A STUDY ON FREE ABELIAN GROUPS AND ITS PROPERTIES

Meera Rose Joseph[1]
[1]Assistant Professor,
Nirmala College, Muvattupuzha

Abstract: An Abelian group, also called a commutative group is a group in which the result of applying the group operation to two group elements does not depend on their order. In this paper we discuss about Free abelian groups and its properties, where Free abelian groups are special cases of Free modules as abelian groups are nothing but modules over the ring. we are taking into account how a Free abelian group is related to Free group and Torsion free abelian group. Also we consider one of the important application of Free abelian groups that is "The proof of Fundamental Theorem of Finitely Generated Abelian Groups"

Keywords: Free abelian group, Free modules, Free group, Torsion free abelian group

1. Introduction

A group is an ordered pair (G,*), where G is a nonempty set and * is a binary operation on G such that the following properties hold:
(1) For all a, b, c ∈ G, a * (b * c) = (a * b) * c. (associative law)
(2) There exists e ∈ G such that, for any a ∈ G , a*e = a = e*a.
(Existence of an identity)
(3) For each a ∈ G, there exits b ∈ G, there exists b ∈ G such that
a * b = e = b * a. (existence of an inverse).
An Abelian group is a group with the property that a * b = b * a. Again a Free Abelian group is an Abelian group with a basis where a basis is a subset of the elements such that every group element can be found by adding or subtracting a finite number of basis elements and such that every element, its expression as a linear combination of basis elements is unique. Number of elements in the basis may be finite or infinite. We mainly discuss about free abelian groups with finite basis and also some of the properties of Free Abelian groups. Proof of Fundamental Theorem of finitely generated abelian group is proved with the help of free abelian groups.

2. Free Abelian Groups

Definition 2.1 A free abelian group is an abelian group that has a basis in the sense that every element of the group can be written in one and only one way as a finite linear combination of elements of the basis with integer coefficients.

Definition 2.2 Let G be an abelian group. A set B ⊂ G is a basis of G if
(1) B generates G
(2) If for some $x_1, ...x_k \in B$ and $n_1, ...n_k \in Z$ we have $n_1x_1+.........n_kx_k = 0$ then $n_1 = ... = n_k = 0$.

EXAMPLES:

1) Integers under addition form a free abelian group with basis 1. Each integer can be formed by using addition or substraction to combine some number of copies of the number 1 and each integer has a unique representation as an integer multiple of the number 1.
2) Integer lattices also form a free abelian group. The two-dimensional integer lattice,consisting of the points in the plane with integer Cartesian co-ordinates form a free abelian group under vector addition with the basis (0, 1), (1, 0). if we say $e_1 = (1, 0)$ and $e_2 = (0, 1)$, then the element (4, 3) can be written as $(4, 3) = 4e_1 + 3e_2$ where multiplication is defined so that $4e_1 = e_1 + e_1 + e_1 + e_1$.In this basis, there is no way to write (4, 3), but with different basis such as (1, 0), (1, 1).
∴ (4, 3) = (1, 0) + 3(1, 1).
3) The trivial group 0 is also considered to be free abelian, with basis the empty set. It may be interpreted as a direct product of zero copies of Z.

3. Properties of free abelian group

Definition 3.1 **Rank** of a free abelian group: Rank of a free abeilan group is defined to be the cardinality of the basis of a free abelian group. Every two bases of the same free abelian group have the same cardinality. In particular, a free abelian group is finitely generated if and only if its rank is a finite number n, in which case the group is isomorphic to Z_n.This notion of rank can be generated from free abelian groups to abelian groups that are not necessarily free. The rank of an abelian group G is defined as the rank of a free abelian subgroup F of G for which the quotient group G | F is a torsion group.

Theorem 3.1 Let $G \neq 0$ be free abelian group with a finite basis. Then every basis of G is finite and all bases of G have the same number of elements.

Proof:
Let G have a basis $X = x_1, x_2, ...x_r$. Then by the above result

$$G \equiv Z \times Z \times Z ..... Z \text{ for r factors.}$$

Let $2G = \{2g / g \in G\}$.
We know 2G is a subgroup of G. Since $G \equiv Z \times Z \times Z ..... Z$ for r factors.

$$G \mid 2G \equiv (Z \times Z \times Z..... \times Z) \mid (2Z \times 2Z \times ... \times 2Z)$$
$$\equiv Z_2 \times Z_2 \times Z_2..... \times Z_2 \text{ for r factors.}$$

Thus $|G \mid 2G| = 2^r$. So the number of elements in any finite basis is $\log_2 |G \mid 2G|$. Thus any two finite basis have the same number of elements.It remains to show that G cannot have an infinite basis. Let Y be any basis for G and let $y_1, y_2, ...y_s$ be some elements in Y .Let H be the subgroup of G generated by $y_1, y_2, ...y_s$ and let K be the subgroup of G be generated by the remaining elements of Y . Then

$$G \equiv H \times K,$$
$$\text{so } G \mid 2G \equiv (H \times K) \mid (2H \times 2K)$$
$$\equiv (H \mid 2H) \times (K \mid 2K).$$

Since $|H \mid 2H| = 2^S$.
Since we have $| G \mid 2G | = 2^r$, we see that $s \leq r$. Then Y cannot be an infinite set for we could take $s > r$.

The direct product of two free abelian groups is itself free abelian, with basis the disjoint union of the bases of the two groups. More generally the direct product of any finite number of free abelian groups is free abelian. For infinite families of free abelian groups the direct product is not necessarily free abelian.

Theorem 3.2 If G and G′ are free abelian groups, then G × G′ is free abelian.

Proof:
Suppose that G and G′ are free abelian with bases X and X′ respectively.Let $X = (x, 0) \mid x \in X$ and $X = (0, x) \mid x \in X$. We claim that $Y = X \cup X'$ is a basis for $G \times G'$. Let $(g, g) \in G \times G'$ Then $g = n_1x_1 + ... + n_rx_r$ and $g = m_1x_1 + ... + m_sx_s$ for unique choices of n_i and m_j , except for possible zero coefficients. Thus $(g, g) = n_1(x_1, 0) + ... + n_r(x_r, 0) + m_1(0, x_1) + ... + m_s(0, x_s)$ for unique choices of the n_i and m_j , except for possible zero coefficients. This shows that Y is a basis for G × G′, which is thus free abelian.

Theorem 3.3. Let G be a free abelian group of a finite rank n and let H be a subgroup of G. Then H is a free abelian group and rank of H ≤ rank of G.

To prove the theorem we need a lemma.

Lemma 1. If $f : G \to H$ is an epimorphism of abelian groups and H is a free abelian group then $G \equiv H \oplus \ker f$.
Proof :

Since G is a free abelian group of rank n,

$$G \cong Z \times Z \times Z..... Z \text{ (n summands)}$$
$$\cong Z_n.$$

∴ we can assume $G = Z_n$.
We want to show if $H \subseteq Z_n$, then H is a free abelian group and rank $H \leq n$.
We apply induction with respect to n.
If n = 1 then H = kZ for some $k \geq 0$ so H = 0 or $H \cong Z$. ∴ H is free abelian.
Next,assume that for some n every subgroup of Z_n is a free Abelian group of rank ≤ n and let $H \subseteq Z_{n+1}$. Define homomorphism $f : Z_{n+1} \to Z$; $f (m_1,m_2, ...m_{n+1}) = m_{n+1}$.
We have $\ker(f) = (m_1,m_2, ...m_n, 0) \mid m_i \in Z \cong Z_n$.
We have an epimorphism $(f \mid H) : H \to \mathrm{Im}(f \mid H)$. Since $\mathrm{Im}(f \mid H) \subseteq Z$, thus $\mathrm{Im}(f \mid H)$ is a free abelian group and by the above lemma

$$H \cong \mathrm{Im}(f \mid H) \oplus \ker(f \mid H).$$

Then we have $\ker(f \mid H) \subseteq \ker(f)$. But we know $\ker(f) \in Z_n$ and ker(f) is a free abelian group of rank n.By the inductive assumption we get that $\ker(f \mid H)$ is a free abelian group of rank ≤ n. Therefore

$$H \cong \mathrm{Im}(f \mid H) \oplus \ker(f \mid H),$$

where Im(f | H) is a free abelian group of rank ≤ 1 and ker(f | H) is a free abelian group of rank ≤ n + 1.

Definition 3.2 Free Group:

In mathematics, the free group F_S over a given set S consists of all expressions that can be built from members of S, considering two expressions different unless their equality follows from the group axioms.The members of S are called generators of F_S. An arbitrary group G is called free if it is isomorphic to F_S for some subset S of G, that is, if there is a subset S of G such that every element of G can be written in one and only one way as a product of finitely many elements of S and their inverses.

Example: The group (Z,+) of integers is free; we can take S = 1.On the other hand, any nontrivial finite group cannot be free, since the elements of a free generating set of a free group have infinite order.

A free abelian group is not free group except in two cases: a free abelian group having an empty basis or having just 1 element in the basis. Other abelian groups are not free groups because in free groups ab must be different from ba if a and b are different elements of the basis, while in free abelian groups, they must be identical.

Definition 3.3 Torsion Groups :

The torsion subgroup A_T of an abelian group A is the subgroup of A consisting of all elements of finite order. An abelian group A is called a torsion (or periodic) group if every element of A has finite order and is called torsion-free if every element of A except the identity is of infinite order.All free abelian groups are torsion-free, and all finitely generated torsion-free abelian groups are Free Abelian.

4. Proof of Fundamental Theorem Of Finitely Generated Abelian Groups

Definition 4.1. Finitely Generated Abelian Groups

Let G be an abelian group. Then G is called a finitely generated Abelian group if it is generated by a finite subset X of G. Every finite abelian group is finitely generated.

1, The integers (Z , +) is a finitely generated abelian group generated by {1}.

2, The integers modulo n , (Z_n , +) is a finitely generated abelian group.

3 Any direct sum of finitely many finitely generated abelian groups is again a finitely generated abelian group.

The group (Q, +) of rational numbers is not finitely generated.If x_1, x_2, ...x_n are rational numbers, pick a natural number k coprime to all the denominators; then 1/k cannot be generated by x_1, x_2, ...x_n.

Fundamental Theorem Of Finitely Generated Abelian Groups

The theorem states that

"Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form $Z_{(p_1)^{r_1}} \times Z_{(p_2)^{r_2}}$ ..... $Z_{(p_n)^{r_n}} \times Z \times Z \ldots Z$ where the p_i are primes, not necessarily distinct and the r_i are positive integers.The direct product is unique except for possible rearrangement of the factors; that is the number (Betti number of G) of factors Z is unique and the prime powers p_i^{ri} are unique. We prove the fundamental theorem by showing that any finitely generated group is isomorphic to a factor of the form(Z × Z × Z...... Z)|(d_1Z | Z .... d_s Z 0 | ...0) where both numerator and denominator have n factor and d_1 divides d_2, which divides d_3,..... which divides d_s. The prime power decomposition of the theorem will then follow. Before that we need to prove some propositions.

Proposition:4.1

If G is an abelian group generated by n elements then G ≅ F | H where F is a free abelian group of rank n and H is some subgroup of F.

Proposition: 4.2

Let G be a finitely generated abelian group with generated set { a1, a2, ...an} Let φ: Z × Z ×...... Z → G (where there are n factors of Z) be defined by φ (h_1, ...h_n) = $h_1a_1 + h_2a_2 + \ldots\ldots + h_na_n$. Then φ is a homomorphism onto G.

Proposition :4.3

If X = {x_1, x_2, ..., x_n} is a basis for a free abelian group G and t ∈ Z then for i > j the set Y = x_1, x_2, ...x_j − 1, $x_j + tx_i$, x_{j+1}, ...x_r is also a basis for G.

Proposition:4.4

Let G be a non zero free abelian group of finite rank n and let K be a non zero subgroup of G. Then K is free abelian of rank $s \leq n$. Furthermore there exist a basis $\{x_1, x_2, x_3.....x_n\}$ for G and positive integers $d_1, d_2, d_3.....d_s$ where d_i divides d_{i+1} for i = 1, , ...s − 1 such that $d_1x_1, d_2x_2...d_sx_s$ is a basis for K.

Proposition: 4.5

Every finitely generated abelian group is isomorphic to a group of the form $Z_{m1} \times Z_{m2}$ ......$Z_{mr} \times Z \times Z$.......Z, where mi divides m_{i+1} for i = 1, ...r − 1.

Proposition :4.6

The group $Z_m \times Z_n$ is cyclic and is isomorphic to Z_{mn} if and only if m and n are relatively prime ie the gcd of m and n is 1.

Proof of fundamental theorem of finitely generated abelian groups

Proposition 4.5 gives us the form of G in terms of a direct product. By proposition 4.6 the cyclic groups of proposition 4.5 can be broken into prime power factors. Consider the torsion subgroup of G. We know from proposition 4.5 the torsion subgroup of a finitely generated abelian group is the direct product of various Z_n s. Let T represent this direct product. Consider G | T. Then G | T is of the form Z × Z × Z × ..... × Z for some numbers of copies of Z. The rank of G | T is the number of copies of Z in this direct product.Also we know all the bases are of same size. But we know Betti number is the number of copies of Z in this direct product.∴ we can say Betti number is the rank of G | T. Also rank of G | T is unique.∴the Betti number is unique across all such direct product representations of G.Hence the proof.

Conclusion

An abelian group is a group which is commutative. A Free abelian group is a group with a basis. There are free abelian groups with finite and infinite basis.Here we focused on free abelian groups with finite basis. Also we discussed various properties of free abelian groups.One of the application of free abelian groups is in the proof of fundamental theorem of finitely generated abelian groups. A finitely generated abelian group G is an abelian group which is generated by a finite subset of G. Also if F is a free abelian group of rank n and G is a finitely generated abelain group generated by the subset $\{a_1, ...a_n\}$ of G, then G ≅ F | H, where H is a subgroup of F. In short we can conclude that in this project we discussed about the topic Free abelian groups and its properties. Also we gave a proof for the fundamental theorem of finitely generated abelian groups.

References

[1] John B. Fraleigh, A First Course in Abstract Algebra,7th edition, Pearson Education Inc,2003.
[2] MK Sen,Shamik Ghosh,Parthasarath Mukhopadhyay,Topics in Abstract Algebra,Universities Press(India) Private Limited.
[3] Michael Artin,ALGEBRA;Prentice Hall of india Private Limited.
[4] Thomas W.Hungerford,Algebra, Springer International Edition.