πg-Locally Closed Sets

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ABSTRACT: The aimof this paper is to continue the study of generalizations of locally closed sets and investigate the class of πg-locally closed sets in a topological space.

Keywords: πg-closed set, π-open set, θ-g-open set, θ-closed set.

1. Introduction

The initiation of the study of generalized closed sets was done by Levine [9] in 1970. The notion of πg-closed sets as a weak form of generalized closed sets was introduced by Dontchev and Noiri [6] in 2000. The notion of locally closed sets in a topological space was introduced by Bourbaki [4]. Ganster and Reilly [8] further studied the properties of locally closed sets and defined the LC-continuity and LC-irresoluteness. Balachandran et al. [3] introduced the concepts of generalized locally closed sets and GLC-continuous functions and investigated some of their properties. In 1997, Arockiarani et al. [1] studied regular generalized locally closed sets and RGL-continuous functions in a topological space. The aim of this paper is to continue the study of generalizations of locally closed sets and investigate the class of πg-locally closed sets in a topological space.

Throughout this paper, a space (X, τ) denotes a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, cl(A) and int(A) denote the closure of A and the interior of A.

Let P(X) be the power set of X and A set $A \subset (X, \tau)$ is called θ-closed [11] if $A = cl_\theta(A)$, where $cl_\theta(A) = \{x \in X: cl(U) \cap A \neq \phi, U \in \tau \text{ and } x \in U\}$. The complement of a θ-open set is called θ-closed.

Before entering into our work, we recall the following definitions which are prerequisite for this paper.

2. Preliminaries

Definition 2.1 A subset A of a space (X, τ) is called

1. regular open [10] if $A = int(cl(A))$.
2. π -open [12] if it is the finite union of regular open sets.
3. generalized closed (g-closed) [9] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X.

4. πg-closed [6] if cl(A) ⊂ U whenever A ⊂ U and U is π -open in X.

5. θ-generalized closed (θ-g-closed) [7] if $cl_\theta(A)$ ⊂ U whenever A ⊂ U and U is open in X.

6. locally closed [8] if A = S ∩ F where S is open and F is closed in X.

7. generalized locally closed (glc) [3] if A = S ∩ F where S is g-open and F is g-closed in X.

8. θ-generalized locally closed (θglc) [2] if A = S ∩ F where S is θ-g-open and F is θ-g-closed in X.

9. θ-locally closed (θ lc) [2] if A = S ∩ F where S is θ-open and F is θ-closed in X.

10. θlc*-set [7] if A = S ∩ F where S is θ-open and F is closed in X.

11. θlc**-set [7] if A = S ∩ F where S is open and F is θ-closed in X.

12. glc*-set [3] if A = S ∩ F where S is g-open and F is closed in X.

13. glc**-set [3] if A = S ∩ F where S is open and F is g-closed in X.

14. θ-glc**-set [7] if A = S ∩ F where S is θ-g-open and F is closed in X.

15. θ-glc**-set [7] if A = S ∩ F where S is open and F is θ-g-closed in X.

The complements of the above mentioned closed (open) sets are called their respective open (closed) sets.

Remark 2.2 [7] The following diagram holds in a topological space.

θ-closed → θ-g-closed

↓ ↓

closed → g-closed → πg-closed

3. πg-locally closed sets

Definition 3.1 A subset S of a space (X, τ) is said to be πg-locally closed (πglc) if S = G ∩ F where G is πg-open and F is πg-closed in (X, τ).

Definition 3.2 A subset S of a space (X, τ) is called πglc* if there exists a πg-open set G and a closed set F of (X, τ) such that S = G ∩ F.

Definition 3.3 A subset B of a space (X, τ) is called πglc** if there exists an open set G and a πg-closed set F of (X, τ) such that B = G ∩ F.

The collection of all πg-locally closed (resp. πglc*, πglc**) sets of a space (X, τ) will be denoted by πGLC(X, τ) (resp. πGLC*(X, τ) , πGLC**(X, τ)).

From the above definitions we have the following results.

Theorem 3.4 1. Every locally closed set is πglc.

2. Every θ-locally closed set is πglc.

3. Every θglc-set is πglc.

4. Every πglc*-set or πglc** is πglc.

5. Every glc-set is πglc.

6. Every θlc-set is πglc* or πglc**.

7. Every glc*-set is πglc*.

8. Every θlc*-set is πglc*.

9. Every θlc**-set is πglc**.

10. Every θ-glc*-set is πglc*.

11. Every locally closed set is πglc* and πglc**.

However the converses of the above Theorem are not true as seen from the following examples.

Example 3.5 Let X = {a, b, c, d} and τ = {ϕ, X, {a, b}, {a, b, c}, {a, b, d}}. Then locally closed sets are ϕ, X, {c}, {d}, {a, b}, {c, d}, {a, b, c}, {a, b, d} and πglc-sets are P(X). It is clear that {a, c} is πglc-set but it is not locally closed.

Example 3.6 In Example 3.5, θ-locally closed sets are ϕ, X and πglc-sets are P(X). It is clear that {a, b} is πglc-set but it is not θ-locally closed set.

Example 3.7 In Example 3.5, θglc-sets are ϕ, X, {a}, {b}, {a, b}, {c, d}, {a, c, d}, {b, c, d} and πglc-sets are P(X). It is clear that {b, c} is πglc-set but it is not θglc-set.

Example 3.8 Let X = {a, b, c, d, e} and τ = {ϕ, X, {a}, {e}, {a, e}, {c, d}, {a, c, d}, {c, d, e}, {a, c, d, e}, {b, c, d, e}}. Then πglc*-sets are ϕ, X, {a}, {b}, {c}, {d}, {e}, {a, b}, {a, c}, {a, d}, {a, e}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, e}, {a, c, d}, {a, c, e}, {a, d, e}, {b, c, d}, {c, d, e}, {a, b, c, d}, {a, c, d, e}, {b, c, d, e} and πglc-sets are P(X). It is clear that {b, c} is πglc-set but it is not πglc*-set.

Example 3.9 In Example 3.5, πglc-sets are P(X) and glc-sets are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}. It is clear that {b, c} is πglc-set but it is not glc-set.

Example 3.10 In Example 3.5, θlc-sets are ϕ, X and πglc* (or) πglc**-sets are P(X). It is clear that {a, b} is πglc* (or) πglc**-set but it is not θlc-set.

Example 3.11 In Example 3.5, glc*-sets are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {c, d}, {a, b, c}, {a, b, d} and πglc*-sets are P(X). It is clear that {b, c} is πglc*-set but it is not glc*-set.

Example 3.12 In Example 3.5, θlc*-sets are ϕ, X, {c}, {d}, {c, d} and πglc*-sets are P(X). It is clear that {a, d} is πglc*-set but it is not θlc*-set.

Example 3.13 In Example 3.5, θlc**-sets are ϕ, X, {a, b}, {a, b, c}, {a, b, d} and πglc**-sets are P(X). It is clear that {a} is πglc**-set but it is not θlc**-set.

Example 3.14 In Example 3.5, θglc*-sets are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {c, d} and πglc*-sets are P(X). It is clear that {b, c} is πglc*-set but it is not θglc*-set.

Example 3.15 In Example 3.5, locally closed sets are ϕ, X, {c}, {d}, {a, b}, {c, d}, {a, b, c}, {a, b, d} and πglc* and πglc**-sets are P(X). It is clear that {a, c} is both πglc* and πglc**-set but it is not locally closed set.

Theorem 3.16 For a subset S of a space (X, τ) the followings are equivalent:

1. S ∈ πglc*(X, τ).
2. S = P ∩ cl(S) for some πg-open set P.
3. cl(S) − S is πg-closed.
4. S ∪ (X − cl(S)) is πg-open.

Proof. (1) ⇒ (2) Let S ∈ πglc*(X, τ). Then there exists a πg-open set P and a closed set F such that S = P ∩ F. Since S ⊂ P and S ⊂ cl(S) we have S ⊂ P ∩ cl(S). Conversely, since cl(S) ⊂ F, P ∩ cl(S) ⊂ P ∩ F = S which implies that S = P ∩ cl(S).

(2) ⇒ (1): Since P is πg-open and cl(S) is closed P ∩ cl(S) ∈ πglc*(X, τ).

(3) ⇒ (4): Let F = cl(S) − S. Then F is πg-closed by the assumption and X − F = X ∩ (cl(S) − S) c = S ∪ (X − cl(S)). But X − F is πg-open. This shows that S ∪ (X − cl(S)) is πg-open.

(4) ⇒ (3): Let U = S ∪ (X − cl(S)). Then U is πg-open. This implies that X – U is πg-closed and X − U = X − (S ∪ (X − cl(S))) = cl(S) ∩ (X − S) = cl(S) − S. Thus cl(S) − S is πg-closed.

(4) ⇒ (2): Let U = S ∪ (X − cl(S)). Then U is πg-open. Hence, we prove that S = U ∩ cl(S) for some πg-open set U. U ∩ cl(S) = (S ∪ (X − cl(S))) ∩ cl(S) = (cl(S) ∩ S) ∪ (cl(S) ∩ X − cl(S)) = S ∪ ϕ = S. Therefore S = U ∩ cl(S).

(2) ⇒ (4): Let S = P ∩ cl(S) for some πg-open set P. Then we prove that S ∪ (X − cl(S)) is πg-open. S ∪ (X − cl(S)) = (P ∩ cl(S)) ∪ (X − cl(S)) = P ∩ (cl(S) ∪ X − cl(S)) = P ∩ X = P which is πg-open. Thus S ∪ (X − cl(S)) is πg-open.

Definition 3.17 A topological space (X, τ) is called πg-submaximal (resp. g- submaximal [5]) if every dense subset is πg-open (resp. g-open).

Theorem 3.18 A topological space (X, τ) is πg-submaximal if and only if P(X) = πglc*(X, τ).

Proof. Necessity: Let S ∈ P(X) and let V = S ∪ (X − cl(S)). Then V is πg-open and cl(V) = cl(S) ∪ (X − cl(S)) = X. This implies that V is a dense subset of X. Hence S ∈ πglc*(X, τ). Therefore, P(X) = πglc*(X, τ).

Sufficiency: Let S be a dense subset of (X, τ). Then S ∪ (X − cl(S)) = S ⇒ S ∈ πglc*(X, τ) and S is πg-open. This proves that X is πg-submaximal.

Remark 3.19 It follows from definitions that if (X, τ) is g-submaximal, then it is πg-submaximal. But the converse is not true as seen from the following example.

Example 3.20 In Example 3.5, dense sets are X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, g-open sets are ϕ, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d} and πg-open sets are P(X). Then it is πg-submaximal but not g-submaximal.

Theorem 3.21 If Z is πg-closed, π -open set in (X, τ) and A ∈ πglc*(Z, τ /Z) then A ∈ πglc*(X, τ).

Proof. Let A ∈ πglc*(Z, τ /Z). Then A = G ∩ F where G is πg-open and F is closed in (Z, τ /Z). Since F is closed in (Z, τ /Z), F = B ∩ Z for some closed set B of (X, τ). Therefore A = G ∩ B ∩ Z. Then B ∩ Z is closed. Hence A ∈ πglc*(X, τ).

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