$) into A′($L_n^{(\alpha)}$).

It is obvious from definition of $A'(L_n^{(\alpha)})$ that $\mathcal{D}(I) \subset A(L_n^{(\alpha)})$ where $\mathcal{D}(I)$ is space of distributions and convergence in $\mathcal{D}$(I) implies convergence in $A(L_n^{(\alpha)})$. Consequently the restriction of any $f \in A'(L_n^{(\alpha)})$ to $\mathcal{D}$(I) is a member of $\mathcal{D}'$(I).

Since $A(L_n^{(\alpha)}) \subset L_2(I)$ and the dual of L_2(I) is L_2(I). Therefore $L_2(I) \subset A'(L_n^{(\alpha)})$

(iv) $\in'$(I) is a subspace of $A'(L_n^{(\alpha)})$ where $\in'$(I) is the space of all distributions whose support are compact subsets of I.

(v) For each $f \in A'(L_n^{(\alpha)})$ there exist a non negative integer r and a positive constant C such that

$$|(f,\varphi)| \le C \max_{0\le k\le r} \gamma_k(\varphi) \tag{4.1.2}$$

for every $\varphi \in A(L_n^{(\alpha)})$, here r and C depend on f but not on φ.

Finally we have to show that if then $f \in A'(L_n^{(\alpha)})$ then

$$f = \sum_{n=0}^{\infty} (f,\psi_n)\psi_n \tag{4.1.3}$$

where the series converges is $A'(L_n^{(\alpha)})$.

For any $\phi \in A(L_n^{(\alpha)})$ and $f \in A'(L_n^{(\alpha)})$ we have

$$(f,\phi) = (f,\Sigma(\phi,\psi_n)\psi_n) \quad by\ using\ (3.1.8)$$

$$= \Sigma(f,\psi_n)\overline{(\phi,\psi_n)}$$

$$= \Sigma(f,\psi_n)(\psi_n,\phi)$$

$$= (\Sigma(f,\psi_n)\psi_n,\phi)$$

i.e $$f = \sum_{n=0}^{\infty} (f,\psi_n)\psi_n$$

Further by using this expansion we can proved the following important theorem:

A necessary and sufficient condition for f to be in $A'(L_n^{(\alpha)})$ is that there be some nonnegative integer q and a $g \in L_2(I)$ such that

$$f = \eta^q g + \Sigma_{\lambda_n=0}\, c_n\psi_n \tag{4.1.4}$$

Where the c_n *denote complex constants*.

Proof: Let relation (4.1.4) satisfied for some nonnegative integer q. We can show that $\eta^q g \in A'(L_n^{(\alpha)})$ and that $A(L_n^{(\alpha)}) \subset A'(L_n^{(\alpha)})$. Since $\psi_n \in A(L_n^{\alpha})$, then in view of (4.1.3) it follows that $f \in A'(L_n^{(\alpha)})$.

Conversely Let $f = \Sigma F(n)\psi_n \in A'\left({L_n}^{(\alpha)}\right)$. Set $G(n) = \lambda_n^{-q}F(n)$ whenever $\lambda_n \neq 0$, where $q \geq 0$ is such that $\Sigma_{\lambda_n \neq 0}|\lambda_n|^{-2q}|F(n)|^2$ converges; also, set $G(n) = 0$ when $\lambda_n = 0$. Hence, $\sum_{n=0}^{\infty}|G(n)|^2$ converges, and, by the Riesz-Fischer theorem , there exists a $g \in L_2(I)$ such that $G(n) = (g, \psi_n)$. Moreover, since then we may write

$$f = \sum_{n=0}^{\infty} F(n)\,\psi_n = \sum_{\lambda_n \neq 0} \lambda_n q\, G(n)\psi_n + \sum_{\lambda_n = 0} F(n)\psi_n$$

$$= \sum_{n=0}^{\infty} (g, {\lambda_n}^{q}\psi_n)\,\psi_n + \sum_{\lambda_n = 0} F(n)\psi_n$$

$$= \sum_{n=0}^{\infty} (\eta^q g,\, \psi_n)\,\psi_n + \sum_{\lambda_n = 0} F(n)\psi_n$$

$$= \eta^q g + + \sum_{\lambda_n = 0} F(n)\psi_n$$

In particular, for α = 0 we denote our testing function space and space of generalized functions by $A(L_0)$ and $A'(L_0)$ respectively.

"It is most important to mention there that the space $A'(L_0)$ identifies with the space of tempered distributions of positive support **(**see Guillemot-Teissiers [12]**)**"

References

1. Bailey W.N. : On the products of two legendre polynomials with different argument, Proc. London Math. Soc.(2), Vol. 41, pp. 215 – 220 (1938).
2. Ditzan, Z. : Summability of Hermite Polynomial Expansions of Generalized Functions, Proc. Cambridge Philos. Soc. , Vol. 68, pp.129 – 131, (1970)
3. Duran,A.J. : Laguerre Expansions of Tempered Distributions and Generalized Functions, J. Math. Anal. Appl., Vol. 150, pp. 166–180, (1990).
4. Giertz, M. : On the expansion of certain generalized functions in series of orthogonal functions, Proc. London Math. Soc. , 3rd Ser., Vol. 14 , pp. 45 – 52, (1964).
5. Judge, D. : On Zemanian's distributionals eigenfunctions transforms, J. Math. Anal. Appl., Vol. 34, pp. 187–201, (1971)
6. Korevaar, J. : Pansions and the theory of Fourier transforms, Trans. Am. Math. Soc., Vol. 91, pp. 53 – 101 (1959).

7. Pandey, J.N. and Pathak, R.S. : Eigen function expansion of generalized function, Nagoya Math. J. Vol. 72, 1 – 25, (1978)

8. Pathak, R.S. : Summability of Leguerre polynomial expansion of generalized functions, J. Inst. Math. Appl., Vol. 21 , 171 –180, (1978)

9. Pathak, R.S. : Orthogonal series representations for generalized functions, J. Math. Anal. Appl., Vol. 130, pp.187 – 201, (1988).

10. Rudin, W. : Functional Analysis, Tata Mc. Graw Hill Publishing Company, (1974).

11. Schwartz, L. : Theorie des distributions Vols. I and II, Hermann, Paris, 1950-1951.

12. Teissiers, and Guillemot, M. : Developments des Distributions en series de functions orthogonals: Series de Legendre et de Laguerre, Ann. Scuola Norm. Sup. Pisa (3), Vol. 25, pp. 519 – 573, (1971)

13. Walter ,G.G. : Expansions of Distributions, Trans. Amer. Math. Soc., Vol. 116, pp. 492, (1965).

14. Widlund ,O. : On the expansion of generalized functions in series of Hermite functions, Kgl. Tekn. Hogsr. Handel. , No. 173, (1961).

15. Zemanian, A.H. : Distribution Theory and Transform Analysis, Mc. Graw. Hill, New York, (1965).

16. Zemanian, A.H. : Orthonormal series expansions of certain distributions and distributional transforms calculus, J. Math. Anal. Appl., Vol. 14, pp. 263 – 275, (1966).

17. Zemanian, A.H. : The distributional Laplace and Mellin transformations, SIAM. J. Appl. Math., Vol. 14, pp. 41-49, (1966).

18. Zemanian, A.H. : Generalized Integral Transformations, Inter Science Publishers, New York, 1968.