Algebraic Operation of Fuzzy Ideals of a Ring

Dr.D.Sivakumar [1] , Dr.S.Anjal Mose [2] , Mrs.R.Padma Priya [3]

[1] Professor, Department of Mathematics, Annamalai University,Annamalainagar-608002.

[2, 3] Assistant Professor, PG & Research Department of Mathematics,
St.Joseph's College of Arts & Science (Autonomous) ,
Cuddalore-607001

ABSTRACT

In this paper we introduce a generalization of a fuzzy ideal in rings. We have obtain necessary and sufficient condition for a fuzzy ideal of a ring to be a maximal fuzzy ideal. We have also obtain the condition for a fuzzy set P defined on the ring of integers Z to be fuzzy prime ideal on Z.

Key words: Fuzzy ideals, Algebraic operation, Triangular fuzzy number, Ring.

1. INTRODUCTION

Fuzzy mathematics is one of the subset of mathematics; it has divided into two section namely fuzzy logic and fuzzy set theory. Fuzzy mathematics is faster growth in research and development of real life problems with mathematics. In 1965 L.A. Zadeh introduced fuzzy logic,it is used in many application where ever uncertainty is used then fuzzy can be applied in that area. In 1971 Zadeh used Similarity relations and fuzzy orderings[1], D. S. Malik, J. N. Mordeson deals with Fuzzy maximal, radical, and primary ideals of a ring[2]. A. Rosenfeld used fuzzy numbers in Fuzzy groups [3] and W. J. Liu applied the new idea for Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems [4]. T. Kuraoka, N. Kuroki worked On fuzzy quotient rings induced by fuzzy ideals, Fuzzy Sets and Systems[5], R. Kellil using fuzzy numbers On 2-Absorbing Fuzzy Ideals to prove some property[6], R. Kellil introduced an New approaches on some fuzzy algebraic structures[7].presently J. Jacas, J. Recasens worked on Fuzzy numbers and equality relations [8]. M.Demirci used fuzzy numbers in Fuzzy functions and their fundamental properties[9], D.Boixader, J. Jacas, J. Recasens said about Fuzzy equivalence relations and the uses[10] and J. M. Anthony, H. Sherwood formed a new definition for Fuzzy group redefined[1]. Here we used triangular fuzzy number to used some of uncertain algebraic operation are used in some algebraic properties like sum and product of commutative, associative , etc.

2. PRELIMINARIES

In this preliminaries section some of few important definitions related to this paper and well known properties related with our given topics are used in the following

Definition 2.1 Fuzzy set:

A fuzzy set is considered by a membership function mapped into a domain space (i.e) mapping connecting universe of discourse X to the unit period [0, 1] given by $\tilde{A} = \{x, \mu_{\tilde{A}}(x) / x \in X\}$

.Here $\mu_A : X \to [0,1]$ is said the degree of membership function of the fuzzy set A.

Definition 2.2 Normal fuzzy set:

X is called a normal fuzzy set if a fuzzy set A of the universe of discourse then if there exits at-least one $x \in X$ such that $\mu_A(x)=1$

Definition 2.3 Height of a fuzzy set:

The leading membership position obtained by some element in the fuzzy set and it is given by $h(\tilde{A}) = \sup \mu_{\tilde{A}}(x)$

Definition 2.4 Convex fuzzy set:

A fuzzy set $\tilde{A}=\{x,\mu_{\tilde{A}}(x)/x\in X\}$ is said to be convex iff for any $x_1, x_2 \in X$, the membership function of A satisfies the condition $\mu_{\tilde{A}}\{\lambda x_1+(1-\lambda)x_2\} \ge \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \lambda \in [0,1]$

Definition2.5 Triangular Fuzzy number:

A TFN $\tilde{A}_p^i$ is a subset of in R with following membership function as follows:

$$\mu_{\tilde{A}_p^i}(x)=\begin{cases} \dfrac{x-l_1}{(l_2-l_1)}, & l_1 \le x \le l_2 \\ \dfrac{l_3-x}{(l_3-l_2)}, & l_2 \le x \le l_3 \\ 0, & otherwise \end{cases}$$

Where $l_1 \le l_2 \le l_3$

Definition 2.6: **Arithmetic operation of TFN based on $(\alpha)-$cuts**

If $\tilde{A}_p^i$ is a TFN, then $(\alpha)-$cuts is given by

$$A_\alpha = \begin{cases} [A_1(\alpha), A_4(\alpha)] \text{ for } \alpha \in [0,0.5] \\ [A_2(\alpha), A_3(\alpha)] \text{ for } \alpha \in [0.5,1] \end{cases}$$

Here $(i)\dfrac{dA_1(\alpha)}{d\alpha} > 0, \dfrac{dA_2(\alpha)}{d\alpha} < 0, \forall \alpha \in [0,0.5]$

$$(ii)\frac{dA_2(\alpha)}{d\alpha} > 0, \frac{dA_3(\alpha)}{d\alpha} < 0, \forall \alpha \in [0.5,1]$$

With the condition $\alpha \le 1$, It is expressed as

$$\tilde{A}_\alpha = \{[A_1(\alpha), A_2(\alpha), A_3(\alpha)]\}; \alpha, \in [0,1]$$

Hence $A_\alpha = \begin{cases} [\alpha(l_2-l_1)+l_1,] & \alpha \in [0,0.5] \\ [\alpha(l_3-l_2)+l_3], & \alpha \in [0.5,1] \end{cases}$

Definition 2.7 Let (G, ⋆) be a group and e its uniqueness. A fuzzy subset φ of G is a fuzzy subgroup of G iff

1. φ (e) =1,

2. $\varphi(l_1 \star l_2) \ge \min\{\varphi(l_1), \varphi(l_2)\}$;

$\forall l_1; l_2 \in G$,

3. For all $l_1 \in G$; $\varphi(l_1) = \varphi(l_1^{-1})$.

It is called normal if in addition

$\varphi(l_1 \star l_2 \star l_1^{-1}) = \varphi(l_2), \forall l_1 \in G$.

3. OPERATIONS ON FUZZY SETS

Definition 3.1. A fuzzy function on a set S is a mapping f: S × S × S ⟶ [0, 1] such that, for x, y ∈ S there exists a Z∈ S satisfying f(x, y, z) = 1. If f is a fuzzy operation on a set S, then f is

1. Commutative

2. Associative

3. Identity

Results 3.1. If E has an identity e for f and $f(x, e, x) = 1$, $\forall x \in S$, then this identity is unique.

Proof: Here we take two identity of the set S for f, then

$1 = f(e', e, e') = f(e, e', e')$ and $1 = f(e, e', e) = f(e', e, e)$ and so $f(e', e, e') = f(e', e, e) = 1$. But a unique element x such that $f(e', e, x) = 1$. Subsequently, $e = e'$.

Definition 3.2 Let $f : S\times S\times S \to [0, 1]$ be a fuzzy operation on a set S and e be the unique identity of S for f. An element $a\in S$ is symmetric if $f(a, a', e) = f(a', a, e)$ for some $a' \in S$.

Definition 3.3 Let $f : S \times S \times S \to [0, 1]$ be a fuzzy operation on a set S and e be the unique identity of S for f. An element $a' \in S$ if it exists such that

$f(a, a', e) = f(a', a, e) = 1$

is called a symmetric element of $a \in E$.

Definition 3.4 Let $f : S\times S\times S\to [0, 1]$ be a fuzzy operation on a set S. An element $a\in S$ is left (resp. right) regular or cancellable if for any elements $x, y, z \in S$, the equality $f(a, x, z) = f(a, y, z)$ (resp. $f(x, a, z) = f(y, a, z)$) implies $x = y$. It is regular or cancellable if it is left and right regular.

Results 3.2 If f a fuzzy operation on S has an identity e. Any left (resp. right) regular element has atmost one symmetric element.

Proof.

Suppose that x is left regular and has two symmetric elements a' and a''. Then $f(a, a', e) = f(a', a, e) = f(a, a'', e) = f(a'', a, e) = 1$ and so $f(a, a', e) = f(a, a'', e) = 1$ which implies the equality $a' = a''$. For the right symmetry the proof is trivial.

4. FUZZY IDEALS

In this section we can recall the important basic definition related to fuzzy ideals

Definition4.1: **Classical definition**

Let $(R, +, \times)$ be a ring and 0_R its identity for +. A fuzzy subset φ of R is called a fuzzy ideal of R if

1. $\varphi(0_R) = 1$,
2. $\varphi(a) = \varphi(-a)$,
3. $\varphi(a + b) \geq \min\{\varphi(a), \varphi(b)\}$, $\forall a, b \in R$,
4. $\varphi(a \times b) \geq \max\{\varphi(a), \varphi(b)\}$, $\forall a, b \in R$.

By altering the last state in the above definition we get a new definition of a fuzzy ideal of a ring as follows

Definition4.2:

New definition

Let $(R, +, \times)$ be a ring and 0_R its identity for +, a fuzzy subset φ of R is a right fuzzy ideal of R if

1. $\varphi(0_R) = 1$,
2. $\varphi(a) = \varphi(-a)$,
3. $\varphi(a + b) \geq \min\{\varphi(a), \varphi(b)\}$,

$\forall a, b \in R$,

4. φ (a) > 0 $\Longrightarrow$ φ (a × b) > 0,

$\forall a, b \in R$.

It is called proper if φ (1) = 0.

Proposition: 1

If φ is a proper fuzzy ideal of R, then for all θ ∈ [0, 1] the set I = {x ∈ R | φ (x) ≥ θ} is an ideal of the ring (R, +, ×). In addition, if ∀a, b ∈ R, a × b ∈ I implies φ (a × b) = φ (a). φ (b), then I is a prime ideal of the ring R. Let us give an example of a fuzzy prime ideal which is not maximal.

Example:

As an application of the above example, let R be the ring Z/10Z. Then the mapping φ given by

a	0	1	2	3	4	5	6	7	8	9
φ (a)	1	0	0.5	0	0.6	0	0.5	0	0.7	0

is as required in the exceeding example so it is a fuzzy prime ideal. We can also verify it directly. For this, to see that (a.b) ≥ max{φ(a), φ(b)} and φ (a.b) > 0 imply φ (a) > 0 or φ (b) > 0, it suffices to draw the product table and move each cell with the value of element found in the consistent cell by the product. If we change the value of φ (2) = 0.6 in the example to φ (6) = 0.7, the subsequent function is still a fuzzy ideal greater than the preceding one and different from the constant fuzzy ideal 1.

5. FUZZY MAXIMAL AND MINIMAL IDEALS

In the crisp case, the concept of maximal ideal is central to the applications of commutative ring theory to algebraic geometry. Therefore they form an important class of ideals in commutative ring theory. we discuss its fuzzy counterpart, viz; fuzzy maximal [left, right] ideals. proved that fuzzy maximal ideal A of R cannot be defined as a fuzzy ideal different from the characteristic function of R such that B is a fuzzy ideal of R and A ⊂ B ⊆ χR ⇒ B = χR Instead they approached the notion of fuzzy maximal ideals through fuzzy maximal left [and right] ideals.

5.1.Theorem. Let A be a fuzzy set on R. Then A is a fuzzy maximal ideal of R if, and only if, there exist a maximal ideal M of R and a dual atom α ∈ L such that A(x) = 1, if x ∈ M, and = α, otherwise

Remark. 1 The above theorem says that there is a 1-1 correspondence between fuzzy maximal ideals of R and pairs (M, α) where M is a maximal ideal of R and α is a dual atom in L, if L = [0,1], then there is no dual atom in L. Hence, in this case, R has no fuzzy maximal ideals.

Due to the importance of prime ideals in classical ring theory, fuzzy prime has been given much attention. In this section, we discuss the notion of fuzzy prime ideals and important properties. We also present a characterization of all fuzzy prime ideals of Z

5.2. Theorem. A fuzzy set P on Z is a fuzzy prime ideal if, and only if, it is given by P(n) = 1, if p|n = α, otherwise where p is a prime integer or zero and α < 1

Remark.2 For a fixed prime number p, we get the prime ideal PZ of Z. Fixing α in [0, 1) we get a unique fuzzy prime ideal of Z as in the above theorem. We may allow p to vary over all positive prime numbers and α to take various values in [0, 1). Also, the fuzzy subset P of Z given by P(n) = 1, if n = 0 = α, otherwise where α

5.3.Theorem. Let φ and β be two fuzzy ideals of a ring R. For a decomposition of an element x as a finite sum, x = ∑ aibi where ai,bi ∈ R, we set s(x) = {min(φ (ai), β(bi))}. Denote by Sx the set of all the possible decompositions of x as above. If any element admits a finite number of decompositions, φ (x) = max Sx, s(x) is an ideal, called in the sequel the product of the fuzzy ideals φ and β

Proof.

1. It is clear that among the different decompositions of 0 we have the decomposition 0 = 0.1 So $\eta(0) = \min\{\varphi(0), \beta(0)\} = 1$.

2. If $\sum aibi$ is a decomposition of x, then $\sum(-ai)bi$ and $\sum(ai)(-bi)$ are also two decompositions of $-x$. On the other hand, for any decomposition of $-x$ there correspond two decompositions of x. Since $\varphi(-ai) = \varphi(ai)$ and $\beta(-ai) = \beta(ai)$, $\eta(x) = \eta(-x)$.

3. For the third axiom, it suffices to prove it for two decompositions x = ab and $x = a'b'$ of x and two similar decompositions of y as y = cd and $y = c'd'$

CASE:1 Suppose that $\varphi(a) \leq \beta(b)$, $\beta(b') \leq \varphi(a')$, $\varphi(c) \leq \beta(d)$, $\varphi(c') \leq \beta(d')$, $\varphi(a) \geq \varphi(a')$, $\varphi(c) \leq \varphi(c')$ and $\varphi(c) \leq \varphi(a) \leq \varphi(c')$ then $\eta(x) = \varphi(a)$ and $\eta(y) = \varphi(c')$. On the other hand

$A=\min\{\min(\varphi(a), \beta(b)), \min(\varphi(c),$

$\beta(d))\} = \min\{\varphi(a), \varphi(c)\} = \varphi(c)$,

$B=\min\{\min(\varphi(a), \beta(b)), \min(\varphi(c'),$

$\beta(d'))\} = \min\{\varphi(a), \varphi(c')\} = \varphi(a)$,

$C=\min\{\min(\varphi(a'), \beta(b')), \min(\varphi(c), \beta(d)) = \min\{\beta(b'), \varphi(c)\} = \varphi(c)$ or $\beta(b')$,

$D=\min\{\min(\varphi(a'), \beta(b')), \min(\varphi(c'), \beta(d'))\} = \min\{\beta(b'), \varphi(c')\} = \beta(b')$.

As $\varphi(a)$ is great than $\beta(b')$ and $\varphi(c)$ we get

$\eta(x + y) = \varphi(a) \geq \mu(a) = \min\{\eta(x), \eta(y)\}$.

CASE:2 Suppose that $\varphi(a) \leq \beta(b)$, $\beta(b') \leq \varphi(a')$, $\varphi(c) \leq \beta(d)$, $\varphi(c') \leq \beta(d')$, $\varphi(a) \geq \varphi(a')$, $\varphi(c) \leq \varphi(c')$ and $\varphi(a) \leq \varphi(c)$ then $\eta(x) = \varphi(a)$ and $\eta(y) = \varphi(c')$, so $\max\{\eta(x), \eta(y)\} = \varphi(c')$. On the other hand

$A = \min(\varphi(ac), \beta(bd)) \geq \min(\max(\varphi(a), \varphi(c)), \max(\beta(b), \beta(d))) = \{ \min(\varphi(c), \beta(b)) = \beta(b)$ if $\beta(b) \leq \varphi(c)$ $\min(\varphi(c), \beta(b)) = \varphi(c)$ otherwise $\}$

$B = \min(\varphi(ac'), \beta(bd')) \geq \min(\max\varphi(a), \varphi(c')), \max(\beta(b), \beta(d'))) = \varphi(c')$

$C = \min(\varphi(a'c), \beta(b'd)) \geq \min(\max(\varphi(a'), \varphi(c)), \max(\beta(b'), \beta(d))) = \varphi(c)$

$D = \min(\varphi(a'c'), \beta(b'd')) \geq \min(\max(\varphi(a'), \varphi(c')), \max(\beta(b'), \beta(d'))) = \varphi(c')$.

We conclude the above conditions that $\max\{A, B, C, D\} = \varphi(c') \geq \varphi(c') = \max\{\eta(x), \eta(y))$

Proposition 2.

Let φ and β be two fuzzy ideals of a commutative ring R. The set

$I = \{x \in R \mid \varphi(y) \leq \beta(xy)\ \forall y \in R\}$ is an ideal of the ring R.

Proof.

1. For all $y \in R$ we have $\varphi(y) \leq 1 = \beta(0) = \beta(0y)$.

2. Let $x, x' \in I$. Then for all $y \in R$, one has $\varphi(y) \leq \beta(xy)$ and $\varphi(y) \leq \beta(x'y)$, $\varphi(y) \leq \min\{\beta(xy), \beta(x'y)\}$.

But $\min\{\beta(xy), \beta(x'y)\} \leq \beta(xy - x'y)$ and the last expression is just the value $\beta((x-x')y)$ and then $x-x' \in I$.

3. Let now $x \in I$ and $a \in R$. Then for any $y \in R$ we have $\varphi(ay) \geq \max\{\varphi(a), \varphi(y)\}$ so $\varphi(y) \leq \varphi(ay) \leq \beta(x(ay)) = \beta((xa)y)$ so $xa \in I$ and the conclusion follows.

6. Conclusion

We conclude that our aim is to introduce the new definition and examples in generalization of classical set into fuzzy ideal ring in algebra. The present paper shows the generalize the previous proposition to the fuzzy ideals of the form φ_a, where a is belongs to N. This work will helpful for researcher for their future work.

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