# Bicomplex Dirichlet Series

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**Abstract**
In the present paper, we have defined the analogous concept of uniform convergence for bicomplex sequences and series. We have given the analogue of Weirstrass's M-test for uniform and absolute convergence for infinite series of Bicomplex variable. We have defined the Bicomplex Dirichlet Series and have studied its various properties.
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## 1. Introduction

The set of Bicomplex Numbers defined as:

$$C_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0,\ i_1 \neq i_2 \text{ and } i_1^2 = i_2^2 = -1,\ i_1 i_2 = i_2 i_1\}$$

Throughout this paper, the sets of complex and real numbers are denoted by $C_1$ and $C_0$, respectively. For details of the theory of Bicomplex numbers, we refer to **[1]**, **[2]**, **[3]** and **[4]**. We shall use the notations $C(i_1)$ and $C(i_2)$ for the following sets:

$$C(i_1) = \{u + i_1 v : u, v \in C_0\};\ C(i_2) = \{\alpha + i_2\beta : \alpha, \beta \in C_0\}$$

### 1.1 Hyperbolic Numbers:

The set of Hyperbolic Numbers H, defined as $H = \{x + y i_1 i_2 : x, y \in C_0\}$

The set of all hyperbolic numbers with non-negative idempotent components is denoted as $H^+$.

Thus, $H^+ = \{a e_1 + b e_2 : a, b \geq 0\}$

### 1.2 Idempotent Elements:

Besides 0 and 1, there are exactly two non – trivial idempotent elements in $C_2$, denoted as $e_1$ and $e_2$ and defined as

$e_1 = \dfrac{1 + i_1 i_2}{2}$ and $e_2 = \dfrac{1 - i_1 i_2}{2}$. Note that $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$.

### 1.3 Cartesian idempotent set:

$$C_2 = C(i_1) \times_e C(i_1) = C(i_1) e_1 + C(i_1) e_2 = \{\xi \in C_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in C(i_1) \times C(i_1)\}$$

$$C_2 = C(i_2) \times_e C(i_2) = C(i_2) e_1 + C(i_2) e_2 = \{\xi \in C_2 : \xi = \xi_1 e_1 + \xi_2 e_2, (\xi_1, \xi_2) \in C(i_2) \times C(i_2)\}$$

### 1.4 Idempotent Representation of Bicomplex Numbers

**(I)** $C(i_1)$- idempotent representation of Bicomplex Number

Throughout this paper $C(i_1)$-idempotent representation of Bicomplex Number is given by

$$\xi = (x_1 + i_1 x_2) + i_2 (x_3 + i_1 x_4) = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2$$
$$= [(x_1 + x_4) + i_1 (x_2 - x_3)] e_1 + [(x_1 - x_4) + i_1 (x_2 + x_3)] e_2 = {}^1\xi e_1 + {}^2\xi e_2$$

**(II)** $C(i_2)$- idempotent representation of Bicomplex Number

Throughout this paper $C(i_2)$ -idempotent representation of Bicomplex Number is given by

$$\xi = (x_1 + i_2 x_3) + i_1 (x_2 + i_2 x_4) = w_1 + i_1 w_2 = (w_1 - i_2 w_2) e_1 + (w_1 + i_2 w_2) e_2$$
$$= [(x_1 + x_4) - i_2 (x_2 - x_3)] e_1 + [(x_1 - x_4) + i_2 (x_2 + x_3)] e_2 = \xi_1 e_1 + \xi_2 e_2$$

### 1.5 Singular Elements

Non zero singular elements exist in $C_2$. In fact, a Bicomplex number $\xi = z_1 + z_2 i_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$. Set of all singular elements in $C_2$ is denoted as $O_2$.

**1.6 Norm**

The norm in $C_2$ is defined as

$$\|\xi\| = \left\{|z_1|^2 + |z_2|^2\right\}^{1/2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right]^{1/2} = \left[x_1^2 + x_2^2 + x_3^2 + x_4^2\right]^{1/2}$$

$C_2$ becomes a modified Banach algebra, in the sense that $\xi, \eta \in C_2$, we have, in general,

$$\|\xi.\eta\| \le \sqrt{2}\,\|\xi\|\,\|\eta\|$$

**1.7 Complex Dirichlet Series**

In general, a Dirichlet series is a series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \qquad \text{............... } \mathbf{(1.1)}$$

where $\{\lambda_n\}$ is a monotonically increasing and unbounded sequence of real numbers, and $s = \sigma + it$ is a complex variable. When the sequence $\{\lambda_n\}$ of exponent is to be emphasized, such a series is called a **Complex Dirichlet series of type** $\lambda_n$.

If $\lambda_n = n$, then $f(s)$ is a power series in $z = e^{-s}$. If $\lambda_n = \log n$, then

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \text{............... } \mathbf{(1.2)}$$

is called an **Ordinary complex Dirichlet series** (cf. **Hardy [5]**).

**Theorem 1.1 [6]:** If the series $\sum_{n=1}^{\infty} a_n n^{-s}$ is convergent for $s_0 = \sigma_0 + it_0$, it is convergent for $s = \sigma + it$, provided that $\sigma > \sigma_0$.

**Theorem 1.2[6]:** If the series $\sum_{n=1}^{\infty} a_n n^{-s}$ is convergent for $s = s_0$. Then it is uniformly convergent throughout the angular region in the plane of s defined by the inequality $|\arg(s - s_0)| \le \alpha < \frac{\pi}{2}$.

**Theorem 1.3[6]:** If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, where $a_n \ge 0$ for every value of n, and where $\sum_{n=1}^{\infty} a_n n^{-\bar{\sigma}}$ is divergent, the function $f(s)$ is not bounded in the region $\sigma > \bar{\sigma}$, $|t| \ge t_0 > 0$.

**Theorem 1.4[6]:** If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is bounded for $\sigma > \alpha$, then $\sum |a_n|^2 n^{-2\alpha}$ is convergent; if $|f(s)| \le M$, then $\sum |a_n|^2 n^{-2\alpha} \le M^2$.

**Lemma 1.1 [7]:** Let $S_0$ be any given complex number and assume that the Dirichlet series $\sum_{n \le x} \alpha_n n^{-s_0}$ has bounded partial sums, say $\left|\sum_{n \le x} \alpha_n n^{-s_0}\right| \le M$ for all $x \ge 1$. Then for each $s$ with $\mathrm{Re}(s) > \mathrm{Re}(s_0)$

$$\left|\sum_{a < n \le b} \alpha_n n^{-s_0}\right| \le 2Ma^{\mathrm{Re}(s) - \mathrm{Re}(s_0)}\left(1 + \frac{|s - s_0|}{\mathrm{Re}(s) - \mathrm{Re}(s_0)}\right)$$

**1.8 The Function $N(\sigma, T)$ :**

Let $t_0$ be a positive number such that $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is regular for $t \ge t_0$ and $\sigma$ sufficiently large. We denote by $N(\sigma, t)$ the number of zeros $\sigma' + it'$ of $f(s)$ such that $\sigma' > \sigma$, $t_0 < t' < T$.

**Theorem 1.5 [6]:** If $f(s)$ is bounded for $\sigma \ge \alpha$, then

$$N(\sigma, T) = O(T) \qquad (\sigma > \alpha).$$

**Theorem 1.6 [6]:** If $f(s)$ is of finite order for $\sigma \geq \alpha$, then

$$N(\sigma,T) = O(T \log T) \qquad (\sigma > \alpha).$$

**Theorem 1.7[7]: Abel's identity.**
For any complex arithmetical function $a(n)$, let $A(x) = \sum_{n \leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume $f:[y,x] \to C_1$ has a continuous derivative on the interval $[y,x]$, $0<y<x$.

Then we have $\sum_{y<n\leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\, dt$.

**Lemma 1.2[8]:** Let $\xi_n = z_{1n} + z_{2n} i_2 \in C_2 \setminus O_2$ be a sequence of invertible bicomplex numbers. The bicomplex product $\prod_{n=1}^{\infty} \xi_n$ converges if and only if the complex products $\prod_{n=1}^{\infty} {}^1\xi_n$ and $\prod_{n=1}^{\infty} {}^2\xi_n$ converge. Moreover, in case of convergence,

$$\prod_{n=1}^{\infty} \xi_n = \left[\prod_{n=1}^{\infty} {}^1\xi_n\right] e_1 + \left[\prod_{n=1}^{\infty} {}^2\xi_n\right] e_2.$$

**Theorem 1.8[8]:** The Bicomplex Riemann Zeta function $\zeta(\xi) = \sum_{n=1}^{\infty} n^{-\xi}$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{n^{\xi}} = \left[\sum_{n=1}^{\infty} \frac{1}{n^{{}^1\xi}}\right] e_1 + \left[\sum_{n=1}^{\infty} \frac{1}{n^{{}^2\xi}}\right] e_2 \text{ i.e., } \zeta(\xi) = \zeta({}^1\xi) e_1 + \zeta({}^2\xi) e_2$$

if and only if $\mathrm{Re}({}^1\xi) > 1$ and $\mathrm{Re}({}^2\xi) > 1$ or equivalently, if and only if $\mathrm{Re}(z_1) > 1$ and $|\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1$.

**2. Uniform Convergence of Sequence and Series of functions of a Bicomplex Variable:**
**Uniform convergence of sequence of functions:**
A sequence $\{f_n(\xi)\}$ of functions defined in $S \subseteq C_2$ is said to converge uniformly to a function $f(\xi)$ defined in $S$ if given any positive number $\varepsilon$, there corresponds a positive number m, independent of $\xi$, such that

$$\| f_n(\xi) - f(\xi) \| < \varepsilon \qquad \forall\, \xi \in S \text{ and } \forall\, n \geq m$$

**Theorem 2.1:** A necessary and sufficient condition for the uniform convergence of a sequence $\{f_n(\xi)\}$ of functions defined in a set $S \subseteq C_2$ is that to every $\varepsilon > 0$, there corresponds a positive integer m independent of $\xi$ such that $\| f_{n+p}(\xi) - f_n(\xi) \| < \varepsilon$, $\forall n \geq m$, $\forall p \geq 0$ and $\forall \xi \in S$.

**2.1 Uniform Convergence of a Series**

Consider an infinite series $\sum_{n=1}^{\infty} f_n(\xi)$ each term of which is a function of $\xi$ defined in a set $S \subseteq C_2$.

Let $S_n(\xi) = \sum_{i=1}^{n} f_i(\xi)$

The series $\sum_{n=1}^{\infty} f_n(\xi)$ is said to be uniformly convergent if the sequence $\{S_n(\xi)\}$ of partial sums of the series is uniformly convergent.

From the general condition of uniform convergence of a sequence of functions, we deduce the corresponding test for the uniform convergence of a series.

**Theorem 2.2:** A necessary and sufficient condition for the uniform convergence of a series $\sum_{n=1}^{\infty} f_n(\xi)$ of functions defined in a set $S \subseteq C_2$ is that to every $\varepsilon > 0$, there corresponds a positive integer m (independent of $\xi$) such that

$\| f_{n+1}(\xi) + f_{n+2}(\xi) + \dots\dots\dots + f_{n+p}(\xi) \| < \varepsilon$, $\forall n \geq m$, $\forall p \geq 0$ and $\forall \xi \in S$.

**2.2 Weirstrass's M-test for uniform and absolute convergence**

**Theorem 2.3:** Let $\sum_{n=1}^{\infty} f_n(\xi)$ be an infinite series of functions defined in a set $S \subseteq C_2$. If the series $\sum_{n=1}^{\infty} M_n$ of positive terms is convergent and if $\| f_n(\xi) \| \leq M_n$, $\forall \xi \in S$, $\forall n \in N$, then $\sum_{n=1}^{\infty} f_n(\xi)$ is uniformly convergent in $S \subseteq C_2$.

**Corollary 2.1:** Let $\sum_{n=1}^{\infty} f_n(\xi)$ be an infinite series of functions $f_n : S \rightarrow C_2 \sim O_2$.

Suppose $\forall \xi \in S$, $\| f_1(\xi) \|$ is bounded and $\left\| \frac{f_{j+1}(\xi)}{f_j(\xi)} \right\| \leq M < \frac{1}{\sqrt{2}} \quad \forall j > 1$

Then $\sum_{n=1}^{\infty} f_n(\xi)$ is uniformly convergent in $S \subseteq C_2$.

**Proof:** $\| f_n(\xi) \| = \left\| f_1(\xi) \frac{f_2(\xi)\ f_3(\xi) \ldots\ldots\ldots f_n(\xi)}{f_1(\xi)\ f_2(\xi) \ldots\ldots\ldots f_{n-1}(\xi)} \right\|$

$$\leq \sqrt{2} \| f_1(\xi) \| \left\| \frac{f_2(\xi)\ f_3(\xi) \ldots\ldots\ldots f_n(\xi)}{f_1(\xi)\ f_2(\xi) \ldots\ldots\ldots f_{n-1}(\xi)} \right\| \qquad \text{... (2.1)}$$

Since $\| f_1(\xi) \|$ is bounded in $S \subseteq C_2$, there exists $K > 0$, such that $\| f_1(\xi) \| \leq K,\ \forall \xi \in S$.

Hence by **(2.1)**, $\| f_n(\xi) \| \leq \sqrt{2}\ K \left\| \frac{f_2(\xi)\ f_3(\xi) \ldots\ldots\ldots f_n(\xi)}{f_1(\xi)\ f_2(\xi) \ldots\ldots\ldots f_{n-1}(\xi)} \right\|$

$$\leq \sqrt{2}\ K\ (\sqrt{2})^{n-1} \left\| \frac{f_2(\xi)}{f_1(\xi)} \right\| \left\| \frac{f_3(\xi)}{f_2(\xi)} \right\| \ldots\ldots \left\| \frac{f_n(\xi)}{f_{n-1}(\xi)} \right\|$$

$$\leq \sqrt{2}\ K\ (\sqrt{2})^{n-1} M^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \| f_n(\xi) \| \leq \sqrt{2}\ K \sum_{n=1}^{\infty} (M \sqrt{2})^{n-1}$$

As $M < \frac{1}{\sqrt{2}}$, the series $\sum_{n=1}^{\infty} (M\sqrt{2})^{n-1}$ is convergent.

Hence by Weirstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(\xi)$ is uniformly convergent in $S \subseteq C_2$.

**Theorem 2.4:** Let $\sum_{n=1}^{\infty} f_n(\xi)$ be an infinite series of functions defined in a set $S \subseteq C_2$. Let $S_n(\xi) = \sum_{i=1}^{n} f_i(\xi)$.

1. Let the sequence $\{S_n(\xi)\}$ be uniformly bounded in S.

   i.e. $\| S_n(\xi) \| \leq K \quad \forall \xi \in S$ and $\forall n$

2. The series $\sum_{n=1}^{\infty} [u_n(\xi) - u_{n+1}(\xi)]$ is uniformly and absolutely convergent in S.

3. $\{u_n(\xi)\}$ tends uniformly to zero in S.

Then the series $\sum_{n=1}^{\infty} u_n(\xi)\ f_n(\xi)$ is uniformly convergent in S.

**Theorem 2.5:** If the series $\sum_{n=1}^{\infty} f_n(\xi)$ is uniformly convergent in $S \subseteq C_2$ and $\sum_{n=1}^{\infty} [u_n(\xi) - u_{n+1}(\xi)]$ is uniformly and absolutely convergent in $S \subseteq C_2$, Then the series $\sum_{n=1}^{\infty} u_n(\xi) f_n(\xi)$ is uniformly convergent in S.

**Theorem 2.6:** Let $X$ be a Cartesian set determined by $X_1$ and $X_2$.

The sequence $\{f_n(\xi)\}$ is uniformly convergent in $X$ if and only if the sequences $\{{}^1f_n({}^1\xi)\}$ and $\{{}^2f_n({}^2\xi)\}$ both are uniformly convergent in $X_1$ and $X_2$ respectively.

**Proof:** Let the sequence $\{f_n(\xi)\}$ be uniformly convergent in $X$.

Given $\varepsilon > 0$, $\exists m \equiv m(\varepsilon) \in N$ s.t. $\forall n \geq m$, $\forall p \geq 0$ and $\forall \xi \in X$

$$\left\| f_{n+p}(\xi) - f_n(\xi) \right\| < \varepsilon$$

$$\Rightarrow \left[ \frac{\left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right|^2 + \left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right|^2}{2} \right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow \left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right|^2 + \left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right|^2 < 2\varepsilon^2$$

$$\Rightarrow \left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right|^2 < 2\varepsilon^2 \text{ and } \left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right|^2 < 2\varepsilon^2$$

$$\Rightarrow \left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right| < \sqrt{2}\,\varepsilon \text{ and } \left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right| < \sqrt{2}\,\varepsilon$$

$\forall n \geq m$, $\forall p \geq 0$ and $\forall \xi \in X$

$$\left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right| < \sqrt{2}\,\varepsilon \text{ and } \left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right| < \sqrt{2}\,\varepsilon$$

As, $\xi \in X \Leftrightarrow {}^1\xi \in X_1$ and ${}^2\xi \in X_2$, X being the Cartesian set, hence $\forall n \geq m$, $\forall p \geq 0$ and $\forall {}^1\xi \in X_1$, we have

$$\left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right| < \sqrt{2}\,\varepsilon$$

$\Rightarrow \{{}^1f_n({}^1\xi)\}$ is uniformly convergent in $X_1$.

Similarly $\forall n \geq m$, $\forall p \geq 0$ and $\forall {}^2\xi \in X_2$

$$\left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right| < \sqrt{2}\,\varepsilon$$

$\Rightarrow \{{}^2f_n({}^2\xi)\}$ is uniformly convergent in $X_2$.

Conversely, we suppose that the sequences $\{{}^1f_n({}^1\xi)\}$ and $\{{}^2f_n({}^2\xi)\}$ both are uniformly convergent in $X_1$ and $X_2$ respectively.

Given $\varepsilon > 0$, $\exists m_1(\varepsilon), m_2(\varepsilon) \in N$ such that

$$\left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right| < \varepsilon \qquad \forall n \geq m_1,\ \forall p \geq 0 \text{ and } \forall {}^1\xi \in X_1$$

$$\left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right| < \varepsilon \qquad \forall n \geq m_2,\ \forall p \geq 0 \text{ and } \forall {}^2\xi \in X_2$$

Let $m = \max(m_1, m_2)$. Then $\forall n \geq m$, $\forall p \geq 0$ and $\forall \xi \in X$

$$\left\| f_{n+p}(\xi) - f_n(\xi) \right\| = \left[ \frac{\left| {}^1f_{n+p}({}^1\xi) - {}^1f_n({}^1\xi) \right|^2 + \left| {}^2f_{n+p}({}^2\xi) - {}^2f_n({}^2\xi) \right|^2}{2} \right]^{\frac{1}{2}} < \varepsilon$$

The sequence $\{f_n(\xi)\}$ is, therefore, uniformly convergent in $X$.

**Theorem 2.7:** Let $X$ be a Cartesian set determined by $X_1$ and $X_2$. The series $\sum_{n=1}^{\infty} f_n(\xi)$ is uniformly convergent in $X$ if and only if the series $\sum_{n=1}^{\infty} {}^1f_n({}^1\xi)$ and $\sum_{n=1}^{\infty} {}^2f_n({}^2\xi)$ both are uniformly convergent in $X_1$ and $X_2$ respectively.

**3. Bicomplex Dirichlet series:**

The **Bicomplex Dirichlet series** is defined as

$$f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi} \quad \text{... (3.1)}$$

where $\{\alpha_n\}$ is a sequence of bicomplex numbers, $\{\lambda_n\}$ is a strictly monotonically increasing and unbounded sequence of positive real numbers and $\xi \in C_2$ is a bicomplex variable. If $\lambda_n = n$, then $f(\xi) = \sum_{n=1}^{\infty} \alpha_n (e^{-\xi})^n$ is a **power series** in $e^{-\xi}$.

If $\lambda_n = \log n$, then

$$f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi} \quad \text{... (3.2)}$$

is a **Ordinary Bicomplex Dirichlet Series**.

If $\alpha_n = 1$ in equation **(3.2)** $f(\xi) = \sum_{n=1}^{\infty} n^{-\xi}$ represent **Bicomplex Riemann Zeta Function** (cf. **[8]**, **[9]**) in that consequence we named $f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ a **Generalized Bicomplex Riemann Zeta Function** (cf. **[10]**, **[11]**).

Note that, if $\xi \in C_2$ and n be a natural number, then

$$n^{-\xi} = e^{-\xi \log n} = e^{-{}^1\xi \log n} e_1 + e^{-{}^2\xi \log n} e_2 = n^{-{}^1\xi} e_1 + n^{-{}^2\xi} e_2$$

Hence if $\{\alpha_n\}$ is a bicomplex sequence, we have

$$\alpha_n n^{-\xi} = [{}^1\alpha_n n^{-{}^1\xi}]e_1 + [{}^2\alpha_n n^{-{}^2\xi}]e_2$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n n^{-\xi} = [\sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}]e_1 + [\sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}]e_2$$

Let $f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ then $f(\xi) = {}^1f({}^1\xi)e_1 + {}^2f({}^2\xi)e_2$, where, ${}^1f({}^1\xi) = \sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ and ${}^2f({}^2\xi) = \sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$.

Throughout, we denote the abscissae of convergence of ${}^1f({}^1\xi) = \sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ and ${}^2f({}^2\xi) = \sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$ by $\sigma_1$ and $\sigma_2$, and the abscissae of their absolute convergence by $\overline{\sigma}_1$ and $\overline{\sigma}_2$, respectively.

**Definition 3.1: Region of Convergence**

The region $\{\xi \in C_2 : \mathrm{Re}({}^1\xi) > \sigma_1 \text{ and } \mathrm{Re}({}^2\xi) > \sigma_2\}$ is the region of convergence of $f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi}$.

**Definition 3.2: Region of Absolute Convergence**

The region $\{\xi \in C_2 : \mathrm{Re}({}^1\xi) > \overline{\sigma}_1 \text{ and } \mathrm{Re}({}^2\xi) > \overline{\sigma}_2\}$ is the region of absolute convergence of $f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi}$.

**Definition 3.3:**

A **Bicomplex arithmetic function** is a function $f(n)$ defined for all $n \in N$; it is taken to be Bicomplex valued, so that it is a function $f : N \to C_2$, or equivalently a sequence $\{a_n\}$ of Bicomplex numbers $a_n = f(n)$.

**Note 3.1:**

**(a)** If $b > a$ and $\mathrm{Re}({}^1\xi) > \mathrm{Re}({}^1\xi_0)$ and $\mathrm{Re}({}^2\xi) > \mathrm{Re}({}^2\xi_0)$

Then, $\left\| a^{\xi_0 - \xi} \right\| > \left\| b^{\xi_0 - \xi} \right\|$.

**(b)** For, $\xi \in C_2$,

$$\|\xi\| \le \frac{1}{\sqrt{2}} \left[ |{}^1\xi| + |{}^2\xi| \right] < |{}^1\xi| + |{}^2\xi|.$$

We first investigate the regions of various types of convergences for $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$. As is customary, we denote

$\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$; $\xi_0 = z_1^0 + i_2 z_2^0 = x_1^0 + i_1 x_2^0 + i_2 x_3^0 + i_1 i_2 x_4^0$.

**Theorem 3.1:** If $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi = \xi_0$ iff $\sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ converges for ${}^1\xi = {}^1\xi_0$ and $\sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$ converges for ${}^2\xi = {}^2\xi_0$.

**Proof:** Assume that $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi = \xi_0$.

Then there exist a bicomplex number $\zeta$ such that $\lim_{m\to\infty}\sum_{n=1}^{m}\alpha_n n^{-\xi_0} = \zeta = {}^1\zeta e_1 + {}^2\zeta e_2$

Given $\varepsilon > 0$, $\exists\, n_0 \in N$ s.t.

$$\left\| \sum_{n=1}^{m}\alpha_n n^{-\xi_0} - \zeta \right\| < \varepsilon, \ \forall m \ge n_0 \qquad \text{... } \mathbf{(3.3)}$$

Now by the properties of idempotent representation,

$$\sum_{n=1}^{m}\alpha_n n^{-\xi_0} - \zeta = \left(\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - {}^1\zeta\right)e_1 + \left(\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - {}^2\zeta\right)e_2$$

so that

$$\left\| \sum_{n=1}^{m}\alpha_n n^{-\xi_0} - \zeta \right\| = \frac{1}{\sqrt{2}}\left[ \left|\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - {}^1\zeta\right|^2 + \left|\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - {}^2\zeta\right|^2 \right]^{\frac{1}{2}}$$

From equation **(3.3)**

$$\frac{1}{\sqrt{2}}\left[ \left|\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - {}^1\zeta\right|^2 + \left|\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - {}^2\zeta\right|^2 \right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow \left|\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - {}^1\zeta\right| < \sqrt{2}\,\varepsilon \text{ and } \left|\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - {}^2\zeta\right| < \sqrt{2}\,\varepsilon \qquad \forall m \ge n_0$$

Now, $\left|\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - {}^1\zeta\right| < \sqrt{2}\,\varepsilon \ \forall m \ge n_0 \Rightarrow \sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ converges to ${}^1\zeta$ for ${}^1\xi = {}^1\xi_0$.

Similarly from $\left|\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - {}^2\zeta\right| < \sqrt{2}\,\varepsilon,\ \forall m \ge n_0 \Rightarrow \sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$ converges to ${}^2\zeta$ for ${}^2\xi = {}^2\xi_0$.

Conversely let $\sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ converges for ${}^1\xi = {}^1\xi_0$ and $\sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$ converges for ${}^2\xi = {}^2\xi_0$ respectively.

Then, $\exists\, s_1, s_2 \in C(i_1)$ such that

$$\lim_{m\to\infty}\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} = s_1 \text{ and } \lim_{m\to\infty}\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} = s_2$$

Given $\varepsilon > 0$, $\exists\, n_1, n_2 \in N$ s.t.

$$\left|\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - s_1\right| < \varepsilon,\ \forall m \ge n_1 \text{ and } \left|\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - s_2\right| < \varepsilon,\ \forall m \ge n_2$$

Let $n_0 = \max(n_1, n_2)$

Then $\forall m \ge n_0$

$$\left|\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - s_1\right| < \varepsilon \text{ and } \left|\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - s_2\right| < \varepsilon$$

Now

$$\left\| \sum_{n=1}^{m}\alpha_n n^{-\xi_0} - (s_1 e_1 + s_2 e_2) \right\| = \left\| \left(\sum_{n=1}^{m} {}^1\alpha_n n^{-{}^1\xi_0} - s_1\right)e_1 + \left(\sum_{n=1}^{m} {}^2\alpha_n n^{-{}^2\xi_0} - s_2\right)e_2 \right\|$$

$$= \frac{1}{\sqrt{2}}\left[\left|\sum_{n=1}^{m} {}^{1}\alpha_n n^{-{}^{1}\xi_0} - s_1\right|^2 + \left|\sum_{n=1}^{m} {}^{2}\alpha_n n^{-{}^{2}\xi_0} - s_2\right|^2\right]^{\frac{1}{2}} < \frac{1}{\sqrt{2}}\left[\varepsilon^2 + \varepsilon^2\right]^{1/2} = \varepsilon$$

$$\Rightarrow \left\|\sum_{n=1}^{m} \alpha_n n^{-\xi_0} - (s_1 e_1 + s_2 e_2)\right\| < \varepsilon \qquad \forall\, m \geq n_0$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n n^{-\xi} \text{ converges to } s_1 e_1 + s_2 e_2 \text{ for } \xi = \xi_0 .$$

**Theorem 3.2:** For any bicomplex arithmetical function $a(n)$ (cf. **Def. 3.3**), let $A(x) = \sum_{n \leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume $f : [y, x] \to C_2$ has a continuous derivative on the interval $[y, x]$, where $0 < y < x$.

Then we have $\sum_{y<n\leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\, dt$

**Proof:** We know that

$$\sum_{y<n\leq x} a(n) f(n) = \Big[\sum_{y<n\leq x} {}^{1}a(n)\, {}^{1}f(n)\Big] e_1 + \Big[\sum_{y<n\leq x} {}^{2}a(n)\, {}^{2}f(n)\Big] e_2 \qquad \dots \textbf{(3.4)}$$

Since ${}^{1}f$, ${}^{1}a(n)$ and ${}^{2}f$, ${}^{2}a(n)$ satisfies all the requirements of Abel's identity, therefore by **Theorem 1.7**,

$$\sum_{y<n\leq x} {}^{1}a(n)\, {}^{1}f(n) = {}^{1}A(x)\, {}^{1}f(x) - {}^{1}A(y)\, {}^{1}f(y) - \int_y^x {}^{1}A(t)\, {}^{1}f'(t)\, dt$$

and $\sum_{y<n\leq x} {}^{2}a(n)\, {}^{2}f(n) = {}^{2}A(x)\, {}^{2}f(x) - {}^{2}A(y)\, {}^{2}f(y) - \int_y^x {}^{2}A(t)\, {}^{2}f'(t)\, dt$

Now, by **(3.4)**,

$$\begin{aligned}
\sum_{y<n\leq x} a(n) f(n) &= [\, {}^{1}A(x)\, {}^{1}f(x) - {}^{1}A(y)\, {}^{1}f(y) - \int_y^x {}^{1}A(t)\, {}^{1}f'(t)\, dt]\, e_1 \\
&\quad + [\, {}^{2}A(x)\, {}^{2}f(x) - {}^{2}A(y)\, {}^{2}f(y) - \int_y^x {}^{2}A(t)\, {}^{2}f'(t)\, dt]\, e_2 \\
&= [\{ {}^{1}A(x)\, {}^{1}f(x) - {}^{1}A(y)\, {}^{1}f(y)\} e_1 + \{ {}^{2}A(x)\, {}^{2}f(x) - {}^{2}A(y)\, {}^{2}f(y)\} e_2] \\
&\quad - [\{ \int_y^x {}^{1}A(t)\, {}^{1}f'(t)\, dt \} e_1 + \{ \int_y^x {}^{2}A(t)\, {}^{2}f'(t)\, dt \} e_2] \\
&= A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\, d(t)
\end{aligned}$$

**Lemma 3.1:** If $\| A(n) \| \leq M \ \ \forall n$, then for each $\xi$ with $\mathrm{Re}({}^{1}\xi) > \mathrm{Re}({}^{1}\xi_0)$, $\mathrm{Re}({}^{2}\xi) > \mathrm{Re}({}^{2}\xi_0)$ and $b > a > 0$

$$\left\| \int_a^b A(t)\, t^{-\xi+\xi_0-1}\, dt \right\| \leq 2M \left[ \frac{a^{\mathrm{Re}({}^{1}\xi_0 - {}^{1}\xi)}}{\mathrm{Re}({}^{1}\xi - {}^{1}\xi_0)} + \frac{a^{\mathrm{Re}({}^{2}\xi_0 - {}^{2}\xi)}}{\mathrm{Re}({}^{2}\xi - {}^{2}\xi_0)} \right]$$

**Proof:**

$$\left\| \int_a^b A(t)\, t^{-\xi+\xi_0-1}\, dt \right\| = \left\| \Big[\int_a^b {}^{1}A(t)\, t^{-{}^{1}\xi+{}^{1}\xi_0-1}\, dt\Big] e_1 + \Big[\int_a^b {}^{2}A(t)\, t^{-{}^{2}\xi+{}^{2}\xi_0-1}\, dt\Big] e_2 \right\|$$

$$= \frac{1}{\sqrt{2}} \left[ \left| \int_a^b {}^{1}A(t)\, t^{-{}^{1}\xi+{}^{1}\xi_0-1}\, dt \right|^2 + \left| \int_a^b {}^{2}A(t)\, t^{-{}^{2}\xi+{}^{2}\xi_0-1}\, dt \right|^2 \right]^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}} \left[ \left| \int_a^b \left| {}^{1}A(t) \right| \left| t^{-{}^{1}\xi+{}^{1}\xi_0-1} \right| dt \right|^2 + \left| \int_a^b \left| {}^{2}A(t) \right| \left| t^{-{}^{2}\xi+{}^{2}\xi_0-1} \right| dt \right|^2 \right]^{\frac{1}{2}}$$

[ $\| A(n) \| \le M \Rightarrow |{}^1A(n)| \le \sqrt{2}\,M$ and $|{}^2A(n)| \le \sqrt{2}\,M$ ]

$$\le \frac{1}{\sqrt{2}}\left[\left|\int_a^b \sqrt{2}\,M\left|t^{-{}^1\xi+{}^1\xi_0-1}\right|dt\right|^2+\left|\int_a^b \sqrt{2}M\left|t^{-{}^2\xi+{}^2\xi_0-1}\right|dt\right|^2\right]^{\frac{1}{2}}$$

$$= M\left[\left|\int_a^b \left|t^{-{}^1\xi+{}^1\xi_0-1}\right|dt\right|^2+\left|\int_a^b \left|t^{-{}^2\xi+{}^2\xi_0-1}\right|dt\right|^2\right]^{\frac{1}{2}}$$

$$= M\left[\left|\int_a^b t^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)-1}\,dt\right|^2+\left|\int_a^b t^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)-1}\,dt\right|^2\right]^{\frac{1}{2}}$$

$$= M\left[\left|\frac{b^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}-a^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}}{\mathrm{Re}({}^1\xi_0-{}^1\xi)}\right|^2+\left|\frac{b^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}-a^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}}{\mathrm{Re}({}^2\xi_0-{}^2\xi)}\right|^2\right]^{\frac{1}{2}}$$

$$\le M\left[\left[\frac{\left|b^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}\right|+\left|a^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}\right|}{\left|\mathrm{Re}({}^1\xi_0-{}^1\xi)\right|}\right]^2+\left[\frac{\left|b^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}\right|+\left|a^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}\right|}{\left|\mathrm{Re}({}^2\xi_0-{}^2\xi)\right|}\right]^2\right]^{\frac{1}{2}}$$

$$\le M\left[\left[\frac{2\left|a^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}\right|}{\left|\mathrm{Re}({}^1\xi_0-{}^1\xi)\right|}\right]^2+\left[\frac{2\left|a^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}\right|}{\left|\mathrm{Re}({}^2\xi_0-{}^2\xi)\right|}\right]^2\right]^{\frac{1}{2}}$$

$$= 2M\left[\left[\frac{a^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}}{\mathrm{Re}({}^1\xi_0-{}^1\xi)}\right]^2+\left[\frac{a^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}}{\mathrm{Re}({}^2\xi_0-{}^2\xi)}\right]^2\right]^{\frac{1}{2}}$$

$$\le 2M\left[\frac{a^{-\mathrm{Re}({}^1\xi)+\mathrm{Re}({}^1\xi_0)}}{\mathrm{Re}({}^1\xi-{}^1\xi_0)}+\frac{a^{-\mathrm{Re}({}^2\xi)+\mathrm{Re}({}^2\xi_0)}}{\mathrm{Re}({}^2\xi-{}^2\xi_0)}\right]$$

**Apostol [7]** has given an interesting result (cf. **Lemma 1.1**). Here we establish the bicomplex version of this result independently.

**Lemma 3.2:** Let the series $\sum_{n=1}^{\infty}\alpha_n n^{-\xi_0}$ has bounded partial sums, say $\left\|\sum_{n\le x}\alpha_n n^{-\xi_0}\right\| \le M$ for all $x \ge 1$.

Then for each $\xi$ with $\mathrm{Re}({}^1\xi) > \mathrm{Re}({}^1\xi_0)$ and $\mathrm{Re}({}^2\xi) > \mathrm{Re}({}^2\xi_0)$ we have

$$\left\|\sum_{a<n\le b}\alpha_n n^{-\xi}\right\| \le 2\sqrt{2}\,M\left[\left\|a^{-(\xi-\xi_0)}\right\|+\left\|(\xi-\xi_0)\right\|\left[\frac{a^{\mathrm{Re}({}^1\xi_0-{}^1\xi)}}{\mathrm{Re}({}^1\xi-{}^1\xi_0)}+\frac{a^{\mathrm{Re}({}^2\xi_0-{}^2\xi)}}{\mathrm{Re}({}^2\xi-{}^2\xi_0)}\right]\right]$$

**Proof:** Note that $\sum_{a<n\le b}\alpha_n n^{-\xi} = \sum_{a<n\le b}(\alpha_n n^{-\xi_0})\,n^{-(\xi-\xi_0)}$

Let $a(n) = \alpha_n n^{-\xi_0}$, $A(x) = \sum_{n\le x} a(n)$ and let $f(n) = n^{-(\xi-\xi_0)}$.

Now by **Theorem 3.2**,

$$\sum_{a<n\le b}\alpha_n n^{-\xi} = \sum_{a<n\le b} a(n)f(n) = A(b)f(b) - A(a)f(a) - \int_a^b A(t)f'(t)\,dt$$

$$= A(b)\,b^{-(\xi-\xi_0)} - A(a)\,a^{-(\xi-\xi_0)} - \int_a^b A(t)D_t(t^{-(\xi-\xi_0)})\,dt$$

$$\sum_{a<n\le b} \alpha_n n^{-\xi} = A(b)\, b^{-(\xi-\xi_0)} - A(a)\, a^{-(\xi-\xi_0)} + (\xi-\xi_0)\int_a^b A(t)\, t^{-\xi+\xi_0-1} dt$$

$$\left\|\sum_{a<n\le b} \alpha_n n^{-\xi}\right\| = \left\|A(b)\, b^{-(\xi-\xi_0)} - A(a)\, a^{-(\xi-\xi_0)} + (\xi-\xi_0)\int_a^b A(t)\, t^{-\xi+\xi_0-1} dt\right\|$$

$$\le \left\|A(b)\, b^{-(\xi-\xi_0)}\right\| + \left\|A(a)\, a^{-(\xi-\xi_0)}\right\| + \left\|(\xi-\xi_0)\int_a^b A(t)\, t^{-\xi+\xi_0-1} dt\right\|$$

$$\le \sqrt{2}\left\|A(b)\right\|\left\|b^{-(\xi-\xi_0)}\right\| + \sqrt{2}\left\|A(a)\right\|\left\|a^{-(\xi-\xi_0)}\right\| + \sqrt{2}\left\|(\xi-\xi_0)\right\|\left\|\int_a^b A(t)\, t^{-\xi+\xi_0-1} dt\right\|$$

$$\le \sqrt{2}M\left\|b^{-(\xi-\xi_0)}\right\| + \sqrt{2}M\left\|a^{-(\xi-\xi_0)}\right\| + \sqrt{2}\left\|(\xi-\xi_0)\right\| 2M\left[\frac{a^{Re({}^1\xi_0-{}^1\xi)}}{Re({}^1\xi-{}^1\xi_0)} + \frac{a^{Re({}^2\xi_0-{}^2\xi)}}{Re({}^2\xi-{}^2\xi_0)}\right] \quad \text{[by Lemma 3.1]}$$

$$= \sqrt{2}M\left\|b^{-(\xi-\xi_0)}\right\| + \sqrt{2}M\left\|a^{-(\xi-\xi_0)}\right\| + 2\sqrt{2}M\left\|(\xi-\xi_0)\right\|\left[\frac{a^{Re({}^1\xi_0-{}^1\xi)}}{Re({}^1\xi-{}^1\xi_0)} + \frac{a^{Re({}^2\xi_0-{}^2\xi)}}{Re({}^2\xi-{}^2\xi_0)}\right]$$

$$\le 2\sqrt{2}M\left\|a^{-(\xi-\xi_0)}\right\| + 2\sqrt{2}M\left\|(\xi-\xi_0)\right\|\left[\frac{a^{Re({}^1\xi_0-{}^1\xi)}}{Re({}^1\xi-{}^1\xi_0)} + \frac{a^{Re({}^2\xi_0-{}^2\xi)}}{Re({}^2\xi-{}^2\xi_0)}\right] \quad \text{[by Note 3.1(a)]}$$

$$= 2\sqrt{2}M\left[\left\|a^{-(\xi-\xi_0)}\right\| + \left\|(\xi-\xi_0)\right\|\left[\frac{a^{Re({}^1\xi_0-{}^1\xi)}}{Re({}^1\xi-{}^1\xi_0)} + \frac{a^{Re({}^2\xi_0-{}^2\xi)}}{Re({}^2\xi-{}^2\xi_0)}\right]\right]$$

**Theorem 3.3:** If $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi=\xi_0$ then $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges in the region $\{\xi\in C_2 : Re({}^1\xi) > Re({}^1\xi_0) \text{ and } Re({}^2\xi) > Re({}^2\xi_0)\} = \{\xi\in C_2 : x_1+x_4 > x_1^0+x_4^0 \text{ and } x_1-x_4 > x_1^0-x_4^0\}$ or equivalently in the region $\{\xi\in C_2 : Re(z_1) > Re(z_1^0) \text{ and } |Im(z_2)-Im(z_2^0)| < Re(z_1)-Re(z_1^0)\}$.

**Remark 3.1:** We shall prove this result using two different approaches. In the first proof, we employ the idempotent techniques of bicomplex analysis whereas in the alternative proof we follow the complex analytic approach of **Apostol [7]**.

**Proof:** Assume that $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi=\xi_0$.
Then by **Theorem 3.1**,
$\sum_{n=1}^{\infty}{}^1\alpha_n n^{-{}^1\xi}$ converges for ${}^1\xi={}^1\xi_0$ and $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-{}^2\xi}$ converges for ${}^2\xi={}^2\xi_0$.

Since $\sum_{n=1}^{\infty}{}^1\alpha_n n^{-{}^1\xi}$ converges for ${}^1\xi={}^1\xi_0$.
By **Theorem 1.1**, we infer that
$\sum_{n=1}^{\infty}{}^1\alpha_n n^{-{}^1\xi}$ converges for every ${}^1\xi$ if $Re({}^1\xi) > Re({}^1\xi_0)$.

Similarly $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-{}^2\xi}$ converges for ${}^2\xi={}^2\xi_0$.

We can infer that $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-{}^2\xi}$ converges for every ${}^2\xi$ if $Re({}^2\xi) > Re({}^2\xi_0)$.
Hence,
$\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for every $\xi$ if $Re({}^1\xi) > Re({}^1\xi_0)$ and $Re({}^2\xi) > Re({}^2\xi_0)$.
i.e. $x_1+x_4 > x_1^0+x_4^0 \text{ and } x_1-x_4 > x_1^0 \quad x_4^0$
Further, note that

$$\operatorname{Re}({}^1\xi) > \operatorname{Re}({}^1\xi_0) \text{ and } \operatorname{Re}({}^2\xi) > \operatorname{Re}({}^2\xi_0) \Leftrightarrow x_1 + x_4 > x_1^0 + x_4^0 \text{ and } x_1 - x_4 > x_1^0 - x_4^0$$
$$\Leftrightarrow x_1 > x_1^0 \text{ and } |x_4 - x_4^0| < x_1 - x_1^0 \Leftrightarrow \operatorname{Re}(z_1) > \operatorname{Re}(z_1^0) \text{ and } |\operatorname{Im}(z_2) - \operatorname{Im}(z_2^0)| < \operatorname{Re}(z_1) - \operatorname{Re}(z_1^0)$$

**Alternative proof of the Theorem 3.3:**

Choose any $\xi$ with $\operatorname{Re}({}^1\xi) > \operatorname{Re}({}^1\xi_0)$ and $\operatorname{Re}({}^2\xi) > \operatorname{Re}({}^2\xi_0)$.

By **Lemma 3.2,**

$$\left\| \sum_{a<n\le b} \alpha_n n^{-\xi} \right\| \le 2\sqrt{2}\, M \left[ \left\| a^{-(\xi-\xi_0)} \right\| + \left\| (\xi - \xi_0) \right\| \left[ \frac{a^{\operatorname{Re}({}^1\xi_0 - {}^1\xi)}}{\operatorname{Re}({}^1\xi - {}^1\xi_0)} + \frac{a^{\operatorname{Re}({}^2\xi_0 - {}^2\xi)}}{\operatorname{Re}({}^2\xi - {}^2\xi_0)} \right] \right]$$
$$\le 2\sqrt{2}\, M \left[ a^{\operatorname{Re}({}^1\xi_0 - {}^1\xi)} + a^{\operatorname{Re}({}^2\xi_0 - {}^2\xi)} + \left\| (\xi - \xi_0) \right\| \left[ \frac{a^{\operatorname{Re}({}^1\xi_0 - {}^1\xi)}}{\operatorname{Re}({}^1\xi - {}^1\xi_0)} + \frac{a^{\operatorname{Re}({}^2\xi_0 - {}^2\xi)}}{\operatorname{Re}({}^2\xi - {}^2\xi_0)} \right] \right]$$
$$= a^{\operatorname{Re}({}^1\xi_0 - {}^1\xi)} \left[ 2\sqrt{2}\, M \left[ 1 + \frac{\left\| (\xi - \xi_0) \right\|}{\operatorname{Re}({}^1\xi - {}^1\xi_0)} \right] \right] + a^{\operatorname{Re}({}^2\xi_0 - {}^2\xi)} \left[ 2\sqrt{2}\, M \left[ 1 + \frac{\left\| (\xi - \xi_0) \right\|}{\operatorname{Re}({}^2\xi - {}^2\xi_0)} \right] \right]$$
$$= K_1 a^{\operatorname{Re}({}^1\xi_0 - {}^1\xi)} + K_2 a^{\operatorname{Re}({}^2\xi_0 - {}^2\xi)}$$

Where $K_1$ and $K_2$ are independent of a.

Since $a^{\operatorname{Re}({}^1\xi_0 - {}^1\xi)} \to 0$ and $a^{\operatorname{Re}({}^2\xi_0 - {}^2\xi)} \to 0$ as $a \to \infty$, the Cauchy condition shows that $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges.

**Corollary 3.1:** If $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ diverges for $\xi = \xi_0$ then $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ diverges in the region

$$\{\xi \in C_2 : \operatorname{Re}({}^1\xi) < \operatorname{Re}({}^1\xi_0) \text{ and } \operatorname{Re}({}^2\xi) < \operatorname{Re}({}^2\xi_0)\} = \{\xi \in C_2 : x_1 + x_4 < x_1^0 + x_4^0 \text{ and } x_1 - x_4 < x_1^0 - x_4^0\}$$

or equivalently in the region $\{\xi \in C_2 : \operatorname{Re}(z_1) < \operatorname{Re}(z_1^0) \text{ and } |\operatorname{Im}(z_2) - \operatorname{Im}(z_2^0)| > \operatorname{Re}(z_1) - \operatorname{Re}(z_1^0)\}$.

**Proof:** Direct consequence of **Theorem 3.3**.

**Corollary 3.2:** If the partial sums $\sum_{n\le x} \alpha_n$ are bounded, the series $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges in the region

$$\{\xi \in C_2 : \operatorname{Re}({}^1\xi) > 0 \text{ and } \operatorname{Re}({}^2\xi) > 0\}.$$

**Proof:** Let $\xi$ be an arbitrary bicomplex number with $\operatorname{Re}({}^1\xi) > 0$ and $\operatorname{Re}({}^2\xi) > 0$.

By **Lemma 3.2,**

$$\left\| \sum_{a<n\le b} \alpha_n n^{-\xi} \right\| \le 2\sqrt{2}\, M \left[ \left\| a^{-\xi} \right\| + \left\| \xi \right\| \left[ \frac{a^{-\operatorname{Re}({}^1\xi)}}{\operatorname{Re}({}^1\xi)} + \frac{a^{-\operatorname{Re}({}^2\xi)}}{\operatorname{Re}({}^2\xi)} \right] \right]$$
$$\le 2\sqrt{2}\, M \left[ a^{-\operatorname{Re}({}^1\xi)} + a^{-\operatorname{Re}({}^2\xi)} + \left\| \xi \right\| \left[ \frac{a^{-\operatorname{Re}({}^1\xi)}}{\operatorname{Re}({}^1\xi)} + \frac{a^{-\operatorname{Re}({}^2\xi)}}{\operatorname{Re}({}^2\xi)} \right] \right]$$
$$= a^{-\operatorname{Re}({}^1\xi)} \left[ 2\sqrt{2}\, M \left[ 1 + \frac{\left\| \xi \right\|}{\operatorname{Re}({}^1\xi)} \right] \right] + a^{-\operatorname{Re}({}^2\xi)} \left[ 2\sqrt{2}\, M \left[ 1 + \frac{\left\| \xi \right\|}{\operatorname{Re}({}^2\xi)} \right] \right]$$
$$= K_1 a^{-\operatorname{Re}({}^1\xi)} + K_2 a^{-\operatorname{Re}({}^2\xi)}$$

where $K_1$ and $K_2$ are independent of a.

As $a \to \infty$, we find that $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges if $\operatorname{Re}({}^1\xi) > 0$ and $\operatorname{Re}({}^2\xi) > 0$.

**Theorem 3.4 (The Uniqueness theorem):** If $F(\xi) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^{\xi}}$ and $G(\xi) = \sum_{n=1}^{\infty} \frac{\beta_n}{n^{\xi}}$ are both absolutely convergent in the region $\{\xi \in C_2 : Re({}^1\xi) \geq a \text{ and} Re({}^2\xi) \geq b\}$ such that $F(\xi) = G(\xi)$ for each $\xi$ with $Re({}^1\xi) \geq a$ and $Re({}^2\xi) \geq b$, then $\alpha_n = \beta_n$ for all n.

**Proof:** Let $\gamma_n = \alpha_n - \beta_n$ and let $H(\xi) = F(\xi) - G(\xi) = 0$.

To prove that $\gamma_n = 0 \ \forall \ n \in N$ we assume that $\gamma_n \neq 0$ for some $n$ and obtain a contradiction.

Let $N$ be the smallest number for which $\gamma_n \neq 0$.

Now for $\xi$ with $Re({}^1\xi) \geq a$ and $Re({}^2\xi) \geq b$, $0 = H(\xi) = \sum_{n=N}^{\infty} \frac{\gamma_n}{n^{\xi}} = \frac{\gamma_N}{N^{\xi}} + \sum_{n=N+1}^{\infty} \frac{\gamma_n}{n^{\xi}}$

Hence $\gamma_N = -N^{\xi} \sum_{n=N+1}^{\infty} \frac{\gamma_n}{n^{\xi}}$

$$\|\gamma_N\| = \left\| -N^{\xi} \sum_{n=N+1}^{\infty} \frac{\gamma_n}{n^{\xi}} \right\| = \left\| \sum_{n=N+1}^{\infty} \frac{\gamma_n}{n^{\xi}} N^{\xi} \right\| \leq \frac{1}{\sqrt{2}} \left[ \sum_{n=N+1}^{\infty} \left| \frac{{}^1\gamma_n}{n^{{}^1\xi}} N^{{}^1\xi} \right| + \sum_{n=N+1}^{\infty} \left| \frac{{}^2\gamma_n}{n^{{}^2\xi}} N^{{}^2\xi} \right| \right]$$

$$\|\gamma_N\| \leq \frac{1}{\sqrt{2}} \left[ \sum_{n=N+1}^{\infty} \frac{|{}^1\gamma_n|}{n^{Re({}^1\xi)}} N^{Re({}^1\xi)} + \sum_{n=N+1}^{\infty} \frac{|{}^2\gamma_n|}{n^{Re({}^2\xi)}} N^{Re({}^2\xi)} \right]$$

Now, $\frac{N^{Re({}^1\xi)}}{n^{Re({}^1\xi)}} = \frac{N^{Re({}^1\xi)-a}}{n^{Re({}^1\xi)-a}} \frac{N^a}{n^a} \leq \left(\frac{N}{N+1}\right)^{Re({}^1\xi)-a} \frac{N^a}{n^a}$ $[\because n \geq N+1]$

and $\frac{N^{Re({}^2\xi)}}{n^{Re({}^2\xi)}} = \frac{N^{Re({}^2\xi)-b}}{n^{Re({}^2\xi)-b}} \frac{N^b}{n^b} \leq \left(\frac{N}{N+1}\right)^{Re({}^2\xi)-b} \frac{N^b}{n^b}$

$$\|\gamma_N\| \leq \frac{1}{\sqrt{2}} \left[ \sum_{n=N+1}^{\infty} |{}^1\gamma_n| \left(\frac{N}{N+1}\right)^{Re({}^1\xi)-a} \frac{N^a}{n^a} + \sum_{n=N+1}^{\infty} |{}^2\gamma_n| \left(\frac{N}{N+1}\right)^{Re({}^2\xi)-b} \frac{N^b}{n^b} \right]$$

$$\|\gamma_N\| \leq \frac{1}{\sqrt{2}} \left[ \left(\frac{N}{N+1}\right)^{Re({}^1\xi)-a} N^a \sum_{n=N+1}^{\infty} \frac{|{}^1\gamma_n|}{n^a} + \left(\frac{N}{N+1}\right)^{Re({}^2\xi)-b} N^b \sum_{n=N+1}^{\infty} \frac{|{}^2\gamma_n|}{n^b} \right]$$

As $Re({}^1\xi) \to \infty$ and $Re({}^2\xi) \to \infty$ we get $\gamma_N = 0$, which is a contradiction.

## Absolute and Uniform Convergence:

**Lemma 3.3:** If $N \geq 1$ and $Re({}^1\xi) \geq c_1 > \bar{\sigma}_1$, $Re({}^2\xi) \geq c_2 > \bar{\sigma}_2$ we have

$$\left\| \sum_{n=N}^{\infty} \alpha_n n^{-\xi} \right\| \leq \frac{1}{\sqrt{2}} \left[ N^{-(Re({}^1\xi)-c_1)} \sum_{n=N}^{\infty} |{}^1\alpha_n| n^{-c_1} + N^{-(Re({}^2\xi)-c_2)} \sum_{n=N}^{\infty} |{}^2\alpha_n| n^{-c_2} \right]$$

**Proof:**
$$\left\| \sum_{n=N}^{\infty} \alpha_n n^{-\xi} \right\| = \left[ \frac{1}{2} \left[ \left| \sum_{n=N}^{\infty} {}^1\alpha_n n^{-{}^1\xi} \right|^2 + \left| \sum_{n=N}^{\infty} {}^2\alpha_n n^{-{}^2\xi} \right|^2 \right] \right]^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}} \left[ \left| \sum_{n=N}^{\infty} {}^1\alpha_n n^{-{}^1\xi} \right| + \left| \sum_{n=N}^{\infty} {}^2\alpha_n n^{-{}^2\xi} \right| \right]$$

$$\leq \frac{1}{\sqrt{2}} \left[ \sum_{n=N}^{\infty} |{}^1\alpha_n| |n^{-{}^1\xi}| + \sum_{n=N}^{\infty} |{}^2\alpha_n| |n^{-{}^2\xi}| \right] = \frac{1}{\sqrt{2}} \left[ \sum_{n=N}^{\infty} |{}^1\alpha_n| n^{-Re({}^1\xi)} + \sum_{n=N}^{\infty} |{}^2\alpha_n| n^{-Re({}^2\xi)} \right]$$

$$= \frac{1}{\sqrt{2}} \left[ \sum_{n=N}^{\infty} |{}^1\alpha_n| n^{-c_1} n^{-(Re({}^1\xi)-c_1)} + \sum_{n=N}^{\infty} |{}^2\alpha_n| n^{-c_2} n^{-(Re({}^2\xi)-c_2)} \right]$$

$$\leq \frac{1}{\sqrt{2}} \left[ N^{-(Re({}^1\xi)-c_1)} \sum_{n=N}^{\infty} |{}^1\alpha_n| n^{-c_1} + N^{-(Re({}^2\xi)-c_2)} \sum_{n=N}^{\infty} |{}^2\alpha_n| n^{-c_2} \right]$$

**Theorem 3.5:** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges absolutely in the region $\{\xi\in C_2:\mathrm{Re}({}^1\xi)>\bar{\sigma}_1 \text{ and } \mathrm{Re}({}^2\xi)>\bar{\sigma}_2\}$, then $\lim_{\mathrm{Re}({}^1\xi)\to\infty}\lim_{\mathrm{Re}({}^2\xi)\to\infty} f(\xi)=\alpha_1$ for $-\infty<\mathrm{Im}({}^1\xi)<+\infty$ and $-\infty<\mathrm{Im}({}^2\xi)<+\infty$.

**Proof:** Since $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$

$$\Rightarrow f(\xi)=\alpha_1+\sum_{n=2}^{\infty}\alpha_n n^{-\xi}$$

We need only to prove that $\sum_{n=2}^{\infty}\alpha_n n^{-\xi}\to 0$ as $\mathrm{Re}({}^1\xi)\to\infty$ and $\mathrm{Re}({}^2\xi)\to\infty$.

Choose $c_1>\bar{\sigma}_1$ and $c_2>\bar{\sigma}_2$. For $\mathrm{Re}({}^1\xi)\ge c_1$ and $\mathrm{Re}({}^2\xi)\ge c_2$, **Lemma 3.3**, implies that

$$\left\|\sum_{n=2}^{\infty}\alpha_n n^{-\xi}\right\|\le\frac{1}{\sqrt{2}}\left[2^{-(\mathrm{Re}({}^1\xi)-c_1)}\sum_{n=N}^{\infty}\left|{}^1\alpha_n\right|n^{-c_1}+2^{-(\mathrm{Re}({}^2\xi)-c_2)}\sum_{n=N}^{\infty}\left|{}^2\alpha_n\right|n^{-c_2}\right]$$

$$=\frac{1}{\sqrt{2}}\left[2^{-\mathrm{Re}({}^1\xi)}\left(2^{c_1}\sum_{n=N}^{\infty}\left|{}^1\alpha_n\right|n^{-c_1}\right)+2^{-\mathrm{Re}({}^2\xi)}\left(2^{c_2}\sum_{n=N}^{\infty}\left|{}^2\alpha_n\right|n^{-c_2}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\left[2^{-\mathrm{Re}({}^1\xi)}A+2^{-\mathrm{Re}({}^2\xi)}B\right],\text{ say}$$

$$=\frac{1}{\sqrt{2}}\left[\frac{A}{2^{\mathrm{Re}({}^1\xi)}}+\frac{B}{2^{\mathrm{Re}({}^2\xi)}}\right],\text{ where A and B are independent of } \xi.$$

Note that $\frac{A}{2^{\mathrm{Re}({}^1\xi)}}\to 0$ as $\mathrm{Re}({}^1\xi)\to\infty$ and $\frac{B}{2^{\mathrm{Re}({}^2\xi)}}\to\infty$ as $\mathrm{Re}({}^2\xi)\to\infty$.

This proves the theorem.

**Corollary 3.3:** $\zeta(\xi)=\sum_{n=1}^{\infty}n^{-\xi}\to 1$ as $\mathrm{Re}({}^1\xi)\to\infty$ and $\mathrm{Re}({}^2\xi)\to\infty$.

**Proof:** Straightforward.

**Theorem 3.6:** A series $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges uniformly on every compact subset lying interior in the region of convergence $C=\{\xi\in C_2:\mathrm{Re}({}^1\xi)>\sigma_1 \text{ and } \mathrm{Re}({}^2\xi)>\sigma_2\}$.

**Proof:** It suffices to prove that $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges uniformly on every compact Cartesian set determined by two-closed rectangle.

Let $R=R_1\times_e R_2$ be a Cartesian compact set determined by two closed rectangles $R_1=[\alpha_1,\beta_1]\times[c_1,d_1]$ with $\alpha_1>\sigma_1$ and $R_2=[\alpha_2,\beta_2]\times[c_2,d_2]$ with $\alpha_2>\sigma_2$.

By **Lemma 3.2**,

$$\left\|\sum_{a<n\le b}\alpha_n n^{-\xi}\right\|\le 2\sqrt{2}M\left[\left\|a^{-(\xi-\xi_0)}\right\|+\left\|(\xi-\xi_0)\right\|\left[\frac{a^{\mathrm{Re}({}^1\xi_0-{}^1\xi)}}{\mathrm{Re}({}^1\xi-{}^1\xi_0)}+\frac{a^{\mathrm{Re}({}^2\xi_0-{}^2\xi)}}{\mathrm{Re}({}^2\xi-{}^2\xi_0)}\right]\right] \quad \textbf{... (3.5)}$$

where $\xi_0={}^1\xi_0e_1+{}^2\xi_0e_2$ is any point in the region of convergence C and $\xi$ is any point with $\mathrm{Re}({}^1\xi)>\mathrm{Re}({}^1\xi_0)$ and $\mathrm{Re}({}^2\xi)>\mathrm{Re}({}^2\xi_0)$.

We choose $\xi_0$ with $\mathrm{Im}({}^1\xi_0)=0$ and $\mathrm{Im}({}^2\xi_0)=0$; $\sigma_1<\mathrm{Re}({}^1\xi_0)<\alpha_1$ and $\sigma_2<\mathrm{Re}({}^2\xi_0)<\alpha_2$

Then if $\xi\in R\Rightarrow {}^1\xi\in R_1$ and ${}^2\xi\in R_2\Rightarrow \mathrm{Re}({}^1\xi)\ge\alpha_1$ and $\mathrm{Re}({}^2\xi)\ge\alpha_2$

$\Rightarrow \mathrm{Re}({}^1\xi)-\mathrm{Re}({}^1\xi_0)\ge\alpha_1-\mathrm{Re}({}^1\xi_0)$ and $\mathrm{Re}({}^2\xi)-\mathrm{Re}({}^2\xi_0)\ge\alpha_2-\mathrm{Re}({}^2\xi_0)$

Now $\left|{}^1\xi_0 - {}^1\xi\right| < K_1$ and $\left|{}^2\xi_0 - {}^2\xi\right| < K_2$

$$\Rightarrow \left\| \xi_0 - \xi \right\| < \left[\frac{K_1^2 + K_2^2}{2}\right]^{\frac{1}{2}} = K$$

Where K is a constant depending on $\xi_0$ and R but not on $\xi$.

Then from **(3.5)**

$$\left\| \sum_{a<n\le b} \alpha_n n^{-\xi} \right\| \le 2\sqrt{2}\, M \left[ \left\| a^{-(\xi-\xi_0)} \right\| + K \left[ \frac{a^{-\alpha_1 + Re({}^1\xi_0)}}{\alpha_1 - Re({}^1\xi_0)} + \frac{a^{-\alpha_2 + Re({}^2\xi_0)}}{\alpha_2 - Re({}^2\xi_0)} \right] \right]$$

$$\left\| \sum_{a<n\le b} \alpha_n n^{-\xi} \right\| \le 2\sqrt{2}\, M \left[ a^{-\alpha_1 + Re({}^1\xi_0)} + a^{-\alpha_2 + Re({}^2\xi_0)} + K \left[ \frac{a^{-\alpha_1 + Re({}^1\xi_0)}}{\alpha_1 - Re({}^1\xi_0)} + \frac{a^{-\alpha_2 + Re({}^2\xi_0)}}{\alpha_2 - Re({}^2\xi_0)} \right] \right]$$

$$\left\| \sum_{a<n<b} \alpha_n n^{-\xi} \right\| \le 2\sqrt{2} M \left[ a^{-\alpha_1 + Re({}^1\xi_0)} B_1 + a^{-\alpha_2 + Re({}^2\xi_0)} B_2 \right]$$

$$\left\| \sum_{a<n<b} \alpha_n n^{-\xi} \right\| \le \left[ L_1 a^{-\alpha_1 + Re({}^1\xi_0)} + L_2 a^{-\alpha_2 + Re({}^2\xi_0)} \right]$$

where $L_1$ and $L_2$ are independent of $\xi$.

Since $a^{-\alpha_1 + Re({}^1\xi_0)} \to 0$, $a^{-\alpha_2 + Re({}^2\xi_0)} \to 0$ as $a \to \infty$, the Cauchy condition for uniform convergence is satisfied.

In the following we take,

$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$, $\xi_0 = x_1^0 + i_1 x_2^0 + i_2 x_3^0 + i_1 i_2 x_4^0$

$X_1 = x_1 - x_1^0$, $X_2 = x_2 - x_2^0$, $X_3 = x_3 - x_3^0$, $X_4 = x_4 - x_4^0$

$\theta_1$ and $\theta_2$ are defined as, $X_2 - X_3 = \tan\theta_1 (X_1 + X_4)$, $X_2 + X_3 = \tan\theta_2 (X_1 - X_4)$

given that $\xi - \xi_0 = X_1 + i_1 X_2 + i_2 X_3 + i_1 i_2 X_4$ …**(3.6)**

Under these notation we prove the following theorem

**Theorem 3.7:** If $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges for $\xi = \xi_0$ then $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges uniformly in the region

$$\{\xi \in C_2 : \left|\arg({}^1\xi - {}^1\xi_0)\right| \le \delta < \frac{\pi}{2} \text{ and } \left|\arg({}^2\xi - {}^2\xi_0)\right| \le \delta < \frac{\pi}{2}\}.$$

or equivalently, in the region $\{\xi \in C_2 : |\theta_1| \le \delta < \frac{\pi}{2} \text{ and } |\theta_2| \le \delta < \frac{\pi}{2}\}$

**Proof:** Suppose that $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges for $\xi = \xi_0$.

Then $\sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ and $\sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$ converges for ${}^1\xi = {}^1\xi_0$ and ${}^2\xi = {}^2\xi_0$ respectively.

By **Theorem 1.2**, the series $\sum_{n=1}^{\infty} {}^1\alpha_n n^{-{}^1\xi}$ converges uniformly in $\left|\arg({}^1\xi - {}^1\xi_0)\right| \le \delta < \frac{\pi}{2}$ and

$\sum_{n=1}^{\infty} {}^2\alpha_n n^{-{}^2\xi}$ converges uniformly in $\left|\arg({}^2\xi - {}^2\xi_0)\right| \le \delta < \frac{\pi}{2}$.

Hence $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges uniformly in $\left|\arg({}^1\xi - {}^1\xi_0)\right| \le \delta < \frac{\pi}{2}$ and $\left|\arg({}^2\xi - {}^2\xi_0)\right| \le \delta < \frac{\pi}{2}$.

Hence if $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges for $\xi = \xi_0$ then $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges uniformly in

$$\{\xi \in C_2 : \left|\arg({}^1\xi \;\; {}^1\xi_0)\right| \le \delta < \frac{\pi}{2} \text{ and } \left|\arg({}^2\xi - {}^2\xi_0)\right| \le \delta < \frac{\pi}{2}\}.$$

Now, by **(3.6)**,

$$\arg({}^1\xi - {}^1\xi_0) = \arg[\{X_1 + X_4\} + i_1\{X_2 - X_3\}] = \theta_1$$

$$\Rightarrow \tan\theta_1 = \frac{X_2 - X_3}{X_1 + X_4} \Rightarrow X_2 - X_3 = \tan\theta_1 (X_1 + X_4)$$

Similarly,

$$\arg({}^2\xi - {}^2\xi_0) = \arg[\{X_1 - X_4\} + i_1\{X_2 + X_3\}] = \theta_2$$

$$\Rightarrow \tan\theta_2 = \frac{X_2 + X_3}{X_1 - X_4} \Rightarrow X_2 + X_3 = \tan\theta_2\ (X_1 - X_4)$$

Hence, $\{\xi \in C_2 : |\arg({}^1\xi\ \ {}^1\xi_0)| \le \delta < \frac{\pi}{2}$ and $|\arg({}^2\xi - {}^2\xi_0)| \le \delta < \frac{\pi}{2}\} = \{\xi \in C_2 : |\theta_1| \le \delta < \frac{\pi}{2}$ and $|\theta_2| \le \delta < \frac{\pi}{2}\}$.

**Corollary 3.4:** If the series $f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges for $\xi = \xi_0$, and has the sum $f(\xi_0)$, then $f(\xi) \to f(\xi_0)$ when $\xi \to \xi_0$ along any path which lies entirely within the region $\{\xi \in C_2 : |\arg({}^1\xi - {}^1\xi_0)| \le \delta < \frac{\pi}{2}$ and $|\arg({}^2\xi - {}^2\xi_0)| \le \delta < \frac{\pi}{2}\}$.

**Proof:** Straightforward.

**Theorem 3.8:** If $\alpha_n$ is bounded then the series $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges absolutely in the region

$$\{\xi \in C_2 : \mathrm{Re}({}^1\xi) > 1 \text{ and } \mathrm{Re}({}^2\xi) > 1\}$$

Or equivalently, in the region $\{\xi \in C_2 : \mathrm{Re}(z_1) > 1$ and $|\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1\}$.

**Proof:** Since $\alpha_n$ is bounded, $\exists K > 0$ s.t. $\|\alpha_n\| \le K, \quad \forall n \in N$

$$\|\alpha_n\| \le K$$

$$\Rightarrow |{}^1\alpha_n|^2 + |{}^2\alpha_n|^2 \le 2K^2$$

$$\Rightarrow |{}^1\alpha_n| \le \sqrt{2}K \text{ and } |{}^2\alpha_n| \le \sqrt{2}K \qquad ...(3.7)$$

Now, $\|\alpha_n n^{-\xi}\| = \|{}^1\alpha_n n^{-{}^1\xi} e_1 + {}^2\alpha_n n^{-{}^2\xi} e_2\|$

$$\le \frac{1}{\sqrt{2}}[|{}^1\alpha_n n^{-{}^1\xi}| + |{}^2\alpha_n n^{-{}^2\xi}|] \qquad \text{[By Note 3.1(b)]}$$

$$= \frac{1}{\sqrt{2}}[|{}^1\alpha_n||n^{-{}^1\xi}| + |{}^2\alpha_n||n^{-{}^2\xi}|]$$

$$= \frac{1}{\sqrt{2}}[|{}^1\alpha_n| n^{-(x_1+x_4)} + |{}^2\alpha_n| n^{-(x_1-x_4)}]$$

$$\le \frac{\sqrt{2}K}{\sqrt{2}}[n^{-(x_1+x_4)} + n^{-(x_1-x_4)}] \qquad \text{[By (3.7)]}$$

$$= K[n^{-(x_1+x_4)} + n^{-(x_1-x_4)}]$$

$$\Rightarrow \|\alpha_n n^{-\xi}\| \le K[n^{-(x_1+x_4)} + n^{-(x_1-x_4)}]$$

$$\Rightarrow \sum \|\alpha_n n^{-\xi}\| \le K[\sum n^{-(x_1+x_4)} + \sum n^{-(x_1-x_4)}]$$

Since the series $\sum n^{-(x_1+x_4)}$ and $\sum n^{-(x_1-x_4)}$ converge if $x_1 + x_4 > 1$ and $x_1 - x_4 > 1$ respectively.

$\Rightarrow \sum \|\alpha_n n^{-\xi}\|$ is convergent if $\mathrm{Re}({}^1\xi) = x_1 + x_4 > 1$ and $\mathrm{Re}({}^2\xi) = x_1 - x_4 > 1$.

Again,

$$\mathrm{Re}({}^1\xi) = x_1 + x_4 > 1 \text{ and } \mathrm{Re}({}^2\xi) = x_1 - x_4 > 1$$

$$\Leftrightarrow x_1 > 1 \text{ and } |x_4| < x_1 - 1 \Leftrightarrow \mathrm{Re}(z_1) > 1 \text{ and } |\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1$$

This completes the proof.

**Remark 3.2:** In particular, if $\alpha_n = 1, \forall n$, the series $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ becomes the **Bicomplex Riemann Zeta function** (cf. **[8]**, **[9]**) and **Theorem 1.8** comes out as a particular case of **Theorem 3.8**.

**Theorem 3.9:** If $\alpha_n = O(n^k)$ then the series $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges absolutely in the region

$$\{\xi \in C_2 : \mathrm{Re}(^1\xi) > 1+k \text{ and } \mathrm{Re}(^2\xi) > 1+k\}$$

Or equivalently, in the region $\{\xi \in C_2 : \mathrm{Re}(z_1) > 1+k \text{ and } |\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1 - k\}$.

**Proof:** Since, $\alpha_n = O(n^k)$

$\Rightarrow \|\alpha_n\| \leq A n^k$

$\Rightarrow \|{}^1\alpha_n\| \leq \sqrt{2} A n^k$ and $\|{}^2\alpha_n\| \leq \sqrt{2} A n^k$

Now $\|\alpha_n n^{-\xi}\| \leq A n^k [n^{-(x_1+x_4)} + n^{-(x_1-x_4)}]$

$\Rightarrow \|\alpha_n n^{-\xi}\| \leq A[n^{-(x_1+x_4-k)} + n^{-(x_1-x_4-k)}]$

$\Rightarrow \sum \|\alpha_n n^{-\xi}\| \leq A[\sum n^{-(x_1+x_4-k)} + \sum n^{-(x_1-x_4-k)}]$

Since the series $\sum n^{-(x_1+x_4-k)}$ and $\sum n^{-(x_1-x_4-k)}$ converge if $x_1 + x_4 - k > 1$ and $x_1 - x_4 - k > 1$ respectively.

$\Rightarrow \sum \|\alpha_n n^{-\xi}\|$ is convergent if $\mathrm{Re}(^1\xi) = x_1 + x_4 > 1+k$ and $\mathrm{Re}(^2\xi) = x_1 - x_4 > 1+k$.

Again,

$\mathrm{Re}(^1\xi) = x_1 + x_4 > 1+k$ and $\mathrm{Re}(^2\xi) = x_1 - x_4 > 1+k$

$\Leftrightarrow x_1 > 1+k$ and $|x_4| < x_1 - 1 - k$.

$\Leftrightarrow \mathrm{Re}(z_1) > 1+k$ and $|\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1 - k$.

This completes the proof.

**Remark 3.3:** If $k = 0$, we get **Theorem 3.8**.

**Theorem 3.10:** If $\alpha_n = O(n^k)$ then the series $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges absolutely and uniformly in the region

$$\{\xi \in C_2 : \mathrm{Re}(^1\xi) > 1+k+\varepsilon \text{ and } \mathrm{Re}(^2\xi) > 1+k+\varepsilon\}$$

Or equivalently, in the region $\{\xi \in C_2 : \mathrm{Re}(z_1) > 1+k+\varepsilon \text{ and } |\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1 - k - \varepsilon\}$.

**Proof:** Under the same assumption as made in **Theorem 3.9,** we obtain $\|\alpha_n n^{-\xi}\| \leq A[n^{-(x_1+x_4-k)} + n^{-(x_1-x_4-k)}]$.

When $x_1 + x_4 - k > 1+\varepsilon$ and $x_1 - x_4 - k > 1+\varepsilon$

$$\|\alpha_n n^{-\xi}\| \leq A[n^{-(1+\varepsilon)} + n^{-(1+\varepsilon)}] = 2K n^{-(1+\varepsilon)}$$

Since $\sum n^{-(1+\varepsilon)}$ is convergent for every $\varepsilon > 0$, by **Weirstrass M-test [Theorem 2.3]**, $\sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ converges absolutely and uniformly.

Again, $x_1 + x_4 > 1+k+\varepsilon$ and $x_1 - x_4 > 1+k+\varepsilon$

$\Leftrightarrow x_1 > 1+k+\varepsilon$ and $|x_4| < x_1 - 1 - k - \varepsilon$

i.e. $\mathrm{Re}(^1\xi) > 1+k+\varepsilon$ and $\mathrm{Re}(^2\xi) > 1+k+\varepsilon$

$\Leftrightarrow \mathrm{Re}(z_1) > 1+k+\varepsilon$ and $|\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1 - k - \varepsilon$.

This completes the proof.

**Theorem 3.11:** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi=\xi_0$ then $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges absolutely and uniformly in the region $\{\xi\in C_2 : Re({}^1\xi)>1+\varepsilon+Re({}^1\xi_0)$ and $Re({}^2\xi)>1+\varepsilon+Re({}^2\xi_0)\}$.

**Proof:** Since $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi=\xi_0$

$$\Rightarrow \lim_{n\to\infty}\alpha_n n^{-\xi_0}=0, \exists K>0 \text{ s.t. } \|\alpha_n n^{-\xi_0}\|\le K \quad \forall n\in N$$

Now, $$\|\alpha_n n^{-\xi}\|=\left\|(\alpha_n n^{-\xi})\frac{n^{-\xi_0}}{n^{-\xi_0}}\right\|=\left\|(\alpha_n n^{-\xi_0})\frac{n^{-\xi}}{n^{-\xi_0}}\right\|=\|(\alpha_n n^{-\xi_0})\, n^{-(\xi-\xi_0)}\|$$

$$\le\sqrt{2}\,\|\alpha_n n^{-\xi_0}\|\,\|n^{-(\xi-\xi_0)}\|\le\sqrt{2}\,K\,\|n^{-(\xi-\xi_0)}\|$$

$$\le K\,[\,|n^{-({}^1\xi-{}^1\xi_0)}|+|n^{-({}^2\xi-{}^2\xi_0)}|\,]=K\,[\,n^{-(Re({}^1\xi)-Re({}^1\xi_0))}+n^{-(Re({}^2\xi)-Re({}^2\xi_0))}\,]$$

$$\|\alpha_n n^{-\xi}\|\le K\,[\,n^{-(Re({}^1\xi)-Re({}^1\xi_0))}+n^{-(Re({}^2\xi)-Re({}^2\xi_0))}\,]$$

When, $Re({}^1\xi)-Re({}^1\xi_0)>1+\varepsilon$ and $Re({}^2\xi)-Re({}^2\xi_0)>1+\varepsilon$

$$\|\alpha_n n^{-\xi}\|\le K\,[\,n^{-(1+\varepsilon)}+n^{-(1+\varepsilon)}\,]=2Kn^{-(1+\varepsilon)}$$

Since, $\sum n^{-(1+\varepsilon)}$ is convergent for every $\varepsilon>0$, by **Weirstrass M-test [Theorem 2.3]**, $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges absolutely and uniformly.

**Theorem 3.12:** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges for $\xi=\xi_0$ then $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ converges absolutely in the region

$$\{\xi\in C_2 : Re({}^1\xi)>1+Re({}^1\xi_0) \text{ and } Re({}^2\xi)>1+Re({}^2\xi_0)\}.$$

**Proof:** Under the same assumption made in **Theorem 3.11**, we obtain

$$\|\alpha_n n^{-\xi}\|\le K\,[\,n^{-(Re({}^1\xi)-Re({}^1\xi_0))}+n^{-(Re({}^2\xi)-Re({}^2\xi_0))}\,]$$

$$\Rightarrow \sum\|\alpha_n n^{-\xi}\|\le K\,[\,\sum n^{-(Re({}^1\xi)-Re({}^1\xi_0))}+\sum n^{-(Re({}^2\xi)-Re({}^2\xi_0))}\,]$$

Since the series $\sum n^{-(Re({}^1\xi)-Re({}^1\xi_0))}$ and $\sum n^{-(Re({}^2\xi)-Re({}^2\xi_0))}$ converge if $Re({}^1\xi)-Re({}^1\xi_0)>1$ and $Re({}^2\xi)-Re({}^2\xi_0)>1$ respectively.

$\Rightarrow \sum\|\alpha_n n^{-\xi}\|$ is convergent if $Re({}^1\xi)>1+Re({}^1\xi_0)$ and $Re({}^2\xi)>1+Re({}^2\xi_0)$

Hence $\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is converges absolutely in the region $\{\xi\in C_2 : Re({}^1\xi)>1+Re({}^1\xi_0)$ and $Re({}^2\xi)>1+Re({}^2\xi_0)\}$.

**Boundedness of $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ in the Region of Absolute Convergence:**

The function $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is bounded in any region properly included in the region of absolute convergence.

For $$\|f(\xi)\|=\left\|\sum_{n=1}^{\infty}\alpha_n n^{-\xi}\right\|\le\sum_{n=1}^{\infty}\|\alpha_n n^{-\xi}\|\le\frac{1}{\sqrt{2}}\left[\sum_{n=1}^{\infty}\left|{}^1\alpha_n n^{-{}^1\xi}\right|+\sum_{n=1}^{\infty}\left|{}^2\alpha_n n^{-{}^2\xi}\right|\right]$$

$$=\frac{1}{\sqrt{2}}\left[\sum_{n=1}^{\infty}|{}^1\alpha_n|n^{-(x_1+x_4)}+\sum_{n=1}^{\infty}|{}^2\alpha_n|n^{-(x_1-x_4)}\right]\le\frac{1}{\sqrt{2}}\left[\sum_{n=1}^{\infty}|{}^1\alpha_n|n^{-\alpha}+\sum_{n=1}^{\infty}|{}^2\alpha_n|n^{-\beta}\right]$$

for $Re({}^1\xi)\ge\alpha>\overline{\sigma}_1$ and $Re({}^2\xi)\ge\beta>\overline{\sigma}_2$.

If the series $\sum_{n=1}^{\infty}|{}^1\alpha_n|n^{-\overline{\sigma}_1}$ and $\sum_{n=1}^{\infty}|{}^2\alpha_n|n^{-\overline{\sigma}_2}$ are convergent we can take $\alpha=\overline{\sigma}_1$ and $\beta=\overline{\sigma}_2$, and the function is bounded in the region of absolute convergence.

But in general the region of absolute convergence is not a region where $f(\xi)$ is bounded, even if we exclude the neighbourhood of singularities on the line $Re({}^1\xi)=\overline{\sigma}_1$ and $Re({}^2\xi)=\overline{\sigma}_2$. To be precise, we have

**Theorem 3.13:**

**(1)** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is such that $\alpha_n\in H^+\ \forall n\in N$ and $\sum_{n=1}^{\infty}{}^1\alpha_n n^{-\bar{\sigma}_1}$ is divergent, then $f(\xi)$ is not bounded in the region $A=\{\xi\in C_2:\mathrm{Re}({}^1\xi)>\bar{\sigma}_1,\left|\mathrm{Im}({}^1\xi)\right|\geq\alpha>0\}$.

**(2)** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is such that $\alpha_n\in H^+\ \forall n\in N$ and $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-\bar{\sigma}_2}$ is divergent, then $f(\xi)$ is not bounded in the region $B=\{\xi\in C_2:\mathrm{Re}({}^2\xi)>\bar{\sigma}_2,\left|\mathrm{Im}({}^2\xi)\right|\geq\beta>0\}$.

**(3)** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is such that $\alpha_n\in H^+\ \forall n\in N$ and $\sum_{n=1}^{\infty}{}^1\alpha_n n^{-\bar{\sigma}_1}$, $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-\bar{\sigma}_2}$ both are divergent, then, $f(\xi)$ is not bounded in the region $C=A\cup B$.

**Proof: (1)** Since $\alpha_n\in H^+\ \forall n\in N\Rightarrow{}^1\alpha_n\geq 0$ and ${}^2\alpha_n\geq 0\ \forall n\in N$

For $\sum_{n=1}^{\infty}{}^1\alpha_n n^{-\bar{\sigma}_1}$ is divergent

By **Theorem 1.3**, ${}^1f({}^1\xi)$ is not bounded in the region

$\mathrm{Re}({}^1\xi)>\bar{\sigma}_1,\left|\mathrm{Im}({}^1\xi)\right|\geq\alpha>0$

Hence $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is not bounded in the region $A=\{\xi\in C_2:\mathrm{Re}({}^1\xi)>\bar{\sigma}_1,\left|\mathrm{Im}({}^1\xi)\right|\geq\alpha>0\}$.

**(2)** Since $\alpha_n\in H^+\ \forall n\in N\Rightarrow{}^1\alpha_n\geq 0$ and ${}^2\alpha_n\geq 0\ \forall n\in N$

For $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-\bar{\sigma}_2}$ is divergent

By **Theorem 1.3**, ${}^2f({}^2\xi)$ is not bounded in the region $\mathrm{Re}({}^2\xi)>\bar{\sigma}_2,\left|\mathrm{Im}({}^2\xi)\right|\geq\beta>0$

Hence $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is not bounded in the region $B=\{\xi\in C_2:\mathrm{Re}({}^2\xi)>\bar{\sigma}_2,\left|\mathrm{Im}({}^2\xi)\right|\geq\beta>0\}$.

**(3)** Since $\alpha_n\in H^+\ \forall n\in N\Rightarrow{}^1\alpha_n\geq 0$ and ${}^2\alpha_n\geq 0\ \forall n\in N$

For $\sum_{n=1}^{\infty}{}^1\alpha_n n^{-\bar{\sigma}_1}$ and $\sum_{n=1}^{\infty}{}^2\alpha_n n^{-\bar{\sigma}_2}$ are divergent

By **Theorem 1.3**, ${}^1f({}^1\xi)$ and ${}^2f({}^2\xi)$ are not bounded in the region

$\mathrm{Re}({}^1\xi)>\bar{\sigma}_1,\left|\mathrm{Im}({}^1\xi)\right|\geq\alpha>0$ and $\mathrm{Re}({}^2\xi)>\bar{\sigma}_2,\left|\mathrm{Im}({}^2\xi)\right|\geq\beta>0$ respectively.

Hence the Bicomplex Dirichlet series $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is not bounded in the region $C=A\cup B$.

**Theorem 3.14:** If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is bounded for $\{\xi\in C_2:\mathrm{Re}({}^1\xi)>\alpha \text{ and } \mathrm{Re}({}^2\xi)>\beta\}$, then $\sum\left\|\alpha_n n^{-(\alpha e_1+\beta e_2)}\right\|^2$ is convergent; if $\left\|f(\xi)\right\|\leq M$, then $\sum\left\|\alpha_n n^{-(\alpha e_1+\beta e_2)}\right\|^2\leq 2M^2$.

**Proof:** Given $f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is bounded for $\{\xi\in C_2:\mathrm{Re}({}^1\xi)>\alpha \text{ and } \mathrm{Re}({}^2\xi)>\beta\}$,

$\Rightarrow{}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n n^{-{}^1\xi}$ and ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n n^{-{}^2\xi}$ are bounded for

$\mathrm{Re}({}^1\xi)>\alpha$ and $\mathrm{Re}({}^2\xi)>\beta$ respectively.

Now by **Theorem 1.4**,

$\sum_{n=1}^{\infty}\left|{}^1\alpha_n\right|^2 n^{-2\alpha}$ and $\sum_{n=1}^{\infty}\left|{}^2\alpha_n\right|^2 n^{-2\beta}$ are convergent.

$$\Sigma\left\|\alpha_n n^{-(\alpha e_1+\beta e_2)}\right\|^2 = \frac{1}{2}\left\{\Sigma\left|{}^1\alpha_n n^{-\alpha}\right|^2 + \Sigma\left|{}^2\alpha_n n^{-\beta}\right|^2\right\}$$

Since $\sum_{n=1}^{\infty}\left|{}^1\alpha_n\right|^2 n^{-2\alpha}$ and $\sum_{n=1}^{\infty}\left|{}^2\alpha_n\right|^2 n^{-2\beta}$ are convergent

Hence $\Sigma\left\|\alpha_n n^{-(\alpha e_1+\beta e_2)}\right\|^2$ is convergent.

$\|f(\xi)\| \le M \Rightarrow \left|{}^1f({}^1\xi)\right| \le \sqrt{2}M$ and $\left|{}^2f({}^2\xi)\right| \le \sqrt{2}M$

$\left|{}^1f({}^1\xi)\right| \le \sqrt{2}M \Rightarrow \sum_{n=1}^{\infty}\left|{}^1\alpha_n\right|^2 n^{-2\alpha} \le 2M^2$

$\left|{}^2f({}^2\xi)\right| \le \sqrt{2}M \Rightarrow \sum_{n=1}^{\infty}\left|{}^2\alpha_n\right|^2 n^{-2\beta} \le 2M^2$

Now, $\sum_{n=1}^{\infty}\left|{}^1\alpha_n\right|^2 n^{-2\alpha} + \sum_{n=1}^{\infty}\left|{}^2\alpha_n\right|^2 n^{-2\beta} \le 4M^2 \Rightarrow \Sigma\left\|\alpha_n n^{-(\alpha e_1+\beta e_2)}\right\|^2 \le 2M^2$.

**The Zeros and Zero Free Region of $f(\xi) = \sum_{n=1}^{\infty}\alpha_n n^{-\xi}$:**

In this section, we study a particular $f(\xi) = \sum_{n=1}^{\infty}\alpha_n n^{-\xi}$, which can be viewed as a power series. Let $\alpha_n = 0$ except when n is a power of 2 and $\alpha_{2^n} = \beta_n$. To be precise,

$\alpha_1 = 0,\ \alpha_2 = \beta_1, \alpha_3 = 0, \alpha_4 = \beta_2,\ \alpha_5 = 0,\ \alpha_6 = 0, \alpha_7 = 0, \alpha_8 = \beta_3, \ldots$..

Then, $f(\xi) = \sum_{n=1}^{\infty}\alpha_n n^{-\xi} = \sum_{n=1}^{\infty}\beta_n\left(2^n\right)^{-\xi} = \sum_{n=1}^{\infty}\beta_n\left(2^{-\xi}\right)^n = \sum_{n=1}^{\infty}\beta_n\zeta^n$.

Evidently, the series can be viewed as a power series as well as Bicomplex Dirichlet Series.

To each zero $\zeta_v$ of the power series corresponds an infinite number of sequences of zeros

$$\eta = -\frac{\log\zeta_v + 2(me_1 + ne_2)\pi i_1}{\log 2} \qquad (m,n = 0, \pm 1, \pm 2, \ldots)$$

$$\Rightarrow \quad {}^1\eta = -\frac{\log{}^1\zeta_v + 2m\,\pi i_1}{\log 2} \text{ and } {}^2\eta = -\frac{\log{}^2\zeta_v + 2n\,\pi i_1}{\log 2}$$

If ${}^1\xi_0$ and ${}^2\xi_0$ are the zeros of smallest modulus (other than zero) then ${}^1f({}^1\xi)$ and ${}^2f({}^2\xi)$ have no zero to the right of the line

$$\alpha = -\frac{\log\left|{}^1\xi_0\right|}{\log 2} \text{ and } \beta = -\frac{\log\left|{}^2\xi_0\right|}{\log 2}, \text{ respectively.}$$

Hence $\{\ \xi \in C_2 : \mathrm{Re}({}^1\xi) > \alpha$ and $\mathrm{Re}({}^2\xi) > \beta\ \}$ is the zero free region of $f(\xi) = \sum_{n=1}^{\infty}\alpha_n n^{-\xi}$.

**Definition 3.4: The function $N(\sigma, \sigma', T_1, T_2)$**

Denote by $N(\sigma, T_1)$, the number of zeros $a + i_1 b$ of ${}^1f({}^1\xi) = \sum_{n=1}^{\infty}{}^1\alpha_n\, n^{-{}^1\xi}$ such that $a > \sigma = \mathrm{Re}({}^1\xi)$, $\alpha < b < T_1$, where $\alpha$ be a positive number such that ${}^1f({}^1\xi)$ is regular for $\mathrm{Im}({}^1\xi) \ge \alpha$ and $\sigma$ sufficiently large.

Denote by $N(\sigma', T_2)$, the number of zeros $c + i_1 d$ of ${}^2f({}^2\xi) = \sum_{n=1}^{\infty}{}^2\alpha_n\, n^{-{}^2\xi}$ such that $c > \sigma' = \mathrm{Re}({}^2\xi)$, $\beta < d < T_2$, where $\beta$ be a positive number such that ${}^2f({}^2\xi)$ is regular for $\mathrm{Im}({}^2\xi) \ge \beta$ and $\sigma'$ sufficiently large.

The number of zeros of $f(\xi) = \sum_{n=1}^{\infty}\alpha_n n^{-\xi}$ is obviously given by $N(\sigma, \sigma', T_1, T_2) = N(\sigma, T_1).N(\sigma', T_2)$.

**Theorem 3.15:** Let ${}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n\, n^{-{}^1\xi}$ and ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n\, n^{-{}^2\xi}$ be bounded for $\mathrm{Re}({}^1\xi)=\sigma\geq\alpha$ and $\mathrm{Re}({}^2\xi)=\sigma'\geq\beta$, respectively. Then $N(\sigma,\sigma',T_1,T_2)=O(T_1T_2)$ $(\sigma>\alpha \text{ and } \sigma'>\beta)$.

**Proof:** Since ${}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n\, n^{-{}^1\xi}$ and ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n\, n^{-{}^2\xi}$ are bounded for $\sigma\geq\alpha$ and $\sigma'\geq\beta$ respectively.

Then, by **Theorem 1.5**

$N(\sigma,T_1)=O(T_1)$ $(\sigma>\alpha)$

$N(\sigma',T_2)=O(T_2)$ $(\sigma'>\beta)$

$N(\sigma,\sigma',T_1,T_2)=N(\sigma,T_1)N(\sigma',T_2)=O(T_1T_2)$ $(\sigma>\alpha,\sigma'>\beta)$

**Theorem 3.16:** If ${}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n\, n^{-{}^1\xi}$ and ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n\, n^{-{}^2\xi}$ are of finite order for $\mathrm{Re}({}^1\xi)=\sigma\geq\alpha$ and $\mathrm{Re}({}^2\xi)=\sigma'\geq\beta$ respectively.

Then, $N(\sigma,\sigma',T_1,T_2)=O(T_1T_2\log T_1\log T_2)$ $(\sigma>\alpha \text{ and } \sigma'>\beta)$.

**Proof:** Since ${}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n\, n^{-{}^1\xi}$ and ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n\, n^{-{}^2\xi}$ are of finite order for $\sigma\geq\alpha$ and $\sigma'\geq\beta$, respectively, we have, by **Theorem 1.6**

$N(\sigma,T_1)=O(T_1\log T_1)$ $(\sigma>\alpha)$

$N(\sigma',T_2)=O(T_2\log T_2)$ $(\sigma'>\beta)$

So that, $N(\sigma,\sigma',T_1,T_2)=N(\sigma,T_1)N(\sigma',T_2)=O(T_1T_2\log T_1\log T_2)$.

## Euler Product:

**Definition 3.5:**

An arithmetic function $f:N\to C_2$ is a **multiplicative function** if $f(1)=1$, and $f(mn)=f(m)f(n)$ whenever $m,n\in N$ are co-prime.

An arithmetic function $f:N\to C_2$ is **completely multiplicative** if $f(1)=1$, and $f(mn)=f(m)f(n)$ for any $m,n\in N$.

**Theorem 3.17:**

**(1)** $f:N\to C_2$ is multiplicative iff both ${}^1f:N\to{}^1C_2$ and ${}^2f:N\to{}^2C_2$ are multiplicative.

**(2)** $f:N\to C_2$ is completely multiplicative iff both ${}^1f:N\to{}^1C_2$ and ${}^2f:N\to{}^2C_2$ are completely multiplicative.

**Proof:**

**(1)** Suppose that $f:N\to C_2$ is multiplicative. Thus,

$f(1)=1$ and $f(mn)=f(m)f(n)$ whenever $m,n\in N$ are co-prime.

$\Leftrightarrow$ ${}^1f(1)e_1+{}^2f(1)e_2=1e_1+1e_2$ and

${}^1f(mn)e_1+{}^2f(mn)e_2={}^1f(m)\,{}^1f(n)e_1+{}^2f(m)\,{}^2f(n)e_2$ whenever $m,n\in N$ are co-prime

$\Leftrightarrow$ ${}^1f(1)=1$ and ${}^2f(1)=1$;

${}^1f(mn)={}^1f(m)\,{}^1f(n)$ and ${}^2f(mn)={}^2f(m)\,{}^2f(n)$ whenever $m,n\in N$ are co–prime

$\Leftrightarrow$ ${}^1f:N\to{}^1C_2$ and ${}^2f:N\to{}^2C_2$ are multiplicative.

**(2)** Next, suppose $f:N\to C_2$ is completely multiplicative.

This is equivalent to the statements

$f(1)=1$ and $f(mn)=f(m)f(n)$ $\forall m,n\in N$

$\Leftrightarrow$ ${}^1f(1)e_1+{}^2f(1)e_2=1e_1+1e_2$ and

${}^1f(mn)e_1+{}^2f(mn)e_2={}^1f(m)\,{}^1f(n)e_1+{}^2f(m)\,{}^2f(n)e_2$ $\forall m,n\in N$

$\Leftrightarrow {}^1f(1) = 1$ and ${}^2f(1) = 1$;

${}^1f(mn) = {}^1f(m)\,{}^1f(n)$ and ${}^2f(mn) = {}^2f(m)\,{}^2f(n)$ $\forall\, m, n \in N$

$\Leftrightarrow {}^1f : N \to {}^1C_2$ and ${}^2f : N \to {}^2C_2$ are completely multiplicative.

**Theorem 3.18:** If $f : N \to C_2$ is multiplicative and $\sum_{n=1}^{\infty} f(n)n^{-\xi}$ is absolutely convergent, then

$$\sum_{n=1}^{\infty} f(n)n^{-\xi} = \prod_{p} \Big[ \sum_{j=0}^{\infty} f(p^j)\,p^{-j\xi} \Big].$$

**Proof:** $\sum_{n=1}^{\infty} f(n)n^{-\xi} = [\sum_{n=1}^{\infty} {}^1f(n)n^{-{}^1\xi}]e_1 + [\sum_{n=1}^{\infty} {}^2f(n)n^{-{}^2\xi}]e_2$

In the complex plane we have $\sum_{n=1}^{\infty} f(n)\,n^{-s} = \prod_{p} \sum_{j=0}^{\infty} f(p^j)\, p^{-js}$

Since $f : N \to C_2$ is multiplicative

$\Rightarrow {}^1f : N \to {}^1C_2$ and ${}^2f : N \to {}^2C_2$ are multiplicative.

Also, $\sum_{n=1}^{\infty} f(n)n^{-\xi}$ is absolutely convergent

$\Rightarrow \sum_{n=1}^{\infty} {}^1f(n)\,n^{-{}^1\xi}$ and $\sum_{n=1}^{\infty} {}^2f(n)\,n^{-{}^2\xi}$ are absolutely convergent.

Therefore $\sum_{n=1}^{\infty} f(n)n^{-\xi} = [\prod_{p} \sum_{j=0}^{\infty} {}^1f(p^j)\, p^{-j\,{}^1\xi}]e_1 + [\prod_{p} \sum_{j=0}^{\infty} {}^2f(p^j)\, p^{-j\,{}^2\xi}]e_2$

Hence, by **Lemma 1.2**,

$$\sum_{n=1}^{\infty} f(n)n^{-\xi} = \prod_{p} \Big[ \sum_{j=0}^{\infty} {}^1f(p^j)\, p^{-j\,{}^1\xi}\Big]e_1 + \Big[\sum_{j=0}^{\infty} {}^2f(p^j)\, p^{-j\,{}^2\xi}\Big]e_2$$

$$= \prod_{p} \Big[ \sum_{j=0}^{\infty} f(p^j)p^{-j\xi} \Big].$$

**Theorem 3.19:** If $f : N \to C_2$ is completely multiplicative and $\sum_{n=1}^{\infty} f(n)n^{-\xi}$ is absolutely convergent, then

$$\sum_{n=1}^{\infty} f(n)\,n^{-\xi} = \prod_{p} (1 - f(p)p^{-\xi})^{-1}.$$

**Proof:** By **Theorem 3.18,**

$$\sum_{n=1}^{\infty} f(n)n^{-\xi} = \prod_{p} \Big[ \sum_{j=0}^{\infty} f(p^j)\,p^{-j\xi} \Big]$$

Since $f : N \to C_2$ is completely multiplicative

$\Rightarrow f(p^j) = [f(p)]^j$

Now,

$$\sum_{n=1}^{\infty} f(n)n^{-\xi} = \prod_{p} \Big[ \sum_{j=0}^{\infty} [f(p)]^j\, p^{-j\xi} \Big]$$

$$= \prod_{p} \Big[ \sum_{j=0}^{\infty} [f(p)\,p^{-\xi}]^j \Big]$$

$$= \prod_{p} (1 - f(p)p^{-\xi})^{-1}.$$

**4. Bicomplex Dirichlet Series of type $\lambda_n$ :**

The **Bicomplex Dirichlet series type $\lambda_n$** is defined as

$$f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$$

where $\{\alpha_n\}$ is a sequence of bicomplex numbers, $\{\lambda_n\}$ is a strictly monotonically increasing and unbounded sequence of positive real numbers and $\xi \in C_2$ is a bicomplex variable.

As, $\alpha_n e^{-\lambda_n \xi} = ({}^1\alpha_n e^{-\lambda_n {}^1\xi}) e_1 + ({}^2\alpha_n e^{-\lambda_n {}^2\xi}) e_2$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi} = \sum_{n=1}^{\infty} {}^1\alpha_n e^{-\lambda_n {}^1\xi} \; e_1 + \sum_{n=1}^{\infty} {}^2\alpha_n e^{-\lambda_n {}^2\xi} \; e_2$$

Now we denote the sum function of the series $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$, $\sum_{n=1}^{\infty} {}^1\alpha_n e^{-\lambda_n {}^1\xi}$ and $\sum_{n=1}^{\infty} {}^2\alpha_n e^{-\lambda_n {}^2\xi}$ by $f(\xi)$, ${}^1f({}^1\xi)$ and ${}^2f({}^2\xi)$ respectively.

Thus $f(\xi) = {}^1f({}^1\xi) e_1 + {}^2f({}^2\xi) e_2$

We denote the abscissae of convergence of ${}^1f({}^1\xi) = \sum_{n=1}^{\infty} {}^1\alpha_n e^{-\lambda_n {}^1\xi}$ and ${}^2f({}^2\xi) = \sum_{n=1}^{\infty} {}^2\alpha_n e^{-\lambda_n {}^2\xi}$ by $\sigma_1$ and $\sigma_2$, and the abscissae of absolute convergence by $\overline{\sigma}_1$ and $\overline{\sigma}_2$, respectively.

**Theorem 4.1**: A Bicomplex Dirichlet series $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ converges for $\xi = \xi_0$ iff $\sum_{n=1}^{\infty} {}^1\alpha_n e^{-\lambda_n {}^1\xi}$ converges for ${}^1\xi = {}^1\xi_0$ and $\sum_{n=1}^{\infty} {}^2\alpha_n e^{-\lambda_n {}^2\xi}$ converges for ${}^2\xi = {}^2\xi_0$.

**Theorem 4.2:** If $f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ converges for $\xi = \xi_0$ then $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ converges in the region

$$\{\xi \in C_2 : Re({}^1\xi) > Re({}^1\xi_0) \text{ and } Re({}^2\xi) > Re({}^2\xi_0)\} = \{\xi \in C_2 : x_1 + x_4 > x_1^0 + x_4^0 \text{ and } x_1 - x_4 > x_1^0 - x_4^0\}$$

or equivalently in the region $\{\xi \in C_2 : Re(z_1) > Re(z_1^0) \text{ and } |Im(z_2) - Im(z_2^0)| < Re(z_1) - Re(z_1^0)\}$.

**Corollary 4.1:** If $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ diverges for $\xi = \xi_0$ then $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ diverges in the region

$$\{\xi \in C_2 : Re({}^1\xi) < Re({}^1\xi_0) \text{ and } Re({}^2\xi) < Re({}^2\xi_0)\} = \{\xi \in C_2 : x_1 + x_4 < x_1^0 + x_4^0 \text{ and } x_1 - x_4 < x_1^0 - x_4^0\}$$

or equivalently in the region $\{\xi \in C_2 : Re(z_1) < Re(z_1^0) \text{ and } |Im(z_2) - Im(z_2^0)| > Re(z_1) - Re(z_1^0)\}$.

**Theorem 4.3:** The Bicomplex Dirichlet series $f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ converges in the region

$$R = \{\xi \in C_2 : Re({}^1\xi) > \sigma_1 \text{ and } Re({}^2\xi) > \sigma_2\}.$$

**Definition 4.1: Region of Convergence of Bicomplex Dirichlet Series**

The region of convergence of Bicomplex Dirichlet series $f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ is the region

$\{\xi \in C_2 : Re({}^1\xi) > \sigma_1 \text{ and } Re({}^2\xi) > \sigma_2\}$ denoted as $R$.

**Definition 4.2: Absolute Convergence of Bicomplex Dirichlet Series**

The Bicomplex Dirichlet series $\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi}$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} \left\| \alpha_n e^{-\lambda_n \xi} \right\|$ is convergent.

Now,

$$\left\| \alpha_n e^{-\lambda_n \xi} \right\| = \left\| \left( {}^1\alpha_n e^{-\lambda_n {}^1\xi} \right) e_1 + \left( {}^2\alpha_n e^{-\lambda_n {}^2\xi} \right) e_2 \right\|$$

$$= \frac{1}{\sqrt{2}} \left[ \left| {}^1\alpha_n e^{-\lambda_n {}^1\xi} \right|^2 + \left| {}^2\alpha_n e^{-\lambda_n {}^2\xi} \right|^2 \right]^{1/2}$$

$$\le \frac{1}{\sqrt{2}}[\left|{}^{1}\alpha_n e^{-\lambda_n {}^{1}\xi}\right| + \left|{}^{2}\alpha_n e^{-\lambda_n {}^{2}\xi}\right|] = \frac{1}{\sqrt{2}}[\left|{}^{1}\alpha_n\right|\left|e^{-\lambda_n {}^{1}\xi}\right| + \left|{}^{2}\alpha_n\right|\left|e^{-\lambda_n {}^{2}\xi}\right|].$$

$$\left\|\alpha_n e^{-\lambda_n \xi}\right\| \le \frac{1}{\sqrt{2}}[\left|{}^{1}\alpha_n\right| e^{-\lambda_n (x_1 + x_4)} + \left|{}^{2}\alpha_n\right| e^{-\lambda_n (x_1 - x_4)}].$$

Therefore,

$$\sum_{n=1}^{\infty}\left\|\alpha_n e^{-\lambda_n \xi}\right\| \le \frac{1}{\sqrt{2}}[\sum_{n=1}^{\infty}\left|{}^{1}\alpha_n\right| e^{-\lambda_n (x_1 + x_4)} + \sum_{n=1}^{\infty}\left|{}^{2}\alpha_n\right| e^{-\lambda_n (x_1 - x_4)}]$$

**Theorem 4.4:** The Bicomplex Dirichlet series $f(\xi) = \sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges absolutely in the region

$$A = \{ \xi \in C_2 : Re({}^{1}\xi) > \bar{\sigma}_1 \text{ and } Re({}^{2}\xi) > \bar{\sigma}_2 \}.$$

**Definition 4.3: Region of Absolute Convergence of Bicomplex Dirichlet Series**

The region of absolute convergence of Bicomplex Dirichlet series $f(\xi) = \sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ is the region

$\{\xi \in C_2 : Re({}^{1}\xi) > \bar{\sigma}_1 \text{ and } Re({}^{2}\xi) > \bar{\sigma}_2\}$ is denoted as $\bar{R}$.

**Definition 4.4: Region of Conditional Convergence**

A region in which the Dirichlet series is convergent but not absolutely convergent will be called the Region of conditional convergence of the Dirichlet series.

In the following we take,

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ \xi_0 = x_1^0 + i_1 x_2^0 + i_2 x_3^0 + i_1 i_2 x_4^0$$

$$X_1 = x_1 - x_1^0,\ X_2 = x_2 - x_2^0,\ X_3 = x_3 - x_3^0,\ X_4 = x_4 - x_4^0$$

$\theta_1$ and $\theta_2$ are defined as $X_2 - X_3 = \tan\theta_1 (X_1 + X_4)$, $X_2 + X_3 = \tan\theta_2 (X_1 - X_4)$

Under these notation we prove the following theorem

**Theorem 4.5**: If $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges for $\xi = \xi_0$ then $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges uniformly in the region

$$U = \{\xi \in C_2 : \left|\arg({}^{1}\xi - {}^{1}\xi_0)\right| \le \delta < \frac{\pi}{2} \text{ and } \left|\arg({}^{2}\xi - {}^{2}\xi_0)\right| \le \delta < \frac{\pi}{2}\} = \{\xi \in C_2 : \left|\theta_1\right| \le \delta < \frac{\pi}{2} \text{ and } \left|\theta_2\right| \le \delta < \frac{\pi}{2}\}$$

**Corollary 4.2:** If the series $f(\xi) = \sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ is convergent for $\xi = \xi_0$, and has the sum $f(\xi_0)$, then $f(\xi) \to f(\xi_0)$ when $\xi \to \xi_0$ along any path which lies entirely within the region $\{\xi \in C_2 : \left|\arg({}^{1}\xi - {}^{1}\xi_0)\right| \le \delta < \frac{\pi}{2} \text{ and } \left|\arg({}^{2}\xi - {}^{2}\xi_0)\right| \le \delta < \frac{\pi}{2}\}$.

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