Graphical Comparison for Convergence of Gauss-Seidel and Jacobi Iterative Methods

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Abstract: In this paper, the graphical comparison between Gauss-Seidel and Jacobi iteration method has been shown. It is observed that neither of the iterative methods always converges. Although, it is possible to obtain a divergent sequence of approximations by applying Gauss-Jacobi or Gauss-Seidel iterative method to a system of simultaneous linear equations. It is observed that convergence of the sequence of approximations, strict diagonal dominance of the coefficient matrix is necessary condition before applying any iterative methods.

Keywords: Iterative methods, convergence and divergence, diagonal dominance matrix.

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Introduction

A square matrix A is diagonally dominant if$|a_{ii}| \geq \sum_{k=1}^{j}$ i.e. in which every element of leading diagonal should be greater or equal to the absolute values of all other elements in corresponding row.If the inequality is strict then the matrix is strictly diagonally dominantand if the inequality is greater than or equal to, then the matrix is weakly diagonal dominant. The convergence rate of various iterative methods tells about how fast the error$|a^n - a|$ tends to zero, as the number of iterations (n) increases. The necessary condition for convergence is represented as$a^{n+1} = Sa^n + d$ to converge is$\gamma(S) = \max_{1<i<k} |\delta(S)| < 1$,where$\gamma(S)$is spectral radius of (S).

Jacobi –Iterative method

Jacobi-iterative method is a type of indirect method used for solving an equation of matrix on a matrix, which do not has zeroes along the main diagonal of it.Each diagonal element is solved.In this method before any new updation is used in calculations,all the values of unknown variables are used. That means starting with initial approximations as$x_1^0, x_2^0,\ x_3^0,\ x_4^0,\ x_5^0, \ldots x_n^0$. We have to calculate next approximation as:

$$x_1^1 = \frac{c_1-(a_{12}x_2^0+\cdots+a_{1k}x_k^0)}{a_{11}} \ , x_2^1 = \frac{c_2-(a_{21}x_1^0+\cdots+a_{2k}x_k^0)}{a_{22}} ,\ldots\ldots\ldots\ldots\ldots, x_k^1 = \frac{c_k-(a_{k1}x_1^0+\cdots+a_{kk-1}x_{k-1}^0)}{a_{kk}}.$$

Continuing this processafter $n$iteration

$$x_1^{n+1} = \frac{c_1-(a_{12}x_2^n+\cdots+a_{1k}x_k^n)}{a_{11}} , x_2^{n+1} = \frac{c_2-(a_{21}x_1^n+\cdots+a_{2k}x_k^n)}{a_{22}} ,\ldots\ldots\ldots\ldots\ldots,$$

$$x_k^{n+1} = \frac{c_k-(a_{k1}x_1^0+\cdots+a_{kk-1}x_{k-1}^n)}{a_{kk}}$$

In generally,$x_i^{n+1} = \frac{c_i - \sum_{j\neq 1} a_{j1}x_j^n}{a_{kk}}$.

This method is also called simultaneous displacement method.

Gauss-Seidel Method

It is modification of Jacobi method. This modification more convenient than Jacobi method and often requires minimum number of iterations to provide the same degree of accuracy. With the Jacobi method, the values of obtained in the nth approximation remain unchanged until the entirenth approximation has been calculated. With the Gauss- Seidel method, on the other hand, we make use of the latestvalues of each as soon as they are known, which simply means, once we have determined from the first equation, that obtained value is employed in the next equation to calculate the new values.

$x_1^1 = \frac{c_1-(a_{12}x_2^0+\cdots+a_{1k}x_k^0)}{a_{11}}$, $x_2^1 = \frac{c_2-(a_{21}x_1^1+\cdots+a_{2k}x_k^0)}{a_{22}}$,… … … …,$x_k^1 = \frac{c_k-(a_{k1}x_1^1+\cdots+a_{kk-1}x_{k-1}^1)}{a_{kk}}$.

Continuing this process,after $n$iteration we obtain:

$x_1^{n+1} = \frac{c_1-(a_{12}x_2^n+\cdots+a_{1k}x_k^n)}{a_{11}}$, $x_2^{n+1} = \frac{c_2-(a_{21}x_1^n+\cdots+a_{2k}x_k^n)}{a_{22}}$,… … … …,

$x_k^{n+1} = \frac{c_k-(a_{k1}x_1^{n+1}+\cdots+a_{kk-1}x_{k-1}^{n+1})}{a_{kk}}$.

In general form,$x_i^{n+1} = \frac{c_i-\sum_{j<i}a_{j1}x_j^{n+1}+\sum_{j>i}a_{j1}x_j^n}{a_{kk}}$.

Convergence of iterative methods

Theorem: If K is diagonally dominant, that means, $|a_{ii}| > \sum_{i\neq j}|a_{i,j}| for\ 1 \leq i \leq n$, then Gauss-Seidel converges to a solution.

Proof: Denote $K = (D - L) - U$ and let

$a_j = \sum_{i=1}^{j-1}|a_{j,i}|$, $b_j = \sum_{i=j+1}^{n}|a_{j,i}| and R_j = \frac{b_j}{a_{jj}-a_j}$.

Because,$K$is diagonally dominant,

$R_j = \frac{b_j}{a_{jj}-a_j} < \frac{a_{jj}-a_j}{a_{jj}-a_j} = 1$,$\forall 1 \leq j \leq n$So, $R = Max_{1\leq j\leq n}$.

Remaining case is to show that,

$\|b\|_\infty = Max_{\|x\|_\infty=1}\|Bx\|_\infty \leq R < 1$, where, $B = C^{-1}M = (D-L)^{-1}U$

Let,$\|x\|_\infty = 1\ and y = Bx, then \|y\|_\infty = Max_{1\leq i\leq n}|y_i| = |y_k|$ for some k

Then, $y = Bx = (D-L)^{-1}Ux$

$(D-L)y = Ux => Dy = Ly + Ux$

$y = D^{-1}(Ly + Ux)$.

Then,$y_k = \frac{1}{a_{kk}}(-\sum_{i=1}^{k-1} a_{ki}y_i - \sum_{i=k+1}^{n} a_{ki}x_i)$

and$\|y\|_\infty = |y_k| \leq \frac{1}{a_{kk}}(a_k\|y\|_\infty + b_k\|x\|_\infty$

This shows that,$\forall x$ with $\|x\|_\infty = 1$,

$\|Bx\|_\infty = \|y\|_\infty \leq \frac{b_j}{a_{jj}-a_j} = R_k < 1$.

Thus,$\|b\|_\infty = Max_{\|x\|_\infty=1}\|Bx\|_\infty \leq R < 1$.

Example 1:Solve the equation using Jacobi method

$4a - b - d = 0$, $-a + 4b - c - e = 5$, $-b + 4c - f = 0$,$-a + d - e = 6$
$-b - d + 4e - f = -2$, $-c - e + 4f = 6$

Now by taking the initial approximationsas$a^k = b^k = c^k = d^k = e^k = f^k = 0$, Starting with these values, proceeding according to the Jacobi method, we obtained the values of unknowns shown in the table 1.

Example 2: Solve the equation by Gauss-Seidel Method.

$4a - b - d = 0$,$-a + 4b - c - e = 5$, $-b + 4c - f = 0$,$-a + d - e = 6$,

$-b-d+4e-f=-2$,$-c-e+4f=6$.

Taking the initial approximations as$a^k=b^k=c^k=d^k=e^k=f^k=0$,we obtained the values of unknowns shown in the table 2.

Table 1: Iteration result for Jacobi Method

Iteration	a	b	c	d	e	f
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.0000000	1.250000	0.000000	1.500000	-0.50000	1.500000
2	0.6875000	1.125000	0.687500	1.375000	0.562500	1.375000
3	0.6250000	1.7344	0.625000	1.812575	0.46875	1.812575
4	0.886756	1.679688	0.886782	1.773438	0.839900	1.773438
5	0.823282	1.898554	0.823282	1.926865	0.806641	1.926865
6	0.956355	1.883301	0.956355	1.917481	0.938668	1.917481
7	0.950196	1.962695	0.950196	1.973606	0.929566	1.973606
8	0.984075	1.957490	0.984875	1.969941	0.977477	1.969941
9	0.981858	1.986497	0.981858	1.990388	0.974343	1.990388
10	0.994199	1.984515	0.994199	1.989050	0.991796	1.989050
11	0.993391	1.995041	0.993391	1.996499	0.990684	1.996499
12	0.997887	1.994354	0.997887	1.996011	0.997011	1.996011
13	0.997593	1.99816	0.997593	1.998725	0.996595	1.998728
14	0.999230	1.997945	0.999230	1.998547	0.998912	1.998547
15	0.999123	1.999343	0.999123	1.999536	0.998760	1.999536
16	0.999720	1.999252	0.999720	1.999471	0.999604	1.999471
17	0.999681	1.999761	0.999681	1.999831	0.999549	1.999831
18	0.999898	1.999728	0.999898	1.999808	0.999856	1.999808

This completes the table and the solution is $(a,b,c,d,e,f)=(0.999898,1.999728,0.999898,1.999808,0.999856,1.999808)$ respectively.

Table2: Iteration result for Gauss-Seidel Method:

Iteration	a	b	c	d	e	f
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	1.250000	0.312500	1.500000	0.187500	1.625000
3	0.687500	1.546875	0.79296	1.718750	0.722656	1.878906
4	0.816406	1.833008	0.927979	1.884766	0.899188	1.956792
5	0.929444	1.939153	0.973986	1.957198	0.963276	1.984316
6	0.974078	1.977835	0.990538	1.984339	0.986623	1.994290
7	0.990544	1.991926	0.996554	1.994292	0.995127	1.997920
8	0.996555	1.997059	0.998745	1.997921	0.998226	1.999243
9	0.998745	1.998929	0.999543	1.999243	0.999354	1.999724
10	0.999543	1.999610	0.999834	1.999724	0.999765	1.999900
11	0.999834	1.999858	0.999940	1.999900	0.999915	1.999964

Hence the solution is

$(a,b,c,d,e,f)=0.999834,1.999858,0.999940,1.999900,0.999915,1,9999640)$respectively.

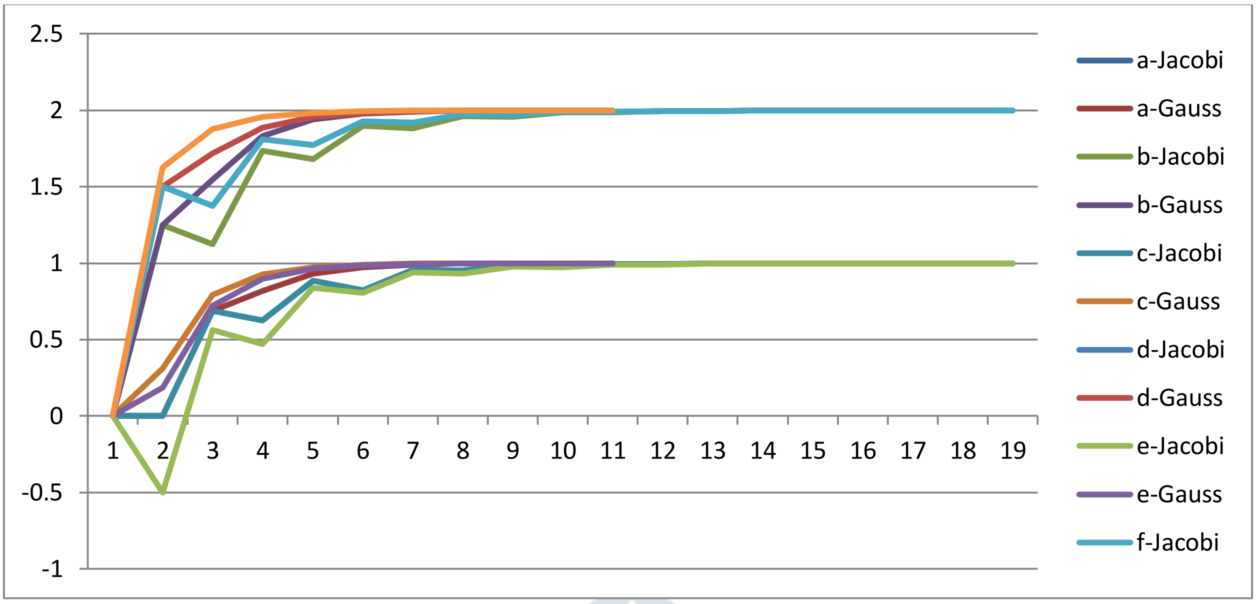

Figure 1

Conclusion

On the basis of our results, we may reach to the conclusion that there are so many iterative methods for solving system of linear equations, which can be compared graphically. In this paper, we have shown graphical comparison between Jacobi iterative method and Gauss-Seidel method, whose key condition is strictly diagonal dominance of the coefficient matrix, whose fulfillment results in convergence, otherwise divergence will take place.Analysis of these two iterative methods for thesystem of linear equations showed that Gauss-Seidel method is more rapid in convergence than the Jacobi method.

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