New Properties of Upper and Lower *z*-Perfectly Continuous Multifunctions

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Abstract

New properties and result of upper and lower *z*-perfectly continuous multifunction are formulated. The relationship with the Hausdroff space with the upper (lower) *z*-perfectly continuous multifunctions is discussed. The distinctiveness of upper (lower) *z*-perfectly continuous multifunction is illustrated through examples.

Keywords: strongly continuous multifunctions, upper (lower) completely continuous multifunctions, upper (lower) *D*-supercontinuous, upper (lower) quasi *z*-supercontinuous, upper (lower) *z*-supercontinuous, upper (lower) almost supercontinuous, upper (lower) almost cl-superfectly

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1 Introduction

The notion of perfectly continuous function was first introduced by Noiri [24] and its properties are further established in [16]. The class of strongly continuous function as defined by Levine [22] is contained in the class of function (class of cl-supercontinuous) as defined by R. Vamanamurthy [27].

In this paper, we further study the strong and weak variants of continuity of function and multifunctions (as defined in [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [20], [23], [24], [25], [27], [30]).

This paper extends the concept of upper (lower) *z*-perfectly continuous multifunctions. the preliminaries and basic definations are discussed in Section 2.

The concept of upper (lower) *z*-perfectly continuous multifunctions is stated in Section 3. In Section 4 the properties of upper *z*-perfectly continuous multifunctions are extended to the union of multifunctions, restriction to a subspace and passage to the graph multifunctions. Section 5 deals with the properties and result of lower *z*-perfectly continuous multifunction.

2 Preliminaries and Basic Definitions

Definition 2.1 ([14]). A multifunction $\phi : X \multimap Y$ from a topological space X into a topological space Y is said to be

(1) strongly continuous if $\phi^{-1}_{-}(B)$ is clopen in X for every subset $B \subset Y$.

(2) upper perfectly continuous if $\phi^{-1}_{-}(V)$ is clopen in X for every open set $V \subset Y$.

(3) lower perfectly continuous if $\phi^{-1}_{+}(V)$ is clopen in X for every open set $V \subset Y$.

(4) upper completely continuous if $\phi^{-1}_{-}(V)$ is regular open in X for every open set $V \subset Y$.

(5) lower completely continuous if $\phi^{-1}_{+}(V)$ is regular open in X for every open set $V \subset Y$.

Definition 2.2. A multifunction $\phi : X \multimap Y$ form a topological space X into a topological space Y is said to be

(1) upper z-supercontinuous [4] if for each $x \in X$ and each open set V containing $\phi(x)$, there exists a cozero set U containing x such that $\phi(U) \subset V$.

(2) lower z-supercontinuous [4] if for each $x \in X$ and each open set V with $\phi(x) \cap V \neq \varphi$, there exists a cozero set U containing x such that $\phi(z) \cap V \neq \varphi$ for each $z \in U$.

(3) upper D-suppercontinous [2] if for each $x \in X$ and each open set V containing $\phi(x)$, there exists an open F_σ-set U containing x such that $\phi(U) \subset V$.

(4) lower D-supercontinuous [2] if for each $x \in X$ and each open set V with $\phi(x) \cap V \neq \varphi$, there exists an open F_σ-set U containing x such that $\phi(z) \cap V \neq \varphi$ for each $z \in U$.

Definition 2.3. A multifunction $\phi : X \multimap Y$ from a topological space X into a topological space Y is said to be

(a) upper (lower) perfectly continuous (respectively almost perfectly continuous, respectively quasi perfectly continuous, respectively δ-perfectly continuous) [10] if $\phi^{-1}_{-}(U)$ ($\phi^{-1}_{+}(U)$) is clopen in X for every open (respectively regular open, respectively θ-open, respectively δ-open) subset U of Y.

(b) upper quasi z-supercontinuous ([13]) if for each $x \in X$ and each θ-open set V containing $\phi(x)$, there exists a cozero set (regular F_σ-set) U containing x such that $\phi(U) \subset V$.

(c) lower quasi z-supercontinuous (D_δ-supercontinuous) ([13]) if for each $x \in X$ and each θ-open set V with $\phi(x) \cap V \neq \varphi$, there exists a cozero set (regular F_σ-set) U containing x such that $\phi(z) \cap V \neq \varphi$ for each $z \in U$.

Definition 2.4. A multifunction $\phi : X \multimap Y$ from a topological space X into a topological space Y is said to be

(a) upper (almost) cl-supercontinuous ([9], [11], [15]) (respectively z- supercontinuous ([4], [17]), respectively D_δ-supercontinuous [5], respectively strongly θ-continuous [21]) at $x \in X$ if for each open (regular open) set V with $\phi(x) \subset V$, there exists a clopen set (respectively cozero set, respectively regular F_σ-set, respectively θ-open set) U containing x such that $\phi(U) \subset V$.

(b) lower (almost) cl-supercontinuous ([9], [11], [15]) (respectively z- supercontinuous ([4], [17]), respectively D_δ-supercontinuous [5], respectively strongly θ-continuous [21]) at $x \in X$ if for each open (regular open) set V with $\phi(x) \cap V \neq \varphi$, there exists a clopen set (respectively cozero set, respectively regular F_σ-set, respectively θ-open set) U containing x such that $\phi(z) \cap V \neq \varphi$; for each $z \in U$.

(c) upper supercontinuoue (δ-continuous) [1] if for each $x \in X$ and each open (regular open) set V containing $\phi(x)$, there exists a regular open set U containing x such that $\phi(U) \subset V$.

(d) lower supercontinuous (δ-continuous) [1], if for each $x \in X$ and each open (regular open) set V with $\phi(x) \cap V \neq \varphi$, there exists a regular open set U containing x such that $\phi(z) \cap V \neq \varphi$, for each $z \in U$.

Definition 2.5. A multifunction $\phi : X \multimap Y$ from a topological space X into a topological space Y is said to be

(i) upper (almost) completely continuous [12] if $\phi^{-1}(V)$ is regular open in X for every (regular) open set $V \subset Y$.

(ii) lower (almost) completely continuous [12] if $\phi^{-1}(V)$ is regular open in X for every (regular) open set $V \subset Y$.

(iii) upper (almost) perfectly continuous [12] if $\phi^{-1}(V)$ is clopen in X for every (regular) open subset V of Y.

(iv) lower (almost) perfectly continuous [12] if $\phi^{-1}(V)$ is clopen in X for every (regular) open subset V of Y.

(v) upper (almost) z-supercontinuous ([4], [13]) at $x \in X$ if for each open (regular open) set V with $\phi(x) \subset V$, there exists a cozero set U containing x such that $\phi(U) \subset V$.

(vi) lower (almost) z-supercontinous ([4], [13]) at $x \in X$ if for each open (regular open) set V with $\phi(x) \cap V \neq \varphi$, there exists a cozero set U containing x such that $\phi(z) \cap V \neq \varphi$ each $z \in U$.

(vii) upper supercontinuous (δ-continuous) [3] if for each $x \in X$ and each open (regular open) set V containing $\phi(x)$, there exists a regular open set U containing x such that $\phi(U) \subset V$.

(viii) lower supercontinuous (δ-continuous) [3] if for each $x \in X$ and each open (regular open) set V with $\phi(x) \cap V \neq \varphi$, there exists a regular open set U containing x such that $\phi(z) \cap V \neq \varphi$ for each $z \in U$.

(ix) upper almost cl-supercontinuous [11] if for each $x \in X$ and each regular open set V in Y containing $\phi(x)$ there exists a clopen set U in X containing x such that $\phi(U) \subset V$.

(x) lower almost cl-supercontinuous [11] if for each $x \in X$ and each regular open set V in Y with $\phi(x) \cap V \neq \varphi$, there exists a clopen set U in X containing x such that $\phi(x) \cap V \neq \varphi$ for each $x \in U$.

3 Upper and lower z-perfectly continuous

Definition 3.1 ([26]). A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is

(a) [26] upper z-perfectly continuous at $x \in X$ if for each cozero set V with $\phi(x) \subset V$, $\exists$ a clopen set U in X containing x such that $\phi(U) \subset V$.

The multifunction is said to be upper z-perfectly continuous if it is upper z- perfectly continuous at each $x \in X$.

(b) [26] lower z-perfectly continuous at $x \in X$ if for each cozero set V with $\phi(x) \cap V \neq \varphi$, there exists a clopen set U in X containing x such that $\phi(x) \cap V \neq \varphi$ for each $x \in U$.

The multifunction is said to be lower z-perfectly continuous if it is lowe z- perfectly continuous at each $x \in X$.

Examples

Example 3.1. Let $X = R$ set of real number with upper limit topology τ and let $Y = \mathrm{R}$ with usual topology U. Define a multifunction $\phi : (X, \tau) \rightarrow (Y, U)$ by $\phi(x) = x \ \forall \ x \in X$. Then clearly ϕ is upper (lower) cl-supercontinuous. But for $\phi^{-1}(A, b) = (A, b) = \phi^{-1}(A, b)$ is not clopen in X. So ϕ is not upper (lower) perfectly continuous.

Example 3.2. Let $X = \mathrm{R}$, set of real number with usual topology U and let Y be endowed with the topology $\tau = \{\varphi, \{1\}, R\}$. Then the identity mapping $\phi : (X, U) \rightarrow (Y, \tau)$ defined by $\phi(x) = \{x\} \ \forall x \in X$ is upper z-perfectly continuous but not continuous.

4 Properties of upper z-perfectly continuous multi- functions

Theorem 4.1. *Let $\phi : X \multimap Y$ be a multifunction and $\psi : X \multimap X \times Y$ defined by $\psi(x) = (x, \phi(x))$ for each $x \in X$, be the graph function. If ψ is upper z-perfectly continuous then so is ϕ.*

Proof. Suppose that the graph function $\psi : X \to X \times Y$ is z-perfectly continuous. Consider the projection map $p_y : X \times Y \to Y$.

Since it is continuous it is z-continuous, so upper z-continuous. Hence by [26], the function $\phi = p_y \circ \psi$ is upper z-perfectly continuous. □

Definition 4.2. The graph Γ_ϕ of multifunction $\phi : X \to Y$ is said to be clopen z- closed with respect to X if for each $(x, y) \notin \Gamma_\phi$ there exists a clopen set U containing x and a cozero set V containing y such that $(U \times V) \cap \Gamma_\phi = \varphi$.

Definition 4.3 ([19]). A topological space X is called functionally Hausdorff if any two distinct points of X can be separated by disjoint cozero sets, if every pair of distinct points, x and y in X, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Theorem 4.4. *If $\phi : X \multimap Y$ and $\psi : Y \multimap Z$ are two upper z-perfectly continuous multifunctions then $\phi \cup \psi : X \multimap Y$ defined by $(\phi \cup \psi)(x) = \phi(x) \cup \psi(x)$ for each $x \in X$, is upper z-perfectly continuous.*

Proof. Let U be a cozero set in Y. Since ϕ and ψ are upper z-perfectly continuous so $\phi^{-1}(U)$ and $\phi^{-1}(U)$ are clopen sets in X.

Since $(\phi \cup \psi)^{-1}(U) = \phi^{-1}(U) \cap \psi^{-1}(U)$ and since finite intersection of clopen sets is clopen, $(\phi \cup \psi)^{-1}(U)$ is clopen in X. Thus $\phi \cup \psi$ is upper z-perfectly continuous. □

In general intersection of two upper z-perfectly continuous multifunction, need not be upper z-perfectly continuous. However, in the forthcoming theorem we formulate a sufficient condition for the intersection of two multifunction to be upper z-perfectly continuous.

Theorem 4.5. *Let $\phi, \psi : X \multimap Y$ be upper z-perfectly continuous multifunction into a functionally Hausdorff space Y such that $\phi(x)$ is compact for each $x \in X$ and the graph Γ_ψ of ψ is clopen z-closed with respect to X. Then the multifunction $\phi \cap \psi$ defined by $(\phi \cap \psi)(x) = \phi(x) \cap \psi(x)$ for each $x \in X$ is upper z-perfectly continuous.*

Proof. Let $x_0 \in X$ and V be a cozero set containing $\phi(x_0) \cap \psi(x_0)$.

It is sufficient to find a clopen set U containing x_0 such that $(\phi \cap \psi)(U) \subset V$. If $V \supset \phi(x_0)$, then since ϕ is upper z-perfectly continuous, $\exists$ a clopen set U containing x_0 such that $\phi(U) \subset V$

If not then consider the set $K = \phi(x_0) \setminus V$ which is compact. Now for each $y \in K$, $y \in Y \setminus \psi(x_0)$. This implies that $(x_0, y) \in X \times Y \setminus \Gamma_\psi$. Since the graph of ψ is clopen z-closed with respect to X, these exists a clopen set U_y containing x_0 and a cozero set V_y containing y such that $\Gamma_y \cap (U_y \times V_y) = \varphi$. Therefore for each $x \in U_y$, $\psi(x) \cap V_y = \varphi$.

Since K is compact, there exist finitely many points $y_1, y_2, \ldots, y_n$ in K such that

$$K \subset \bigcup_{i=1}^{n} V_{y_i}.$$

Let $W = \bigcup_{i=1}^{n} V_{y_i}$ Then $V \cup W$ is an open set containing $\phi(x_0)$Since ϕ is upper z-perfectly continuous, there exist a clopen set U_0 containing x_0 such that $\phi(U_0) \subset V \cup W$. Let $U = U_0 \cap \left(\bigcap_{i=1}^{n} U_{y_i}\right)$. Then U is a clopen set containing x_0. Hence for each $z \in U$, $\phi(z) \subset V \cup W$ and $\psi(z) \cap W = \phi$.

Therefore $(\phi(z) \cap \psi(z)) \cap w = \phi$ for each $z \in U$.

This proves that $\phi \cap \psi$ is upper z-perfectly continuous at x_0. □

Corollary 4.6. *Let $\psi : X \multimap Y$ be a multifunction from a space X into a functionally compact Hausdorff space Y such that the graph Γ_ψ of ψ is clopen z-closed with respect X. Then ψ is upper z-perfectly continuous.*

Proof. Let the multifunction $\phi : X \multimap Y$ be defined by $\phi(x) = Y$ for each $x \in X$. Now applying the previous theorem we get the desired result. □

Theorem 4.7. *If $\phi : X \multimap Y$ is upper z-perfectly continuous multifunction and $\psi : Y \multimap Z$ is upper z-supercontinuous multifunction then their composition $\psi \circ \phi$ is upper cl-supercontinuous*

Proof. Let V be an open set in Z. Since ψ is upper z-supercontinuous, $\psi^{-1}(V)$ is z-open set in Y and so $\psi^{-1}(V) = \bigcup_{\alpha \in \Lambda} V_\alpha$, where each V_α is a cozero set.

Since ϕ is upper z-perfectly continuous, each $\phi^{-1}(V_\alpha)$ is a clopen set.

Hence $(\psi \circ \phi)^{-1}(V) = \varphi^{-1}(\psi^{-1}(V)) = \phi^{-1}\left(\bigcup_{\alpha \in \Lambda} V_\alpha\right) = \bigcup_{\alpha \in \Lambda} \phi^{-1}(V_\alpha)$ is cl-open. □

So $\psi \circ \phi$ is upper cl-supercontinuous.

5 Properties of lower z-perfectly continuous multi- function

Theorem 5.1. *Let $\phi : X \multimap Y$ be a multifunction and $\psi : X \multimap X \times Y$ define by $\psi(x) = x, \phi(x))$ for each $x \in X$, be the graph function. If ψ is lower z-perfectly continuous then so* *Proof.* Suppose that the graph function $\psi : X \multimap X \times Y$ is lower z-perfectly continuous. Consider the projection map $p_y : X \times Y \multimap Y$. Since it is continuous, it is z-continuous, so lower z-continuous. Hence by Theorem 5.3 the function $\phi = p_y \circ \psi$ is lower z-perfectly continuous. □

Theorem 5.2. *If $\phi : X \multimap Y$ and $\psi : X \multimap Y$ are lower z-perfectly continuous multifunction,* *then the multifunction* $\phi \cup \psi : X \multimap Y$ *defined by* $(\phi \cup \psi)(x) = \phi(x) \cup \psi(x)$ *for EACH* $x \in X$ *is lower z-perfectly continuous.*

Proof. Let U be a cozero set in Y. Since ϕ and ψ are lower z-perfectly continuous, So $(\phi^{-1}_{+}(U))$ and $\psi^{-1}_{+}(U)$ are clopen set in X. Since $(\phi \cup \psi)^{-1}_{+}(U) = \phi^{-1}_{+}(U) \cap \psi^{-1}_{+}(U)$ and since finite intersection of clopen sets is clopen, $(\phi \cup \psi)^{-1}_{+}(U)$ is clopen in X.
Thus $\phi \cup \psi$ is lower z-perfectly continuous. □

Theorem 5.3. *If* $\phi : X \multimap Y$ *is lower z-perfectly continuous multifunction AND* $\psi : Y \multimap Z$ *is lower z-supercontinuous multifunction then their composition* $\psi \circ \phi$ *is lower cl-supercontinuous*

Proof. Let V be an open set in Z. Since ψ is lower z-supercontinuous, $\psi^{-1}_{+}(V)$ is z-open set in Y and so $\psi^{-1}_{+}(V) = \bigcup_{\alpha \in \Lambda} V_\alpha$, where each V_α is a cozero set.

Since ϕ is lower z-perfectly continuous, each $\varphi^{-1}_{+}(V_\alpha)$ is clopen set.

Hence $(\psi \circ \phi)^{-1}_{+}(V) = \phi^{-1}_{+}(\phi^{-1}_{+}(V)) = \phi^{-1}_{+}\left(\bigcup_{\alpha \in \Lambda} V_\alpha\right) = \bigcup_{\alpha \in \Lambda} \phi^{-1}_{+}(V_\alpha)$ is cl-open. □

So $\psi \circ \phi$ is lower cl-supercontinuous.

Proposition 5.4. *If* $\phi : X \multimap Y$ *is lower z-perfectly continuous multifunction AND* $\psi : Y \multimap Z$ *is lower ALMOST z-supercontinuous, then their composition* $\psi \circ \phi$ *is lower ALMOST cl-supercontinuous.*

Proof. Let V be a regular open set of Z. In view of lower almost z-supercontinuity of ψ, $\psi^{-1}_{+}(V)$ is cozero in Y and since ϕ is lower z-perfectly continuous, therefore $\phi^{-1}_{+}(\psi^{-1}_{+}(V))$ is clopen in X. Now $(\psi \circ \phi)^{-1}_{+}(V) = \phi^{-1}_{+}(\psi^{-1}_{+}(V))$ is clopen in X.

So $\psi \circ \phi$ is lower almost cl-supercontinuous. □

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