Spectral theorem on compact self adjoint operators

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ABSTRACT

Spectral Theorem provides spectral decomposition, Eigen value decomposition of the underlying vector space on which the operator acts. Here, we have tried to work on the formulation of an operator explicitly, operator being self adjoint and compact defined on Hilbert space.

INTRODUCTION

Spectral Theory for a self adjoint operator is quite complicated. But it becomes easier if the operator at hand is compact. Consider on operator. $T : H \rightarrow H$ with H being a Hilbert space. The complete spectral decomposition of T can be stated in a quite elementary fashion. Spectral Theorem is a generalization of the familiar theorem from Linear algebra asserting that a self adjoint n x n matrix A can be diagnolized , there is a diagonal matrix D and unitary matrix U st.

$$A = UDU^{-1}$$

In particular a compact self adjoint operator can be unitarily diagonalized. Actually, spectral theory is an inclusive term for theories extending the Eigen vector and Eigen value theory of single matrix to a much broader theory of operators in a variety of mathematical spaces.

This project is concerned with studying the “spectral representation of compact self adjoint operators. Here, we have tried to concentrate on both finite dimensional spaces and infinite dimensional spaces. Although, the proofs for both the cases have been discussed in standard books, but we have tried to prove it with slightly different approach. The concept is relatively straight forward for operators on finite dimensional spaces but will require some modifications for operators on infinite dimensional spaces.

Here, we started with proving the existence of unit vector x_0 of H with $\|Tx_0\| = \|T\|$. Then we proved the fact that T has an Eigen value $\|T\|$ or $-\|T\|$, where T is a compact and self adjoint operator defined on Hilbert space H. Then we found the representation of T in the main proof.

For infinite dimensional spaces, we started with family of projections. We represented T in terms of Riemann Steiltjes Integral. Considering the fact that Eigen values of T can both be positive as well as negative. We discussed both the cases and concluded the result. But before going further, we shall require a few facts concerning the terms and definitions used in the Theorem , which are as follows

Important Definitions

1. **Compact Operator :**

   A Linear operator $A: X \rightarrow Y$ is said to be compact if the set $Cl\{ Ax : \| x \| \leq 1\}$ is compact in Y.

2. **Self Adjoint Operator :**

   A bounded operator A on a Hilbert space H is said to be self adjoint if $A^* = A$, where A^* is adjoint of A.

3. Kernel of an Operator :

If $A : X \rightarrow Y$ b a Linear operator then

$N(A) = \{ x \in X : Ax = 0\}$ is called kernel of A or null space of A.

4. Eigen Spectrum: The set of all Eigen values of A is called Eigen spectrum of A. It is denoted by σ_{eig} (A) that is

$\sigma_{eig}(A) = [\lambda \in K : \exists\, x \neq 0 \text{ st } A x = \lambda x]$

a) $\lambda < \mu$ and $E_\lambda \leq E_\mu$ implies $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

b) Lt $E_\lambda x = 0$ as $\lambda \rightarrow -\infty$

c) Lt $E_\lambda x = x$ as $\lambda \rightarrow \infty$

5. Orthonormal sequence :

A sequence $< x_n >$ is X is said to be orthonormal sequence whose terms form an orthonormal set i.e it follows two conditions :

a) $\forall\, x, y \in X$, $< x, y > = 0$

b) $\forall\, x \in X$, $< x, y > = 1$

6. Weakly convergent sequence

A sequence $< x_n >$ in X is said to coverge to an element

$x \in X$ if $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ for every $f \in X'$

written as $x_n \rightarrow x$ weakly.

7. Reflexive normed space :

A normed linear space X is said to be reflexive if the canonical Isometry J : $X \rightarrow X''$ defined by (Jx) f = f (x)

$\forall\, x \in X, f \in X'$ is surjective.

8. Separable space :

A metric space is said to be separable if it has a countable dense subset.

9. Projection: A linear operator $P : X \rightarrow X$ is called a projection operator or simple projection if $Px = x\ \forall\, x \in R(P)$

SPECTRAL THEOREM

Here, we will discuss the spectral theorem of compact self adjoint operators. Firstly, we will give statement of the theorem. For proving the theorem, we require some additional results which will be discussed in the subsequent sections.

Statement of Theorem:

Let $T : H \rightarrow H$ be a compact and self adjoint operator on a Hilbert space H. Then there is a finite or infinite sequence $\{\lambda_n\}_{n=1}^{N}$ ($n \in Z^+$ Or $N = \infty$) of real eigen values $\lambda_n \neq 0$ and a corresponding orthonormal sequence $\{e_n\}_{n=1}^{N}$ in H

such that

(a) $Te_n = \lambda_n e_n\ \forall$ n with $1 \leq n \leq N$

(b) $N(T) = \text{span}(\{e_n\}_{n=1}^{N}$

(c) if $N = \infty$ then $\lambda_n = 0$ as $n \to \infty$

Before going further, we will prove some very important results for the theorem which are given in the form of lemma:

2.1 Some Important Results

Lemma 1: If X be a reflexive normed space and X is separable. Then every bounded sequence $<x_n>$ in X has a subsequence which is weakly convergent.

Proof : Let $< x_n >$ be a bounded sequence in X.

$\therefore \exists$ a +ve number B s.t $\| x_n \| \leq B \; \forall \; n$.

Now X' is separable, $\exists$ a countable dense subset of X'

Let that set be $\{ f_1, f_2, ... \}$

Now $| f_1 (x_j) | \leq \| f_1 \| \| x_j \| \leq \| f_1 \| B , \; \forall \; j$

Which implies $< f_1 (x_j)>$ is a bounded sequence.

Hence by Bolzano Weirstrass Theorem , every bounded sequence has a convergent subsequence, that is, $\exists$ a subsequence $<x_{1,j}>$ of $<x_j>$

Lt $<f_1 (x_1 y)>$ exists as $j \to \infty$.

$\Rightarrow <f_1 (x_{1,j}) >$ is convergent and hence bounded.

Again applying the above argument, $\exists$ a sub sequence $\langle x_{2,j} \rangle$ of $<x_{1,j}>$ such that $<f_2 (x_{2,j})>$ is covergent. This argument is repeated to form sequence $\langle x_{3,,j} \rangle$ $<x_{4,j}>$

.... which are successive subsequences of each other. Finally let $y_m › = x_{m,m}$ and

Let $<y_m>$ be the 'diagonol sequence' $\forall$ m $\geq$ 1

Now, $<y_m>$ is a subsequence of $<x_j>$

and Lt $f_n (y_m)$ exists as $m \to \infty$ for all n $\geq$1

Let Lt $f_n(y_m)$ = f where fϵ X'

For a given $\in$ > 0, we can write $\|f_n - f\| < \frac{\epsilon}{10B}$

i.e. $< f_n (y_m)>$ is convergent and hence a cauchy's sequence.

$\therefore$ sequence $< f_n (y_k) >_{k=1}^{\infty}$ is a cauchy's sequence.

$\therefore$ for a given $\in$ > 0 we can find k $\geq$1

$|| f_n (y_k) - f_n (y_{k'}) || < \frac{\epsilon}{10} \forall k, k' \geq K.$

Consider,

$| f(y_k) - f (y_k') = | f(y_k) - f_n(y_k) + f_n (y_k) - f_n(y_k') + f_n (y_k') - f(y_k') |$

$= | (f-f_n) (y_k) + (f_n(y_k) - f_n (y_k')) + (f_n-f) (y_k') |$

$\leq | (f-f_n) (y_k) | + | f_n(y_k) - f_n (y_k') | + | (f_n-f) (y_k')|$

$< \|f-f_n\| \|y_k\| + \frac{\epsilon}{10} + \|f - f_n\| \|y_k'\|$

$< \frac{\epsilon}{10B} B + \frac{\epsilon}{10} + \frac{\epsilon}{10B} B < \in$

$\Rightarrow | f (y_k) - f (y_k') | < \in \forall k, k' \geq K$

$\Rightarrow < f(y_k) >$ is a cauchy's sequence.

Limit exists and suppose Lt $f(y_k) = g(f)$ as $k \to \infty$

Then g is a map s.t $g: X' \to K$ is Linear map

and $|g(f)| = \text{Lt} \, |f(y_k)| \leq \text{Lt Sup} \, \|f\| \, \|y_k\| \leq B. \, \|f\|$

i.e. $|g(f)| \leq B \, \|f\| \Rightarrow$ g is bounded linear functional , $g \in X''$ and $\|g\| \leq B$.

Now since X is reflexive, $\exists \, x \in X$ s.t $g = g_X$

where $g_x(f) = f(x) \; \forall \, f \in X'$

Hence $f(x) = g_x(f) = g(f) =$ Lt $f(y_k)$ as $k \to \infty$

i.e Lt $f(y_k) = f(x)$ as $k \to \infty$

ic $\exists$ a subsequence $< y_k >$ of sequence $< x_n >$ in X which is weakly convergent.

Lemma 2: If $T : H \to H$ be compact and self adjoint operator on Hilbert Space H. Then T has an eigen vector with eigen value $\|T\|$ or $-\|T\|$

Proof : Let us suppose $\lambda = \|T\|$

Let $\lambda > 0$, Because if $\|T\| = \lambda = 0 \Rightarrow T = 0$

So, we suppose any $Y \varepsilon H - \{0\}$ is an eigen vector

By Lemma 2, there is a point $x_0 \varepsilon S_1$ s.t $\|Tx_0\| = \lambda$

Now , because T is self adjont, we have

$< T^2 x_0, x_0 > = < Tx_0, T^* x_0 > = < Tx_0, Tx_0 > = \|Tx_0\|^2$ ---- **

Again, By Cauchy's Schwarz Inequality

$|<T^2x_0, x_0>| \leq \|T^2 x_0\| \| x_0 \leq \| T^2 \|.1 \leq \| T \|^2 = \lambda^2$--- ***

Combining ** and ***, There should be equality in ***. Because equality in Cauchy's schwarz is possible only if two vectors are linearly dependent .

$\therefore$ T^2x_0 and x_0 are dependent and Let $T^2x_0 = \alpha x_0$, $\alpha \in k$

Again,$<T^2x_0,x_0> =<\alpha x_0,x_0> = \alpha<x_0,x_0> =\alpha$,Here $x_0 \in S_1$........****

Combining *** and**** , $\alpha=\lambda^2$

$\therefore$ we have $T^2x_0 = \alpha x_0 = \lambda^2 x_0$

Adding $T\lambda x_0$ on both sides

$T^2x_0 + T\lambda x_0 = \lambda^2x_0 + T\lambda x_0$

$T(\lambda x_0+Tx_0) = \lambda(\lambda x_0+T\lambda x_0)$

Again consider $T^2x_0 = \lambda^2x_0 \Rightarrow - T^2x_0= -\lambda^2x_0$

Adding $T\lambda x_0$ on both sides, we get

$T(\lambda x_0- Tx_0) = -\lambda(\lambda x_0- Tx_0)$

We have $T(\lambda x_0 +Tx_0) = \lambda(\lambda x_0+Tx_0)$

and $T(\lambda x_0-Tx_0) = -\lambda(\lambda x_0-Tx_o)$

This implies if $\lambda x_o+Tx_o \neq 0$ then $\lambda x_0 + Tx_0$ is eigen vector of T with eigen value λ and $\lambda x_o-Tx_o \neq 0$ then λx_o-Tx_o is eigen vector of T with eigen value $-\lambda$

Sum of two vector = $2\lambda x_o \neq 0$

Hence, at least one of the vectors must be non zero.

∴ There is an eigen vector with eigen value λ or an eigen vector with eigen value – λ.

Lemma 3: Let $T : H \to H$ be a bounded self adjoint operator on a Hilbert space H. and Let $Y \subset H$, be a subspace

s.t $T(Y) \subset Y$. Then $T(Y^{\perp}) \subset Y^{\perp}$ and $T|_{Y^{\perp}} : Y^{\perp} \to Y^{\perp}$ is a bounded self adjont operator on Hilbert space $Y^{\perp}$ with norm

$\| T_{Y^{\perp}} \| \leq \| T \|$

Proof: Let $z \in Y^{\perp}$ and $y \in Y \Rightarrow T(y) \in T(Y) \subset Y$

∴ $z \in Y^{\perp}$ and $T(y) \in Y$

Consider $<Tz,y> = <z,T^{*}y> = <z, Ty> = 0$

because $Z \in Y^{\perp}$ and $Ty \in Y$

$\Rightarrow$ Tz and y are orthogonal ie $Tz \perp y \in Y \Rightarrow Tz \in Y^{\perp}$ (*)

Now, we have to prove $T(Y^{\perp}) \subset Y^{\perp}$

Because, $Z \in Y^{\perp}$, it implies $T(z) \in T(Y^{\perp})$

But $Tz \in Y^{\perp}$ (By *)

∴ $T(Y^{\perp}) \subset Y^{\perp}$ Remaining results are obvious

Proof of the Theorem

In this section , we start with finding eigen vectors and then restrict the attention to the orthogonal complement of this set of vectors and then prove the required result as given here. We have taken the analogy between the spectral theory of operators on Hilbert spaces and that of operators on finite dimensional spaces about as far as it will go without requiring serious modifications.

Firstly, Let us suppose T = 0, then the theorem is trivial.

So, suppose $T \neq 0$, Then by Lemma 3, there is an Eigen vector e_1 with Eigen value such that

$\lambda_1 = \| T \|$ or $\lambda_1 = - \| T \|$. Since $T \neq 0$, we have $\lambda_1 \neq 0$

Let us assume $\| e_1 \| = 1$

Let $H_1 = \text{span } \{e_1\}^\perp$ Then by Lemma 4, the restriction map $T_{|H1}$ is a self adjoint operator on H_1 and $\| T_{|H1|} \| \leq \| T \| = \lambda$

If $T_{|H1} = 0$, then we stop here. If not, we repeat the above process. After n-1 steps, we found an orthonormal sequence of Eigen vectors $e_1, e_2, \ldots\ldots e_{n-1}$ in H with corresponding real Eigen values $\lambda_1, \lambda_2 \ldots. \lambda_{n-1}$ s .t.

$$| \lambda_1 | \geq | \lambda_2 | \geq \ldots\ldots\ldots\ldots\ldots\ldots \geq | \lambda_{n-1} | > 0$$

Let $H_{n-1} = \text{span } \{e_1, e_2 \ldots. e_{n-1}\}^\perp$ then $T_{|H\, n-1}$ is a self adjoint operator defined on H_{n-1} and $\| T_{IH\, n-1} \| \leq |\lambda_{n-1}|$

Now if $T_{|Hn-1} = 0$, then we stop after this step.

and if $T|_{H_{n-1}} \neq 0$, we continue to nth step and again apply Lemma 3, we find an Eigen vector $e_n \in H_{n-1}$ with eigen value $\lambda_n = \| T|_{H_{n-1}} \|$

or $\lambda_n = - \| T|_{H_{n-1}} \|$. Here, we have $| \lambda_n | \leq | \lambda_{n-1} |$. Let us assume $\| e_n \| = 1$

Because $e_n \in H_{n-1} = \text{span} \{ e_1, e_2 \ldots e_{n-1} \}^{\perp}$ We have

$< e_n, e_k > = 0$ for $k = 1, 2, .. n -1$ i.e. $e_1, e_2, \ldots e_n$ is an orthonormal sequence.

Let $H_n = \text{span} \{e_1, e_2 \ldots e_n\}^{\perp}$. Then again by lemma 4, the restriction map $T|_{H_n}$ is self adjoint operator on H_n

since, $H_n \subset H_{n-1}$, We have $\| T|_{H_n} \| \leq \| T|_{.H_{n-1}} \| = | \lambda_n |$. which is the same situation as above.

Now, if the process stops after step N, we have $T|_{H_N} = 0$

This implies, $\text{span} (\{e_n\}^N_{n=1}) = H_N \subset N(T)$ ……………..(a)

Again if $x \in N(T) \Rightarrow Tx = 0$

for Each n, $\lambda_n <e_n, x> = < \lambda_n e_n, x> = <Te_n, x> = <e_n, Tx>$

$= <e_n, o> = 0$

i.e. $\lambda_n < e_n, x> = 0 \quad \Rightarrow x \perp e_n$

$\Rightarrow x \in \text{span} (\{e_n\}^N_{n=1})^{\perp}$ ……………………………..(b)

combining (a) and (b), $N(T) = \text{span} (\{e_n\}^N_{n=1})^{\perp}$

Now, if the process never steps, we obtain an infinite sequence $\{\lambda_n\}$ of real non –zero eigen values with

$|\lambda_1| \geq |\lambda_2| \geq$.... and a corresponding orthonormal sequence $\{e_n\}^{\infty}_{n=1}$ of eigen vectors s.t. $Te_n = \lambda_n e_n \; \forall \; n \geq 1$.

Now, because T is compact and $\{e_n\}$ bounded and hence bounded sequence has a convergent subsequence. ie $\exists$ a subsequence $1 \leq n_1 < n_2 <$ .... S.t $<Te_{nj}>$ converges in H as $j \rightarrow \infty$.

$\| Te_{nj} - Te_{nj'} \| \rightarrow 0$ as $j, j' \rightarrow \infty$ .... (1)

But $Te_{nj} = \lambda_{nj} e_{nj}$ and these vectors are mutually orthogonal for distinct j's. Hence by Pythagoras formula,. we have $\forall$ j< j'

$\| Te_{nj} - Te_{nj} \|^2 = \| \lambda_{nj} e_{nj} \|^2 + \| \lambda_{nj} e_{nj'} \|^2$

$= \| \lambda_{nj} \|^2 + \| \lambda_{nj'} \|^2$...(2)

Combining (1) and (2) Lt $|\lambda_{nj}|^2 = 0$ as $j \rightarrow \infty$,

since $|\lambda_1| \geq |\lambda_2| >$ ....

$\therefore$ full sequence $< \lambda_n >$ converges to '0' which is third condition of the theorem.

Now, we are left to prove 2nd condition i.e.

$N(T) = \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$, Firstly, Let $x \in N(T) \Rightarrow Tx = 0$

Now $\lambda_n < e_n, x > = <\lambda_n e_n, x> = <Te_n, x> = < e_n, Tx > = 0$

$\Rightarrow x \perp e_n \Rightarrow x \in \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$

$\therefore N(T) \subset \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$ .... (3)

Conversely Let $x \in \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$

then $x \in H_n \ \forall$ n and because $\| T_{|HN} \| \leq | \lambda_n |$

we have $\| Tx \| \leq |\lambda_n| \| x \|$ and Lt $\lambda_n = 0$ as $n \to \infty$

Hence $\| Tx \| = 0$ i.e. $x \in N(T)$

Hence span $(\{e_n\}^{\infty}_{n=1})^{\perp} = N(T)$ ……………………..(4)

Combining (3) and (4),we get

$N(T) = \text{span}\,(\{e_n\}^{\infty}_{n=1})^{\perp}$. i.e., condition is fulfilled.

CONCLUSION

we basically discussed the spectral representation of compact self adjoint operators. The significance and usefulness of this result lies in the fact that we can represent T (ν),$\nu \in$ H in a simple and unique form. Sometimes it is possible and convenient to break up a vector space into special disjoint subspaces. So, we are writing H as, $H = Y + Y^{\perp}$, where we assume $Y = \text{span}\,(\{e_n\}^{N}_{n=1})$

We then proved kernel of T as orthogonal of the set which we proved as combination of the orthonormal elements, that is, we presented Hilbert space as

$$H = Y + N(T)$$

Motivated for the application of this theorem, we here tried to write $v \in H$ in the form $\nu = (\sum \alpha_n e_n) + Z$,

Where $\alpha_n \in k$, $Z \in N(T)$ and for each such vector ν, we have

$$T(\nu) = \sum \lambda_n \alpha_n e_n$$

which is required representation and this presentation is unique.

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