ACYLIC IRREGULAR AND b-IRREGULAR COLOURING OF GRAPHS

[1]K. Anitha, [2]B. Selvam, [3]K. Thirusangu
[1]Department of Mathematics, Sri Sairam Engineering College, Chennai – 600 044, India
[2,3]Department of Mathematics, S.I.V.E.T. College, Gowrivakkam, Chennai – 600 073, India

Abstract : In this paper, we introduce two new irregular colourings namely acyclic irregular colouring and b-irregular colouring and investigate an acyclic irregular chromatic number of line and middle graph of star, bi-star and double star graph and b-irregular chromatic number of K-ary tree, (2,n)-barbell tree and friendship graph.

Index Terms- Acyclic, Irregular, colouring, b-chromatic, Graph.

I.Introduction

The concept of irregular colouring was studied by Avudainayaki, Selvam [1] and Burris [2]. The concept of acyclic colouring was introduced by B.Grunbaum [5] and studied about acyclic chromatic number by [3,4,6]. The b-chromatic number was introduced by Lrving and Manlove in [7]. In this paper, we introduce two new irregular colourings namely acyclic irregular colouring and b-irregular colouring and investigate an acyclic irregular chromatic number of line and middle graph of star, bi-star and double star graph and b- irregular chromatic number of K-ary tree, (2,n)-barbell tree and friendship graph.

II.Terminologies

Definition 2.1 Colouring A proper colouring of a graph G is a function c: V(G) → N having the property that c(u) ≠ c(v) for every pair u, v of adjacent vertices of G. A k-colouring of G uses k colours. The chromatic number $\chi(G)$ is the least positive integer k for which G admits k-colouring.

Definition 2.2 Irregular colouring
For a graph G, a colouring c: V(G)→ {1,2,3,…,k} of the vertices of G for some positive integer k, the colour code of a vertex v of G (with respect to c) is the ordered (k+1)-tuple code(v) = $(a_0, a_1, a_2, \ldots, a_k)$ where a_0 is the colour assigned to v and $1 \le i \le k$, a_i is the number of vertices of G adjacent to v that are coloured i. The colouring c is irregular if the different vertices have different colour codes and the irregular chromatic number is denoted by $\chi_{ir}(G)$

Definition 2.3 Acyclic colouring
A acyclic-colouring of a graph is colouring if every cycle uses at least three colours. The acyclic chromatic number of G is denoted by a(G).

Definition 2.4 b-colouring
The b-colouring of a graph G is a colouring of the vertices where each colour contains a vertex that has a neighbour in all other colour classes. The b-chromatic number $\chi_b(G)$ of a graph G is the largest integer that the graph has b-colouring with b number of colours.

Definition 2.5 Star graph
The complete bigraph $K_{1,n}$ is called star graph.

Definition 2.6 Bistar
The bistar graph $B_{m,n}$ is the graph obtained from K_2 by joining m pendent edges to one end and n pendent edges to the other end of K_2. Let u and v be the vertices of K_2 and $u_1,u_2,\ldots,u_m$ and $v_1,v_2,\ldots,v_n$ be the pendent vertices joined to u and v respectively.

Definition 2.7 Double star graph
Double star $K_{1,n,n}$ is obtained by $K_{1,n}$ by joining a new pendant edges of the existing n pendant vertices.

Definition 2.8 Line graph
In the line graph L(G), vertices are the edges of G and edges aretwovertices of L(G) adjacent whenever the corresponding edges of G are adjacent.

Definition 2.9 Middle graph

The vertex set of the middle graph M(G) is the union V(G) and E(G) and (i) x,y are in E(G) and x,y are adjacent in G, (ii) x is in V(G) and x,y ∈ E(G) are incident in G.

Definition 2.10 K-ary tree

A graph G is called an K-ary tree if G is a rooted tree such that the root has degree k and all the other vertices have degree k+1.

Definition 2.11 (2,n) -Barbell graph

The (2,n)-Barbell graph is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge and it is denoted by $B(K_n, K_n)$,n≥3.

Definition 2.12 Friendship graph

The friendship graph T_n is a set of n triangles having a common central vertex and $V(T_n)=\{v_1,v_2,...,v_{2n+1}\}$ with v_1 as the central vertex and $E(T_n)=\{ v_1v_i / 2\le i\le 2n+11\}U\{ v_{2i}v_{2i+1} / 1\le i\le n\}$ respectively.

III.Main Result

Definition 3.1 Acyclic irregular colouring

An irregular colouring of a graph G is an acyclic irregular colouring if every cycle uses at least three colours. The acyclic irregular chromatic number of G is denoted by $\chi_{air}(G)$.

Definition 3.2 b- irregular colouring

The b-irregular colouring of a graph G is an irregular colouring of the vertices where each colour contains a vertex that has a neighbour in all other colour classes. The b-irregular chromatic number $\chi_{bir}(G)$ of a graph G is the largest integer that the graph has b-irregular colouring with b number of colours.

Algorithm 3.1 We use the following algorithm to determine an acyclic irregular chromatic number of the line graph of star graph.

Procedure (Acyclic irregular colouring of $L(K_{1,n})$,n≥ 2)

Input: $V\leftarrow\{u_1,u_2,\dots,u_n\}$

$E\leftarrow\{ e_1,e_2,...,e_{n(n-1)/2}\}$

if n≥ 2

for i= 1 to n do

$u_i\leftarrow i$

end for

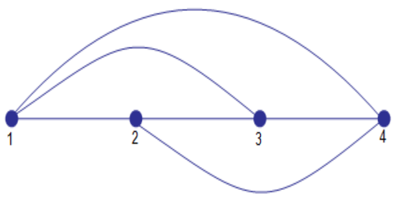
end if

end procedure

Theorem 3.1 For any star graph $K_{1,n}$, $\chi_{air}(L(K_{1,n}))=n$.

Proof- Let $L(K_{1,n})$ be the line graph of star graph. The vertex set and edge set of $L(K_{1,n})$ is as follows; $V(L(K_{1,n}))=\{u_i/ 1\le i\le n\}$, $E(L(K_{1,n}))=\{u_iu_{i+j}/1\le i\le n-1, 1\le j\le n-i\}$. In $L(K_{1,n})$, the vertices $u_1,u_2,\dots,u_n$ form a complete graph of order n. Thus we have $\chi_{air}(K_{1,n})\ge n$. Using algorithm 2.8, we coloured the vertices of $L(K_{1,n})$ by using n colours. In the way of minimum colouring, the line graph of star graph contains no bi-coloured cycle and no same colour codes. Hence $\chi_{air}(L(K_{1,n}))=n$.

Example: Acyclic irregular colouring of line graph of star



Algorithm 3.2

We use the following algorithm to determine an acyclic irregular chromatic number of the line graph of bi-star graph.

Procedure: (Acyclic irregular colouring of $L(B_{n,n})$,n≥ 2)

Input: $V\leftarrow\{v, u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n\}$

$E\leftarrow\{ e_1,e_2,...,e_{n^2+n}\}$

if n≥ 2

v←1

$v_n\leftarrow n+2$

for i= 1 to n do

$u_i \leftarrow i+1$
end for
for i= 1 to n -1do
$v_i \leftarrow i+1$
end for
end if
end procedure

Theorem 3.2 For any bi-star graph $B_{n,n}$, χ_{air} $(L(B_{n,n})=n+2$.

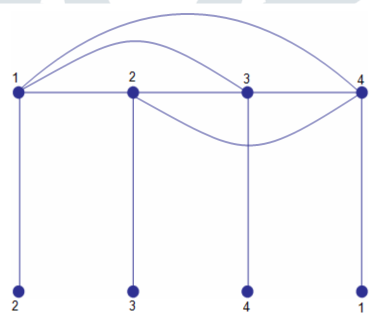
Proof- Let $L(B_{n,n})$ be the line graph of bi-star graph. The vertex set and edge set of $L(B_{n,n})$ is as follows; $V(L(B_{n,n}))=\{v,v_i,u_i/ 1\leq i\leq n\}$, $E(L(B_{n,n}))=\{ u_iu_{i+j}, v_iv_{i+j} / 1\leq i\leq n-1, 1\leq j\leq n-i\}\cup\{vu_i, vv_i /1\leq i\leq n\}$. In $L(B_{n,n})$, the vertices $v,u_1,u_2,\ldots,u_n$ form a complete graph of order n+1. Therefore we need at least (n+1) colours to colour the vertices of $L(B_{n,n})$. The following colouring for $L(B_{n,n})$ is acyclic irregular. For $1\leq i\leq n$, $c(v)=1$, $c(u_i)=i+1$; For $1\leq i\leq n-1$, $c(v_i)=i+1$; $c(vn)=n+2$. In the way of minimum colouring, the line graph of bi-star graph contains no bi-coloured cycle and no same colour codes. Hence χ_{air} $(L(B_{n,n})=n+2$.

Algorithm 3.3 We use the following algorithm to determine an acyclic irregular chromatic number of the line graph of double star graph.
Procedure: (Acyclic irregular colouring of $L(k_{1,n,n}),n\geq 2)$
Input: $V\leftarrow\{ u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$
$E\leftarrow\{ e_1,e_2,\ldots,e_{(n^2+ n)/2}\}$
if n≥ 2
for i= 1 to n do
$u_i \leftarrow i$
end for
for i= 1 to n -1do
$v_i \leftarrow i+1$
end for
$v_n \leftarrow 1$
end if
end procedure

Theorem 3.3 For any double star graph $k_{1,n,n}$, χ_{air} $(L(k_{1,n,n}))=n$.

Proof- Let $L(k_{1,n,n})$ be the line graph of double star graph. The vertex set and edge set of $L(k_{1,n,n})$ is as follows; $V(L(k_{1,n,n}))=\{v_i,u_i/ 1\leq i\leq n\}$, $E(L(k_{1,n,n}))=\{ u_iu_{i+j}/ 1\leq i\leq n-1, 1\leq j\leq n-i\}\cup\{v_iu_i/1\leq i\leq n\}$. In $L(k_{1,n,n})$, the vertices $u_1,u_2,\ldots,u_n$ form a complete graph of order n. Therefore χ_{air} $(L(k_{1,n,n}))\geq n$. The acyclic irregular colouring of $L(k_{1,n,n})$ is as follows; For $1\leq i\leq n$, $c(u_i)=i$; For $1\leq i\leq n-1$, $c(v_i)=i+1$, $c(v_n)=1$. In the way of minimum coloring, the line graph of double star graph contains no bi-colored cycle and no same color codes Hence χ_{air} $(L(k_{1,n,n})=n$.

Example: Acyclic irregular colouring of line graph of double star



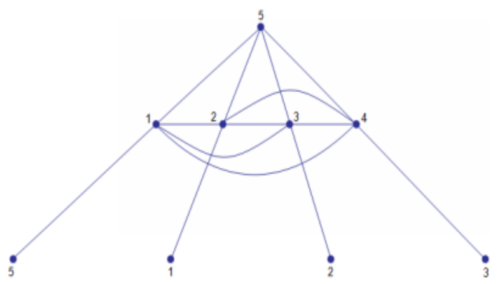
Algorithm 3.4
We use the following algorithm to determine an acyclic irregular chromatic number of the middle graph of star graph.
Procedure: (Acyclic irregular colouring of $M(K_{1,n}),n\geq 2)$
Input: $V\leftarrow\{v,u_1,u_2,\ldots,u_n,w_1,w_2,\ldots,w_n\}$
$E\leftarrow\{ e_1,e_2,\ldots,e_{(n^2+3 n)/2}\}$
if n≥ 2
for i= 1 to n do
$w_i \leftarrow i$
end for
$v \leftarrow n+1$

$u_1 \leftarrow n+1$
for i= 1 to n-1 do
$u_i \leftarrow i$
end for
end if
end procedure

Theorem 3.4 For any star graph $K_{1,n}$, $\chi_{air}(M(K_{1,n}))=n+1$.

Proof- Let $M(K_{1,n})$ be the middle graph of star graph. The vertex set and edge set of $M(K_{1,n})$ is as follows; $V(M(K_{1,n}))=\{v,u_i,w_i /1\leq i\leq n\}$, $E(M(K_{1,n}))=\{u_i/1\leq i\leq n\}\cup\{w_i\ w_{i+j}/1\leq i\leq n-1,\ 1\leq j\leq n-i\}$. In $M(K_{1,n})$, the vertices $v,w_1,w_2,\ldots,w_n$ form a complete graph of order n+1. Thus we have $\chi_{air}(M(K_{1,n}))\geq n+1$. In $M(K_{1,n})$, $deg(w_i)\neq deg(u_i)$, it follows that $code(w_i)\neq code(u_i)$. So $\chi_{air}(M(K_{1,n}))\leq n+1$. Hence $\chi_{air}(M(K_{1,n}))=n+1$.

Example: Acyclic irregular colouring of middle graph of star

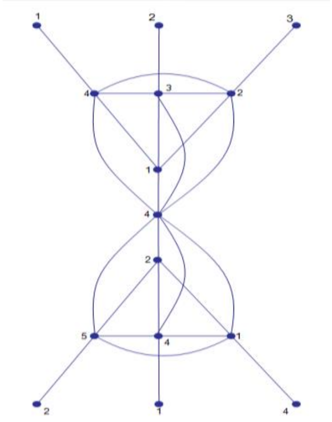


Algorithm 3.5
We use the following algorithm to determine an acyclic irregular chromatic number of the middle graph of bi-star graph.
Procedure (Acyclic irregular colouring of $M(B_{n,n})$,$n\geq 2$)
Input: $V\leftarrow\{u,v,w,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,s_1,s_2,\ldots,s_n,t_1,t_2,\ldots,t_n\}$
$E\leftarrow\{e_1,e_2,\ldots,e_{n^2+5n+2}\}$
if $n\geq 2$
$u\leftarrow 1$
$v\leftarrow 2$
$s_1\leftarrow 1$
$v_1\leftarrow n+2$
$w\leftarrow n+2$
for i= 1 to n do
$t_i\leftarrow i+1$
$u_i\leftarrow i$
end for
for i= 2 to n do
$s_i\leftarrow i+2$
end for
for i= 1to n-1 do
$v_{i+1}\leftarrow i$
end for
end if
end procedure

Theorem 3.5 For any bi-star graph $B_{n,n}$, $\chi_{air}(M(B_{n,n}))=n+2$.

Proof- Let $M(B_{n,n})$ be the middle graph of bi-star graph. The vertex set and edge set of $M(B_{n,n})$ is as follows; $V(M(B_{n,n}))=\{u,v,w,v_i,u_i,t_i,s_i/1\leq i\leq n\}$, $E(M(B_{n,n}))=\{t_it_{i+j},\ s_is_{i+j}\ /\ 1\leq i\leq n-1,\ 1\leq j\leq n-i\}\cup\{uw,\ vw\}\cup\{ws_i,wt_i,ut_i,vs_i,v_is_i,u_it_i\ /1\leq i\leq n\}$. In $M(B_{n,n})$, the vertices $u,w,t_1,t_2,\ldots,t_n$ form a complete graph of order n+2. Therefore we need atleast (n+2) colours to colour the vertices of $M(B_{n,n})$. The following colouring for $M(B_{n,n})$ is acyclic irregular. For $1\leq i\leq n$, $c(u)=1$, $c(v)=2$, $c(s_1)=1$, $c(v_1)=n+2$, $c(w)=n+2$, $c(u_i)=i$, $c(t_i)=i+1$; For $2\leq i\leq n$, $c(s_i)=2+i$; For $1\leq i\leq n-1$, $c(v_{i+1})=i$. In the way of minimum coloring, the middle graph of bi-star graph contains no bi-colored cycle and no same color codes. Hence $\chi_{air}(M(B_{n,n}))=n+2$.

Example: Acyclic irregular colouring of Middle graph of Bi-star



Algorithm 3.6 We use the following algorithm to determine an acyclic irregular chromatic number of the middle graph of double star graph.

Procedure: (Acyclic irregular colouring of $M(k_{1,n,n}),n\geq 2$)

Input: $V\leftarrow\{u,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n, s_1,s_2,\ldots,s_n, t_1,t_2,\ldots,t_n\}$

$E\leftarrow\{e_1,e_2,\ldots,e_{(n^2+9n)/2}\}$

if n≥ 2

for i= 1 to n do

$v_i\leftarrow i$

$s_i\leftarrow i$

$u_i\leftarrow n+1$

end for

for i= 1 to n -1do

$t_i\leftarrow i+1$

end for

$t_n\leftarrow 1$

$u\leftarrow n+1$

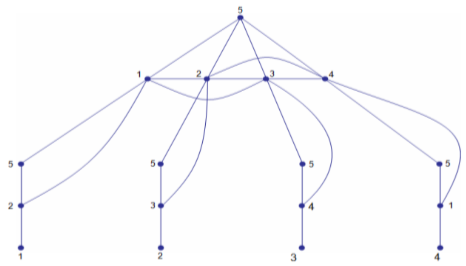
end if

end procedure

Theorem 3.6 For any double star graph $k_{1,n,n}$, $\chi_{air}(L(k_{1,n,n}))=n+1$.

Proof- Let $M(k_{1,n,n})$ be the middle graph of double star graph. The vertex set and edge set of $M(k_{1,n,n})$ is as follows; $V(M(k_{1,n,n}))=\{u,v_i,u_i,s_i,t_i /1\leq i\leq n\}$, $E(M(k_{1,n,n}))=\{ us_i, u_is_i, u_it_i, v_it_i\ s_it_i / 1\leq i\leq n\}\cup\{s_is_{i+j} /1\leq i\leq n-1, 1\leq j\leq n-i\}$. In $M(k_{1,n,n})$, the vertices $u,s_1,s_2,\ldots,s_n$ form a complete graph of order n+1. Therefore $\chi_{air}(M(k_{1,n,n}))\geq n+1$. The acyclic irregular colouring of $M(k_{1,n,n})$ is as follows; For $1\leq i\leq n$, $c(v_i)=c(s_i)=i$, $c(u_i)=n+1$;For $1\leq i\leq n-1$, $c(t_i)=i+1$, $c(t_n)=1$, $c(u)=n+1$. In the way of minimum colouring, the line graph of double star graph contains no bi-coloured cycle and no same colour codes. Hence $\chi_{air}(M(k_{1,n,n}))=n+1$.

Example: Acyclic irregular colouring of Middle graph of double star



Algorithm 3.7

We use the following algorithm to determine b-irregular chromatic number of k-ary tree graph, m≥3.

$V\leftarrow\{v,v_1,v_2,\ldots,v_k, u_1,u_2,\ldots,u_{k^2}\}$

$E\leftarrow\{e_1,e_2,\ldots,e_{k^2+k}\}$

if k≥ 3

$v\leftarrow k+1$

for i=1to k do

$v_i\leftarrow i$

$u_i\leftarrow i+1$

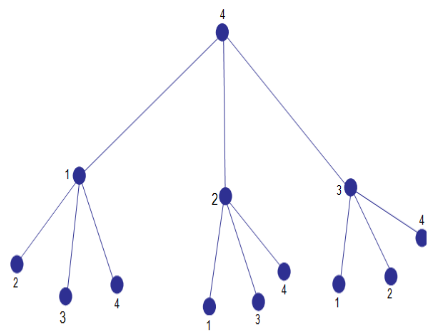
end for

for i=1to k-1do
 for j=1to i do
u_{ki+j} ← j
end for
end for
for i= 2 to k do
 for j=0 to k-i do
u_{ki-j} ← $k + 1 - $ j
end for
end for
end if
end procedure

Theorem 3.7 For k≥3, $\chi_{bir}($ k-ary tree)=k+1.

Proof- From the structure of k-ary tree, the vertex set and edge set of k-ary tree as follows; V(k-ary tree)={v,u_1,u_2,…,u_k^2,v_1,v_2,…,v_k} and E(k-ary tree) = {vv_i/1≤i≤k} ∪ {v_iu_j/1≤i≤k, 1≤j≤k}. Define a function f:V←{1,2,3,...} such that f(u)≠f(v) if uv∈ E. The b-irregular colouring pattern of k-ary tree as follows; f(v)=k+1, f(v_1)=1, for 1≤i≤k, f(v_i)=i and f(u_i)=i+1; for 1≤i≤k-1, 1≤j≤i ,f(u_{ki+j})=j; for 2≤i≤k, 0≤j≤k-i ,f(u_{ki-j})=k+1-j. In the way of minimum b-irregular colouring, we need at most k+1 colours to colour the vertices of k-ary tree. Hence $\chi_{bir}($ k-ary tree)=k+1.

Example: b-irregular colouring of k-ary tree



Algorithm 3.8
We use the following algorithm to determine an b-irregular chromatic number of (2,n)-Barbell graph B(K_n ,K_n), n>2.
Procedure (b-irregular colouring of (2,n)-Barbell graph B(K_n ,K_n), n>2)
Input: V←{v_1,v_2,…,v_n, u_1,u_2,…,u_n}
E←{ e_1,e_2,...,e_{n^2-n+1}}
if n≥ 3
u_1← n+1
for i=1 to n do
v_i ← i
end for
for i= 1 to n-1 do
u_{i+1} ← i
end for
end if
end procedure

Theorem 3.8 For n≥3, $\chi_{bir}(B(K_n, K_n) = n+1$

Proof- Let B(K_n,K_n) be the (2,n)-Barbell graph. Let {v_1,v_2,…,v_n} be the vertices of first complete graph denoted as K^1 and {u_1,u_2,…,u_n} be the vertices of second complete graph denoted as K^2. Define a function f:V←{1,2,3,...} such that f(u)≠f(v) if uv∈ E, The b-irregular colouring pattern of B(K_n,K_n) as follows; f(u_1)= n+1; for 1≤ i≤ n, f(v_i)=i; for 1≤ i≤ n-1, f(u_{i+1})=i. In the way of minimum b-irregular colouring, we used at most (n+1) colours to colour the vertices of (2,n)- Barbell graph and the difference between the number of appearance of each pair of colours does not exceed one. Hence $\chi_{bir}(B(K_n, K_n) = n+1$.

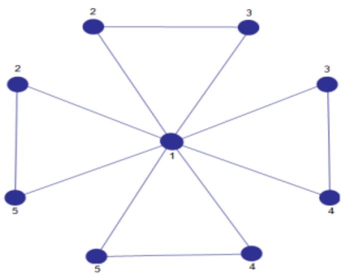
Example: b-irregular colouring of B(K_5,K_5)

Algorithm 3.9
We use the following algorithm to determine the b-irregular chromatic number of the friendship graph T_n, $n \geq 3$.
Procedure (b-irregular colouring of friendship graph T_n, $n \geq 3$)
Input: $V \leftarrow \{v_1, v_2, \ldots, v_{2n+1}\}$
$E \leftarrow \{ e_1, e_2, \ldots, e_{2n}\}$
if n≥3
$v_1 \leftarrow 1$
$v_{2n+1} \leftarrow 2$
for i=1 to n do
$v_{2i} \leftarrow i+1$
end for
for i=1 to n-1 do
$v_{2i+1} \leftarrow i+2$
end for
end if
end procedure

Theorem 2.3 For $n \geq 3$, $\chi_{bir}(T_n) = n+1$.

Proof- Let T_n be a friendship graph. The vertex set and edge set of friendship graph is as follows; $V(T_n)=\{v_i/1 \leq i \leq 2n+1\}$ and $E(T_n)=\{v_1 v_i/2 \leq i \leq 2n+1\}$. Define a function $f:V \leftarrow \{1,2,3,\ldots\}$ such that $f(u) \neq f(v)$ if $uv \in E$. The b-irregular colouring pattern of friendship graph is as follows; $f(v_1)=1$, $f(v_{2n+1})=2$, for $1 \leq i \leq n$, $f(v_{2i})=i+1$; for $1 \leq i \leq n-1$, $f(v_{2i+1})=2+i$. In the way of b-irregular colouring, we need at most n+1 colours to colour the vertices of friendship graph and the difference between the number of appearance of each pair of colours does not exceed one. Hence $\chi_{eir}(T_n) = n+1$.

Example: b- irregular colouring of Friendship graph



III. CONCLUSION

we have determined the acyclic irregular chromatic number for line graph of star, bi star, double star and middle graph of star, bi star, double star graph and b-chromatic number of k-ary tree and (2,n)-barbell graph, friendship graph.

IV. REFERENCES

(1) R.Avudainayaki, B.Selvam, Thirusangu, Irregular Coloring of Some Classes of Graphs, International Journal of Pure and Applied Mathematics , Volume 109, 2016, 119-127.
(2) A.C.Burris, The irregular coloring number of a tree. Discrete Math.141(1995) 279-283.
(3) O.V.Borodin. On acyclic colourings of planar graphs. Discrete Mathematics, 25; 211-236, 1979.
(4) Baum.B.Grun, “Acyclic coloring of planar graphs”,Israel J.Math, 14(3), 390-408, 1973.
(5) Grunbaum, Acyclic Coloring of Planar graphs, Israel.J.Math.14(1973) 390-412
(6) N.Ramya, Muthukumar, On Star and Acyclic Coloring of Graphs, International Journal of Pure and Applied Mathematics, Volume116, 2017, 467-470.
(7) W.Lrving, F.Manlove, the b-chromatic number of a graph, Discrete applied mathematics 91(1999) 127-141.