Gegenbauer polynomials or ultra-spherical polynomials - An Overview

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Abstract

This paper attempts to study ***Gegenbauer polynomials*** Orthogonal polynomials Pn λ−1/2 h λ−1/2; a particular case of the Jacobi polynomials . In the constructive theory of spherical functions the Gegenbauer polynomials play an important role. Apart from constant factors they are certain Jacobi polynomials. For α, β> −1, the indices, the Jacobi polynomials P µ . A new formula expressing explicitly the integrals of ultraspherical polynomials of any degree that has been integrated an arbitrary number of times of ultraspherical polynomials is given.

The tensor product of ultraspherical polynomials is used to approximate a function of more than one variable. Formulae expressing the coefficients of differentiated expansions of double and triple ultraspherical polynomials in terms of the original expansion are stated and proved. The Weierstrass elliptic function is implemented in the Wolfram Language as WeierstrassP[u, {g2, g3}]. Half-periods and invariants can be interconverted using the Wolfram Language commands Weierstrass Invariants[{omega1, omega2}] and Weierstrass Half Periods[{g2, g3}]. The derivative of a Weierstrass elliptic function is implemented as WeierstrassPPrime[u, {g2, g3}], and the inverse Weierstrass function is implemented as Inverse WeierstrassP[p, {g2, g3}]. Inverse WeierstrassP[{p, q}, {g2, g3}] finds the unique value of u for which $p = \wp(u; g_2, g_3)$ and $q = \wp'(u; g_2, g_3)$. A symmetric polynomial on n variables x_1, ..., x_n (also called a totally symmetric polynomial) is a function that is unchanged by any permutation of its variables. In other words, the symmetric polynomials satisfy

$$f(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_n), \quad (1)$$

where $y_i = x_{\pi(i)}$ and π being an arbitrary permutation of the indices 1, 2, ..., n.

For fixed n, the set of all symmetric polynomials in n variables forms an algebra of dimension n. The coefficients of a univariate polynomial $f(x)$ of degree n are algebraically independent symmetric polynomials in the roots of f, and thus form a basis for the set of all such symmetric polynomials.

There are four common homogeneous bases for the symmetric polynomials, each of which is indexed by a partition λ (Dumitriu et al. 2004). Letting l be the length of λ, the elementary functions e_λ, complete homogeneous functions h_λ, and power-sum functions p_λ are defined for $l = 1$

Key words: recurrence relation, weight function, generating function, product linearization, connection coefficient, ultraspherical polynomials

Introduction

The (associated) Legendre function of the first kind $P_n^m(z)$ is the solution to the Legendre differential equation which is regular at the origin. For m, n integers and z real, the Legendre function of the first kind simplifies to a polynomial, called the Legendre polynomial. The associated Legendre function of first kind is given by the Wolfram Language command LegendreP[n, m, z], and the unassociated function by LegendreP[n, z]. 3-parameter class of Askey-Wilson polynomials being expanded in terms of continuous q-ultraspherical polynomials with a product of two 2ϕ2's as coefficients, and an analytic proof will be given for it. Then Gegenbauer's addition formula for ultraspherical polynomials and Rahman's addition formula for q-Bessel functions will be obtained as limit cases. This q-analogue of Gegenbauer's addition formula is quite different from the addition formula for continuous q-ultraspherical polynomials obtained by Rahman and Verma in 1986. Furthermore, the functions occurring as factors in the expansion coefficients will be interpreted as a special case of a system of biorthogonal rational functions with respect to the Askey-Roy q-beta measure. A degenerate case of this biorthogonality are Pastro's biorthogonal polynomials associated with the Stieltjes-Wigert polynomials. The Weierstrass constant is defined as the value $\sigma(1 \mid 1, i)/2$, where $\sigma(z \mid \omega_1, \omega_2)$ is the Weierstrass sigma function with half-periods ω_1 and ω_2. Amazingly, it has the closed form (OEIS A094692), where $\Gamma(z)$ is the gamma function. The case of the Weierstrass elliptic function with invariants $g_2 = -1$ and $g_3 = 0$. The half-periods for this case are $L(1+i)/4$ and $L(-1+i)/4$, where L is the lemniscate constant

The second-order ordinary differential equation

$$\left(1-x^2\right) y'' - 2(\mu+1) x\, y' + (\nu-\mu)(\nu+\mu+1)\, y = 0$$

sometimes called the hyperspherical differential equation . The solution to this equation is

$$y = \left(x^2-1\right)^{-\mu/2} \left[C_1 P_\nu^\mu(x) + C_2 Q_\nu^\mu(x)\right],$$

where $P_\nu^\mu(x)$ is an associated Legendre function of the first kind and $Q_\nu^\mu(x)$ is an associated Legendre function of the second kind.

A number of other forms of this equation are sometimes also known as the ultraspherical or Gegenbauer differential equation, including

$$\left(1-x^2\right) y'' - (2\mu+1) x\, y' + \nu(\nu+2\mu)\, y = 0.$$

The general solutions to this equation are

$$y = \left(x^2 - 1\right)^{(1-2\mu)/4} \left[C_1\, P_{-1/2+\mu+\nu}^{1/2-\mu}(x) + C_2\, Q_{-1/2+\mu+\nu}^{1/2-\mu}(x)\right].$$

If $-1/2 + \mu + \nu$ is an integer, then one of the solutions is known as a Gegenbauer polynomials $C_n^{(\lambda)}(x)$, also known as ultraspherical polynomials.

The form

$$\left(1 - x^2\right) y'' - (2\,m + 3)\,x\,y' + \lambda\,y = 0$$

is also given by Infeld and Hull and Zwillinger . It has the solution

$$y = \left(x^2 - 1\right)^{-(2\,m+1)/4} \left[C_1\, P_{-1/2+\sqrt{(1+m)^2+\lambda}}^{1/2+m}(x) + C_2\, Q_{-1/2+\sqrt{(1+m)^2+\lambda}}^{1/2+m}(x)\right].$$

Objective:

This paper intends to explore **Gegenbauer polynomials** or **ultraspherical polynomials** *C* α–1/2. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials.

Gegenbauer polynomials

The Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are solutions to the Gegenbauer differential equation for integer n. They are generalizations of the associated Legendre polynomials to $(2\,\lambda + 2)$-D space, and are proportional to (or, depending on the normalization, equal to) the ultraspherical polynomials $P_n^{(\lambda)}(x)$.

Following Szegö, in this work, Gegenbauer polynomials are given in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = \beta = \lambda - 1/2$ by

$$C_n^{(\lambda)}(x) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(2\,\lambda)}\, \frac{\Gamma(n + 2\,\lambda)}{\Gamma\left(n + \lambda + \frac{1}{2}\right)}\, P_n^{(\lambda-1/2,\lambda-1/2)}(x)$$

(Szegö 1975, p. 80), thus making them equivalent to the Gegenbauer polynomials implemented in the Wolfram Language as GegenbauerC[n, lambda, x]. These polynomials are also given by the generating function

$$\frac{1}{\left(1 - 2\,x\,t + t^2\right)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)\, t^n.$$

The first few Gegenbauer polynomials are

$C_0^{(\lambda)}(x) = 1$

$C_1^{(\lambda)}(x) = 2\lambda x$

$C_2^{(\lambda)}(x) = -\lambda + 2\lambda(1+\lambda)x^2$

$C_3^{(\lambda)}(x) = -2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3.$

In terms of the hypergeometric functions,

$$C_n^{(\lambda)}(x) = \binom{n+2\lambda-1}{n}\,{}_2F_1\left(-n, n+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)\right)$$

$$= 2^n\binom{n+\lambda-1}{n}(x-1)^n\,{}_2F_1\left(-n, -n-\lambda+\tfrac{1}{2}; -2n-2\lambda+1; \frac{2}{1-x}\right)$$

$$= \binom{n+2\lambda+1}{n}\left(\frac{x+1}{2}\right)^n\,{}_2F_1\left(-n, -n-\lambda+\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \frac{x-1}{x+1}\right).$$

They are normalized by

$$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-1/2}\left[C_n^{(\lambda)}\right]^2 dx = 2^{1-2\lambda}\pi\,\frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma^2(\lambda)\Gamma(n+1)}$$ (

for $\lambda > -1/2$.

Derivative identities

$$\frac{d}{dx}C_n^{(\lambda)}(x) = 2\lambda C_{n-1}^{(\lambda+1)}(x)$$

$$\left(1-x^2\right)\frac{d}{dx}\left[C_n^{(\lambda)}\right] = [2(n+\lambda)]^{-1}\left[(n+2\lambda-1)(n+2\lambda)C_{n-1}^{(\lambda)}(x) - n(n+1)C_{n+1}^{(\lambda)}(x)\right]$$

$$= -n x C_n^{(\lambda)}(x) + (n+2\lambda-1)C_{n-1}^{(\lambda)}(x)$$

$$= (n+2\lambda)x C_n^{(\lambda)}(x) - (n+1)C_{n+1}^{(\lambda)}(x)$$

$$n C_n^{(\lambda)}(x) = x\frac{d}{dx}\left[C_n^{(\lambda)}(x)\right] - \frac{d}{dx}\left[C_{n-1}^{(\lambda)}(x)\right]$$

$$(n+2\lambda)C_n^{(\lambda)}(x) = \frac{d}{dx}\left[C_{n+1}^{(\lambda)}(x)\right] - x\frac{d}{dx}\left[C_n^{(\lambda)}(x)\right]$$

$$\frac{d}{dx}\left[C_{n+1}^{(\lambda)}(x) - C_{n-1}^{(\lambda)}(x)\right] = 2(n+\lambda)C_n^{(\lambda)}(x)$$

$$= 2\lambda\left[C_n^{(\lambda+1)}(x) - C_{n-2}^{(\lambda+1)}(x)\right]$$

Consider the probability $Q_1(n, d)$ that no two people out of a group of n will have matching birthdays out of d equally possible birthdays. Start with an arbitrary person's birthday, then note that the probability that the second person's birthday is different is $(d-1)/d$, that the third person's birthday is different from the first two is $[(d-1)/d]\,[(d-2)/d]$, and so on, up through the nth person. Explicitly,

$$Q_1(n, d) = \frac{d-1}{d}\,\frac{d-2}{d}\cdots\frac{d-(n-1)}{d}$$

$$= \frac{(d-1)(d-2)\cdots[d-(n-1)]}{d^{n-1}}.$$

But this can be written in terms of factorials as

$$Q_1(n, d) = \frac{d!}{(d-n)!\,d^n},$$

so the probability $P_2(n, d)$ that two or more people out of a group of n do have the same birthday is therefore

$$P_2(n, d) = 1 - Q_1(n, d)$$

$$= 1 - \frac{d!}{(d-n)!\,d^n}.$$

In general, let $Q_i(n, d)$ denote the probability that a birthday is shared by exactly i (and no more) people out of a group of n people. Then the probability that a birthday is shared by k or more people is given by

$$P_k(n, d) = 1 - \sum_{i=1}^{k-1} Q_i(n, d).$$

In general, $Q_k(n, d)$ can be computed using the recurrence relation

$$Q_k(n, d) = \sum_{i=1}^{\lfloor n/k \rfloor}\left[\frac{n!\,d!}{d^{ik}\,i!\,(k!)^i\,(n-ik)!\,(d-i)!}\sum_{j=1}^{k-1} Q_j(n-ik,\, d-i)\,\frac{(d-i)^{n-ik}}{d^{n-ik}}\right]$$

However, the time to compute this recursive function grows exponentially with k and so rapidly becomes unwieldy.

If 365-day years have been assumed, i.e., the existence of leap days is ignored, and the distribution of birthdays is assumed to be uniform throughout the year , then the number of people needed for there to be at least a 50% chance that at least two share birthdays is the smallest n such that $P_2(n, 365) \geq 1/2$. This is given by $n = 23$, since

$$P_2(23, 365) = \frac{38\,093\,904\,702\,297\,390\,785\,243\,708\,291\,056\,390\,518\,886\,454\,060\,947\,061}{75\,091\,883\,268\,515\,350\,125\,426\,207\,425\,223\,147\,563\,269\,805\,908\,203\,125}$$

$\approx 0.507297.$

The number n of people needed to obtain $P_2(n, d) \geq 1/2$ for $d = 1$, 2, ..., are 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, ...

The Jacobi polynomials, also known as hypergeometric polynomials, occur in the study of rotation groups and in the solution to the equations of motion of the symmetric top. They are solutions to the Jacobi differential equation, and give some other special named polynomials as special cases. They are implemented in the Wolfram Language as JacobiP[n, a, b, z].

For $\alpha = \beta = 0$, $P_n^{(0,0)}(x)$ reduces to a Legendre polynomial. The Gegenbauer polynomial

$$G_n(p, q, x) = \frac{n!\,\Gamma(n+p)}{\Gamma(2n+p)} P_n^{(p-q,q-1)}(2x-1)$$

and Chebyshev polynomial of the first kind can also be viewed as special cases of the Jacobi polynomials.

Plugging

$$y = \sum_{\nu=0}^{\infty} a_\nu (x-1)^\nu$$

into the Jacobi differential equation gives the recurrence relation

$$[\gamma - \nu(\nu + \alpha + \beta + 1)]\, a_\nu - 2(\nu+1)(\nu+\alpha+1)\, a_{\nu+1} = 0$$

for $\nu = 0$, 1, ..., where

$$\gamma \equiv n(n + \alpha + \beta + 1).$$

Solving the recurrence relation

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{d\,x^n}\left[(1-x)^{\alpha+n} (1+x)^{\beta+n}\right]$$

for $\alpha, \beta > -1$. They form a complete orthogonal system in the interval $[-1, 1]$ with respect to the weighting function

$$w_n(x) = (1-x)^\alpha (1+x)^\beta,$$

and are normalized according to

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

where $\binom{n}{k}$ is a binomial coefficient. Jacobi polynomials can also be written

$$P_n^{(\alpha,\beta)} = \frac{\Gamma(2n+\alpha+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)} G_n\left(\alpha+\beta+1, \beta+1, \tfrac{1}{2}(x+1)\right),$$

where $\Gamma(z)$ is the gamma function and

$$G_n(p, q, x) \equiv \frac{n!\,\Gamma(n+p)}{\Gamma(2n+p)} P_n^{(p-q,q-1)}(2x-1).$$

Jacobi polynomials are orthogonal polynomials and satisfy

$$\int_{-1}^{1} P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)} (1-x)^\alpha (1+x)^\beta \, dx$$
$$= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)} \delta_{mn}.$$

The coefficient of the term x^n in $P_n^{(\alpha,\beta)}(x)$ is given by

$$A_n = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n\, n!\,\Gamma(n+\alpha+\beta+1)}.$$

They satisfy the recurrence relation

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) P_{n+1}^{(\alpha,\beta)}(x)$$
$$= \left[(2n+\alpha+\beta+1)\left(\alpha^2-\beta^2\right) + (2n+\alpha+\beta)_3\, x\right] P_n^{(\alpha,\beta)}(x) -$$
$$2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) P_{n-1}^{(\alpha,\beta)}(x),$$

hypergeometric function of probability

The probability $P_2(n, d)$ can be estimated as

$$P_2(n, d) \approx 1 - e^{-n(n-1)/2d}$$
$$\approx 1 - \left(1 - \frac{n}{2d}\right)^{n-1},$$

where the latter has error

$$\epsilon < \frac{n^3}{6(d-n+1)^2}$$

.

Q_2 can be computed explicitly as

$$Q_2(n,d) = \frac{n!}{d^n}\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{2^i}\binom{d}{i}\binom{d-i}{n-2i}$$

$$= \frac{d!}{d^n(d-n)!}\left[{}_2F_1\left(\tfrac{1}{2}n, \tfrac{1}{2}(1-n); d-n+1; 2\right)-1\right],$$

where $\binom{n}{m}$ is a binomial coefficient and ${}_2F_1(a, b, ; c; z)$ is a hypergeometric function. This gives the explicit formula for $P_3(n, d)$ as

$$P_3(n,d) = 1 - Q_1(n,d) - Q_2(n,d)$$

$$= 1 - d^{-n}\, d!\, {}_2\tilde{F}_1\left(\tfrac{1}{2}n, \tfrac{1}{2}(1-n); 1+d-n; 2\right),$$

where ${}_2\tilde{F}_1(a, b; c; z)$ is a regularized hypergeometric function.

A good approximation to the number of people n such that $p = P_k(n, d)$ is some given value can be given by solving the equation

$$n\, e^{-n/(d\,k)} = \left[d^{k-1}\, k!\, \ln\left(\frac{1}{1-p}\right)\left(1 - \frac{n}{d\,(k+1)}\right)\right]^{1/k}$$

for n and taking $\lceil n \rceil$, where $\lceil n \rceil$ is the ceiling function . For $p = 0.5$ and $k = 1$, 2, 3, ..., this formula gives $n = 1$, 23, 88, 187, 313, 459, 622, 797, 983, 1179, 1382, 1592, 1809, ... (OEIS A050255), which differ from the true values by from 0 to 4. A much simpler but also poorer approximation for n such that $p = 0.5$ for $k < 20$ is given by

$$n = 47\,(k-1.5)^{3/2}$$

, which gives 86, 185, 307, 448, 606, 778, 965, 1164, 1376, 1599, 1832, ... for $k = 3$, ω_1 and ω_2 are labeled such that $\mathrm{I}[\tau] \equiv \mathrm{I}[\omega_2/\omega_1] > 0$, where $\mathrm{I}[z]$ is the imaginary part.

Conclusion

The solutions are Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ or, in terms of hypergeometric functions, as

$$y(x) = C_1\, {}_2F_1\left(-n, n+1+\alpha+\beta, 1+\alpha, \tfrac{1}{2}(x-1)\right)$$
$$+2^{\alpha}(x-1)^{-\alpha}\, C_2\, {}_2F_1\left(-n-\alpha, n+1+\beta, 1-\alpha, \tfrac{1}{2}(1-x)\right).$$

The equation (2) can be transformed to

$$\frac{d^2u}{dx^2} + \left[\frac{1}{4}\,\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\,\frac{1-\beta^2}{(1+x)^2} + \frac{n(n+\alpha+\beta+1)+\frac{1}{2}(\alpha+1)(\beta+1)}{1-x^2}\right]u = 0,$$

where

$$u(x) = (1-x)^{(\alpha+1)/2}\,(1+x)^{(\beta+1)/2}\,P_n^{(\alpha,\beta)}(x),$$

A "cell" of an elliptic function is defined as a parallelogram region in the complex plane in which the function is not multi-valued. Properties obeyed by elliptic functions include

1. The number of poles in a cell is finite.

2. The number of roots in a cell is finite.

3. The sum of complex residues in any cell is 0.

4. Liouville's elliptic function theorem: An elliptic function with no poles in a cell is a constant.

5. The number of zeros of $f(z) - c$, this is impossible.

7. Elliptic functions with a single pole of order 2 with complex residue 0 are called Weierstrass elliptic functions. Elliptic functions with two simple poles having residues a_0 and $-a_0$ are called Jacobi elliptic functions.

8. Any elliptic function is expressible in terms of either Weierstrass elliptic function or Jacobi elliptic functions.

9. The sum of the affixes of roots equals the sum of the affixes of the poles.

10. An algebraic relationship exists between any two elliptic functions with the same periods.

The elliptic functions are inversions of the elliptic integrals. The two standard forms of these functions are known as Jacobi elliptic functions and Weierstrass elliptic functions. Jacobi elliptic functions arise as solutions to differential equations of the form

$$\frac{d^2 x}{d t^2} = A + B x + C x^2 + D x^3,$$

References

1. Analysis Bourbaki, Nicolas , Topological vector spaces, Elements of mathematics, Berlin, New York: Springer-Verlag, ISBN 978-3-540-13627-9
2. Bourbaki, Nicolas , Integration I, Berlin, New York: Springer-Verlag, ISBN 978-3-540-41129-1
3. Braun, Martin , Differential equations and their applications: an introduction to applied mathematics, Berlin, New York: Springer-Verlag, ISBN 978-0-387-97894-9
4. BSE-3 [1994], "Tangent plane", Encyclopedia of Mathematics, EMS Press
5. Choquet, Gustave , Topology, Boston, MA: Academic Press

6. Dennery, Philippe; Krzywicki, Andre , Mathematics for Physicists, Courier Dover Publications, ISBN 978-0-486-69193-0
7. Dudley, Richard M. , Real analysis and probability, The Wadsworth & Brooks/Cole Mathematics Series, Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software, ISBN 978-0-534-10050-6
8. Dunham, William , The Calculus Gallery, Princeton University Press, ISBN 978-0-691-09565-3
9. Evans, Lawrence C. , Partial differential equations, Providence, R.I.: American Mathematical Society, ISBN 978-0-8218-0772-9
10. Folland, Gerald B. , Fourier Analysis and Its Applications, Brooks-Cole, ISBN 978-0-534-17094-3
11. Gasquet, Claude; Witomski, Patrick , Fourier Analysis and Applications: Filtering, Numerical Computation, Wavelets, Texts in Applied Mathematics, New York: Springer-Verlag, ISBN 978-0-387-98485-8
12. Ifeachor, Emmanuel C.; Jervis, Barrie W. , Digital Signal Processing: A Practical Approach , ISBN 978-0-201-59619-9
13. Krantz, Steven G. , A Panorama of Harmonic Analysis, Carus Mathematical Monographs, Washington, DC: Mathematical Association of America, ISBN 978-0-88385-031-2
14. Kreyszig, Erwin , Advanced Engineering Mathematics , Introductory functional analysis with applications, Wiley Classics Library, New York: John Wiley & Sons, ISBN 978-0-471-50459-7, MR 0992618
15. Lang, Serge , Real analysis, Addison-Wesley, ISBN 978-0-201-14179-5
16. Lang, Serge , Real and functional analysis, Berlin, New York: Springer-Verlag, ISBN 978-0-387-94001-4
17. Loomis, Lynn H. , An introduction to abstract harmonic analysis, Toronto-New York–London: D. Van Nostrand Company, Inc., pp. x+190, hdl:2027/uc1.b4250788
18. Narici, Lawrence; Beckenstein, Edward . Topological Vector Spaces. Pure and applied mathematics . Topological Vector Spaces. GTM. 8 , Topological vector spaces, distributions and kernels, Boston, MA: Academic Press
19. Banach, Stefan , "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales , Betrachtungen über einige Gegenstände der Elementargeometrie , Éléments d'histoire des mathématiques , "A general outline of the genesis of vector space theory", Historia Mathematica, 22 : 227–261, doi:10.1006/hmat.1995.1024, MR 1347828
20. Fourier, Jean Baptiste Joseph , Théorie analytique de la chaleur , Die Lineale Ausdehnungslehre - Ein neuer Zweig der Mathematik , Lectures on Quaternions, Royal Irish Academy

21. Möbius, August Ferdinand , Der Barycentrische Calcul : ein neues Hülfsmittel zur analytischen Behandlung der Geometrie , "The axiomatization of linear algebra: 1875–1940", Historia Mathematica, 22 : 262–303, doi:10.1006/hmat.1995.1025
22. Peano, Giuseppe , Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle Operazioni della Logica Deduttiva Formulario mathematico: vct axioms via Internet Archive
23. Kleiner, Bruce; Lott, John . "Notes on Perelman's papers". Geometry & Topology. 12 : 2587–2855. arXiv:math/0605667. doi:10.2140/gt.2008.12.2587. MR 2460872. S2CID 119133773.
24. Huai-Dong Cao; Xi-Ping Zhu . "Hamilton-Perelman's Proof of the Poincaré Conjecture and the Geometrization Conjecture". arXiv:math.DG/0612069.
25. Morgan, John W.; Tian, Gang . Ricci Flow and the Poincaré Conjecture. Clay Mathematics Monographs. 3. Providence, RI: American Mathematical Society. arXiv:math/0607607. ISBN 978-0-8218-4328-4. MR 2334563.: Detailed proof, expanding Perelman's papers.
26. O'Shea, Donal . The Poincaré Conjecture: In Search of the Shape of the Universe. Walker & Company. ISBN 978-0-8027-1654-5.
27. Perelman, Grisha . "The entropy formula for the Ricci flow and its geometric applications". arXiv:math.DG/0211159.
28. Perelman, Grisha . "Ricci flow with surgery on three-manifolds". arXiv:math.DG/0303109.
29. Perelman, Grisha . "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds". arXiv:math.DG/0307245.
30. Szpiro, George . Poincaré's Prize: The Hundred-Year Quest to Solve One of Math's Greatest Puzzles. Plume. ISBN 978-0-452-28964-2.
31. Stillwell, John . "Poincaré and the early history of 3-manifolds". Bulletin of the American Mathematical Society. 49 : 555–576. doi:10.1090/S0273-0979-2012-01385-X. MR 2958930.