DYNAMICS OF CHARGED PARTICLE IN ROTATING ELECTRIC QUADRUPOLE POTENTIAL

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Abstract: In this paper we present charged particle motion in rotating quadrupole field. Quadrupole field configuration is used in many practical systems, e.g., in charged particle accelerator components for focusing of charged particle beam or in ion traps. Rotation of quadrupole field with time introduces a coupling of transverse dynamics. Study of such dynamical system throws light in to the details.

Index Terms **–Rotating Quadrupole Potential, Brouwer's Equation.**

I. INTRODUCTION

Qudrupole configuration of electric field or magnetic field is used in many applications, particularly in the field of charged particle accelerators or ion traps [1-5]. In charged particle beam transport systems, electrostatic of magnetic quadrupoles are used to keep the beam confined in transverse direction or to focus the beam. Inherently, a quadrupole field configuration cause focusing in one transverse plane and defocusing in the other transverse plane. A sequence of focusing and defocusing quadrupole field causes effective focusing of beam in both the transverse planes. Strong focusing, i.e., alternating focusing and defocusing fields, gains new properties suitable for beam line application by exploring field rotations about the optic axis. Literatures are found on electric or magnetic quadrupoles rotating helically in space around the optic axis [6, 11]. Field rotating in time, rather than space, has also been discussed and it has been suggested that such strong focusing method can effectively be explored in beam line applications, particularly in radiofrequency quadrupole accelerator system [12].

II. **Rotating Quadrupole potential**

Keeping in mind the particular application of charged particle beam transport system we use a coordinate system with a preferential longitudinal axis (z) along the beam transport line. We use cylindrical coordinate system (ρ, φ, z) to evaluate the scalar potential ψ fromd which the electric field can be derived by taking space gradient. In the limit where transverse dimension of the device is much less than the longitudinal dimension, as it is generally in case of a beam transport device like electrostatic quadrupoles, the scalar potential follows the two dimensional Laplace equation in transverse (x, y) plane

$$\nabla^2 \psi = 0 \qquad (1)$$

The solution of this equation is given by

$$\psi(\rho, \varphi) = \sum_{n=1}^{\infty} a_n \rho^n \cos(n\varphi) + b_n \rho^n \sin(n\varphi) \qquad (2)$$

Electric field is given by the gradient of scalar potential,

$$E = -\nabla\psi \qquad (3)$$

Here different values of n represents different field forms, e.g., $n = 1$ *and* 2 represent dipole and quadrupole field forms respectively. The quadrupole potential is given by

$$\psi_2(\rho, \varphi) = a_2 \rho^2 \cos(2\varphi) + b_2 \rho^2 \sin(2\varphi)$$

It is rather easier to visualize the motion in Cartesian coordinate system,

$$\psi_2(x, y) = a_2(x^2 - y^2) + 2b_2 xy \qquad (4)$$

$$\vec{E}(x, y) = -\frac{\partial \psi_2}{\partial x}\hat{\imath} - \frac{\partial \psi_2}{\partial y}\hat{\jmath} = (-2a_2 x\,\hat{\imath} + 2a_2 y\,\hat{\jmath}) + (-2b_2 y\,\hat{\imath} + 2b_2 x\,\hat{\jmath}) \qquad (5)$$

We consider the motion of a particle with charge $+q$ and mass m, moving along z-axis with velocity $\vec{v}$ under the influence of this quadrupole electric field. The electric force, given by $q\vec{E}$, for $a_2 \neq 0$ term is having a component along $-\hat{\imath}$ direction that is proportional to x and a component along $\hat{\jmath}$ direction that is proportional to y. This is the *normal quadrupole field*, giving focusing force in x-plane and defocusing force in y-plane. The electric force for $b_2 \neq 0$ term is having a component along $-\hat{\imath}$ direction that is proportional to y and a component along $\hat{\jmath}$ direction that is proportional to x. This force gives rise to coupling of x and y motion. It is obvious that the second term in equation (4), when transformed in a coordinate (x', y') rotated in transverse plane by angle θ, is given by

$$2b_2 xy = b_2\left(x'^2 - y'^2\right)\sin 2\theta - x'y'\cos 2\theta \qquad (6)$$

In case of $\theta = \pi/4$, $2b_2xy = b_2\left(x'^2 - y'^2\right)$ is the same form as the normal quadrupole potential. Therefore, $b_2 \neq 0$ term of field is called *skew quadrupole field*.

Generally, in static quadrupole devices, one of these field configurations is devised in a particular unit, say normal configuration with focusing in x-plane. By changing the polarity of the poles, the unit can act as defocusing in x-plane. A sequence of focusing and defocusing units give a total result of focusing in both planes without coupling the motion of transverse planes.

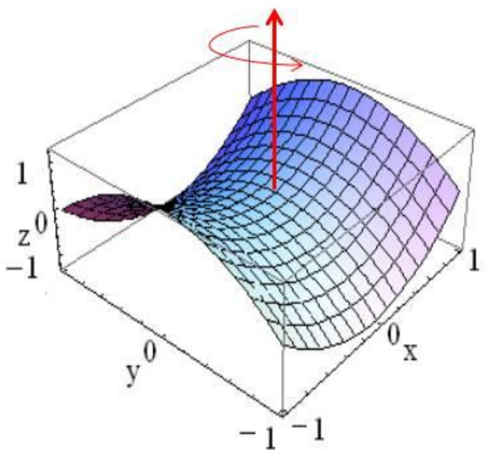


Figure 1: Electrostatic quadrupole potential $a_2(x^2 - y^2)$. The potential is to be rotated around z-asis to get a rotating quadrupole field.

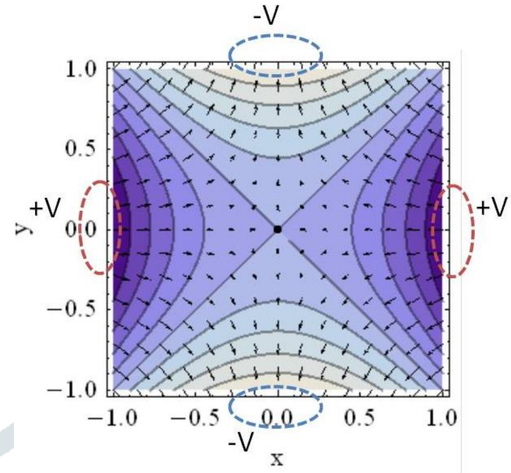


Figure 2. Equipotential contours and electric field of a *normal-quadrupole* configuration. The dotted figures schematically represent the poles.

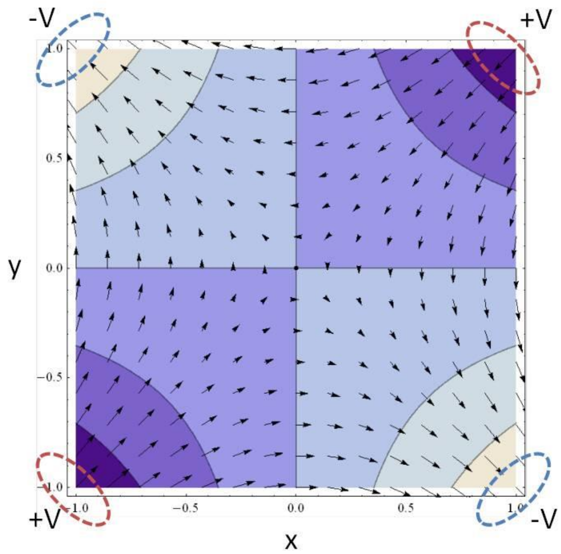


Figure 3: Equipotential contours and electric field of a *skew-quadrupole* configuration. The dotted figures schematically shows the approximate poles.

Now, in this paper, we consider the case where the quadrupole field configuration rotates with time in transverse plane. That means the scalar potential function is to be rotated around the z-axis. Figure 1 shows the quadrupole potential $a_2(x^2 - y^2)$, is to be rotated around z-asis to get a rotating quadrupole field. Figure 2 shows the equipotential lines and electric field configuration of ideal normal quadupole configuration. Ideally, the profile of the poles needs to be hyperbolic, following the equipotential contours. But practically metallic poles of finite transverse dimensions can be approximated to generate dominant quadrupole potential in the *good-field-region* near the z-axis where the charge particle beam travels. In figure 2, the poles are shown schematically by dotted shapes. The poles at $\theta = 0°$ and $180°$ are at $+V$ potential and the poles at $\theta = 90°$ and $270°$ are at $-V$ potential. The skew configuration $2b_2xy$ can be obtained just by rotating the poles by $\theta = 45°$, as shown in figure 3.

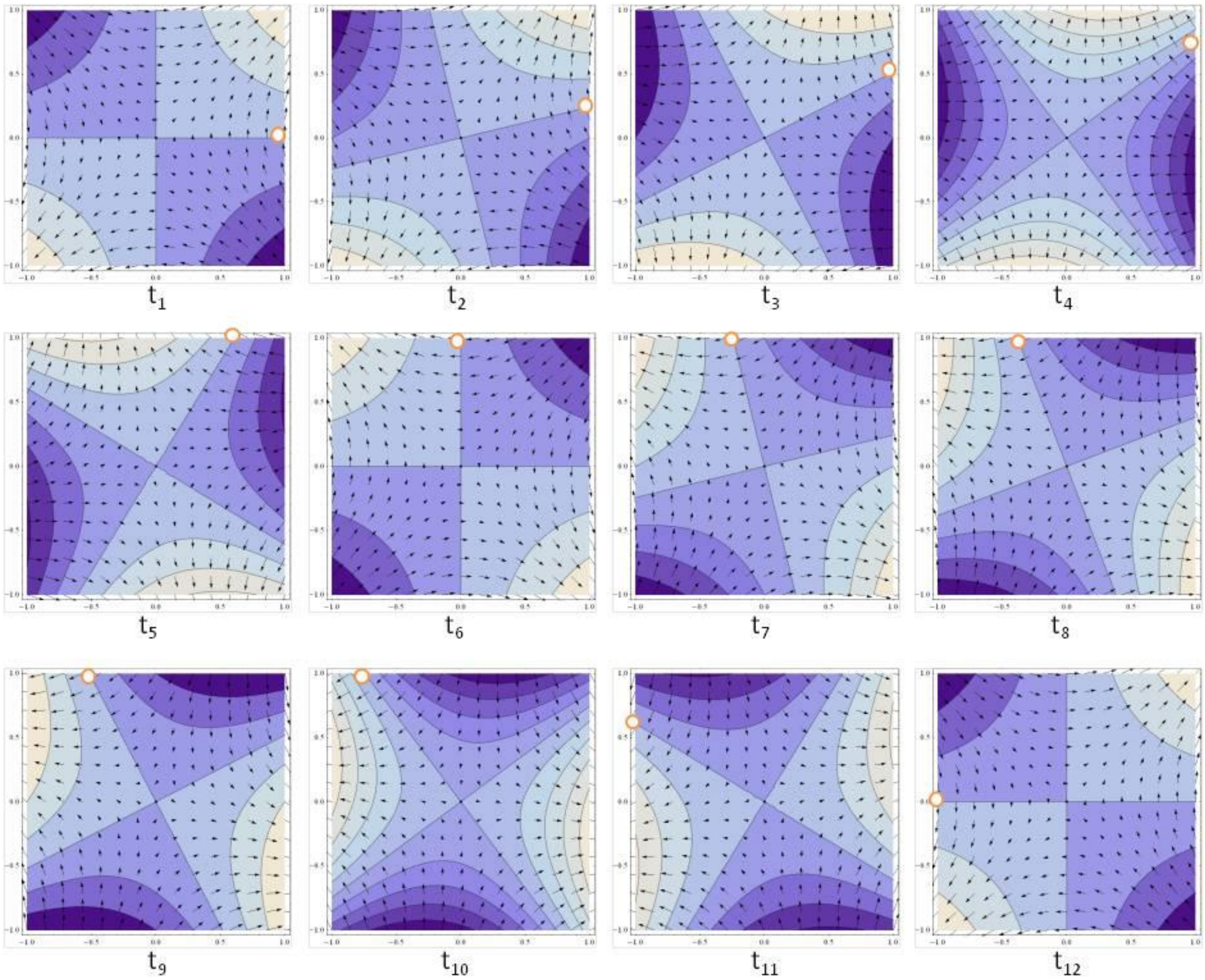


Figure 4: Potential and field pattern at different instance t_1, t_2, t_3, ..., t_{12}, where $0 < \omega t_n < \pi$.

A quadrupole potential rotating with time in transverse plane can be generated with a time varying *normal* and *skew* configuration superimposed on each other. The time varying potential function can be written as,

$$\psi(x,y,t) = a_2 \cos(\omega t)\ (x^2 - y^2) + 2b_2 \cos(\omega t + \varphi_0)\ xy \qquad (7)$$

Here time varying sinusoidal potential with angular frequency $\omega = 2\pi/T$, T being the time period, has been applied to the poles. The sinusoidal potentials applied to *normal* and *skew* sets of poles have a phase difference of φ_0. Figure 4 shows the potential and field pattern at different instances of time $0 < \omega t_n < \pi$. In these plots we have considered $a_2 = 1, 2b_2 = 1$ and $\varphi_0 = \pi/2$.

III. Dynamics and Stability of Motion

The potential and the field, for $\varphi_0 = \pi/2$, are as follows respectively,

$$\psi(x,y,t) = a_2 \cos(\omega t)\ (x^2 - y^2) + 2b_2 \sin(\omega t)\ xy$$

$$\vec{E}(x,y,t) = -(2a_2 x\ \cos(\omega t) + 2b_2 y\ \sin(\omega t))\hat{\imath} + (-2b_2 x\ \sin(\omega t) + 2a_2 y\ \cos(\omega t))\hat{\jmath}$$

Force on a charged particle $\vec{F} = q\vec{E}(x,y,t)$. Hence the equations of motion are as follows:

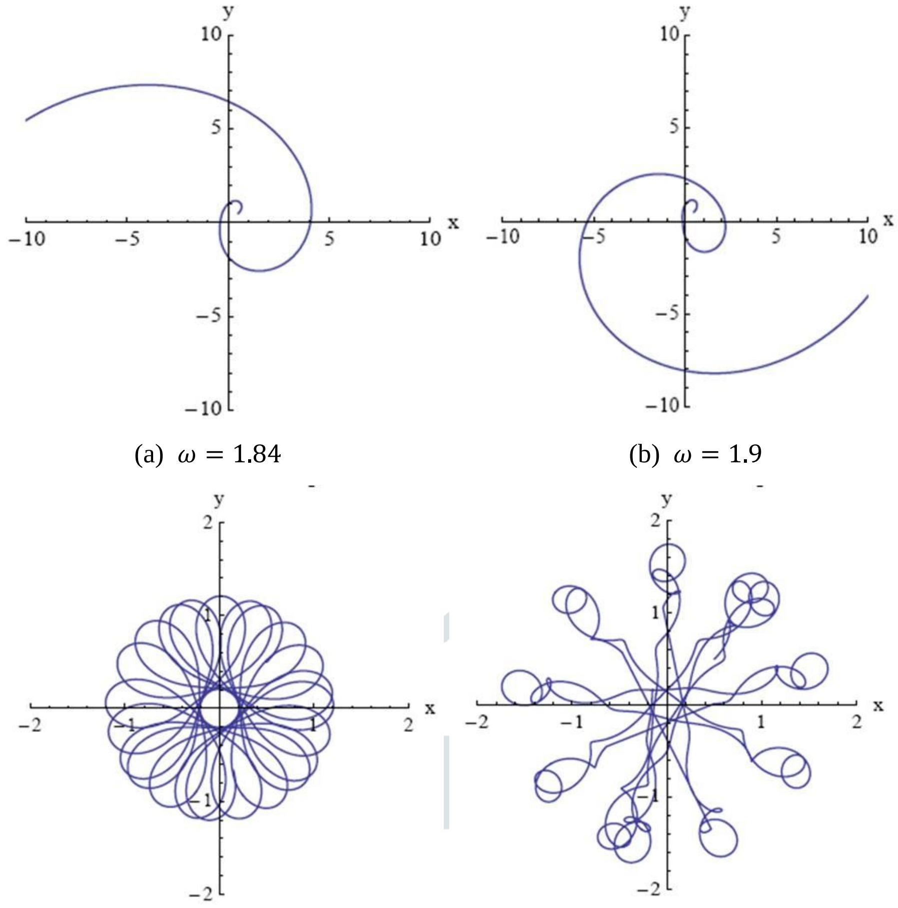
$$\ddot{x} = -\frac{q}{\gamma m_0}(2a_2 x\ \cos(\omega t) + 2b_2 y\ \sin(\omega t))$$

$$\ddot{y} = \frac{q}{\gamma m_0}(-2b_2 x\ \sin(\omega t) + 2a_2 y\ \cos(\omega t))$$

We may normalize the constants as $\frac{2a_2 q}{\gamma m_0} = \frac{2b_2 q}{\gamma m_0} = 1$,

$$\ddot{x} + x\ \cos(\omega t) + y\ \sin(\omega t) = 0 \qquad (8)$$

$$\ddot{y} + x\ \sin(\omega t) - y\ \cos(\omega t) = 0 \qquad (9)$$



(a) $\omega = 1.84$ (b) $\omega = 1.9$

(c) $\omega = 2$ (d) $\omega = 3$

Figure 5: Trajectories of charged particle under rotating quadrupole potential. In case of (a) and (b), the motion is unstable; where as in case of (c) and (d), the motion is stable or confined in the transverse direction. In case (d) the particle executes a small scale circular micro-motion and a large scale circular secular motion.

Equations (8) and (9) are similar to the well known equations derived by Brouwer (1881-1966) for motion of a particle under the influence of a rotating saddle potential, obtained by rotating the graph of $z = (x^2 - y^2)/2$ around z axis with angular velocity ω. The solutions of these equations in parametric form are shown in figure 5. The particle's motion is unstable for small values of < 2 , as shown in case (a) and (b). The motion is stable for all higher values as shown in case (c) and (d). It stable motion, the particle executes a prograde precession, i.e., it moves on a stretched curved trajectory that itself rotates in the same sense as the rotating field.

IV. CONCLUSION

A scheme for rotation of quadrupole potential is investigated mathematically. Transverse dynamics of a charged particle in the rotating quadurpole potential is investigated. The equations of motion are similar to the famous Brouwer's equations for a particle on a rotating slippery surface. Parametric plots of numerical solution of the equations show that the motion is confined in transverse direction for all higher rotational frequencies. For small frequencies the transverse motion is not stable. In case of stable motion the trajectory in transverse plane is a combination of circular micromotion and a prograde precession.

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