Common Fixed Point Theorems for Faintly Compatible and Subsequently Continuous Mappings in Cone Metric Space

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ABSTRACT: In this paper, we prove common fixed point theorems for faintly compatible and subsequently continuous mappings satisfying general contractive condition in cone metric spaces. Our result extend and generalize results of Badshah et.al. [4].

Key words. Cone metric space, faintly compatible, coincidence point, fixed point.

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1. INTRODUCTION

Haung and Zhang [7] have introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space and they showed some fixed point theorems of contractive type mapping on cone metric spaces. Many authors study this subject and many fixed-point theorems are proved. For example [1, 6, 8, 10, 13]. In partial cone metric spaces, the self-distance for any point need not equal to zero. Specially, from the point of sequences, a convergent sequence need not have unique limit. A common fixed-point theorem for commuting mappings gave Jungck [9], which generalizes the Banach's fixed point theorem and he also introduced the concept of compatible maps which is weaker than weakly commuting maps. Further this result was generalized by Pant [11], Amari and Moutawakil [3], Al-Thagafi and Shahzad [2]. In [2] author defend the concept of occasionally weakly compatible which is more general than the concept of weakly compatible maps. Pant et. al. [15] introduced the concept of conditional compatible maps. Faintly compatible maps introduced by Bisht and Shahzad [14] as an improvement of conditionally compatible maps. In this paper, we generalize the result of Badshah et. al. [4].

Definition 1.1. [16] Let E be a real Banach space and P be a subset of E. P is called a cone if
(a) P is closed, nonempty and $P \neq \{0\}$;
(b) $a, b \in R$, $a\, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering "$\leq$" with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $-x \in P^0$, where P^0 stands for the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose E is a Banach space, P is a cone in E with $P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.2 [16] A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying :

(a) 0< d (x, y) for all x, y ∈X and d (x, y) = 0 if and only if x = y,
(b) d (x, y) = d (y, x) for all x, y X
(c) d (x, y) ≤ d (x, z) + d (z, y) for all x ,y ,z∈ X.

Definition 1.3. [16] Let (X, d) be a cone metric space $\{x_n\}$ a sequence in X and x ∈ X. If for any c ∈ E with c >> 0,there is N such that for all n>N, $d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converge to x, i.e. $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as n →∞.

Definition 1.4. [16] Let (X, d) be a cone metric space $\{ x_n \}$ a sequence in X, if for any c∈ E with c>>0, there is N such that for all n, m >N, $d(x_m, x_n) << c$, then $\{ x_n \}$ is called Cauchy sequence in X.

Lemma 1.1. [16] Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{ x_n \}$ and $\{ y_n \}$ be two sequences in X and $y_n \to y, x_n \to x$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Lemma 1.2. [16]Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{ x_n \}$ be a sequence in X. Then $\{ x_n \}$ converge to x if and only if $d(x_n , x) \to 0$ as $n \to \infty$.

Lemma 1.3. [16] Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{ x_n \}$ be a sequence in X. Then $\{ x_n \}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Definition 1.5. [11] Let X be a set and let f, g be two self-mappings of X. A point x in X is called a coincidence point of f and g if and only if fx = gx, we shall call w = fx = gx a point of coincidence point.

Definition 1.6.[11] Two self-maps f and g of a set X are occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 1.7. The pair (f; g) is said to be faintly compatible iff (f; g) is conditionally compatible and (f; g) commutes on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

Lemma 1.4. [12] Let X be a set f, g owc self-mappings of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

2. Main Result.

Theorem 2.1. Let (X, d) be a complete cone metric space and P be a normal cone. Let A, B, S, T, P, Q: X→X be mappings such that

(i) P(X)⊂ST(X) and Q(X)⊂AB(X)

(ii) The pairs (P, AB) and (Q, ST) are faintly compatible and subsequently continuous and satisfying the following condition

$$d(Px, Qy) \leq a_1 d(PSx,QTy)+a_2 d(Ax,PSx)+a_3 d(By,QTy)+a_4 d(PSx,By) + a_5 d(Ax,QTy) \quad (1)$$

for all $x,y \in X$ and $\varphi : R_+ \to R_+$ continuous. Then A, B, S, T, P and Q have unique common fixed point.

Proof: Since the pairs (P, AB) are faintly compatible and subsequently continuous, then there exist sequence $\{z_n\}$ in X where $\lim_{n\to\infty} P(z_n) = \lim_{n\to\infty} AB(z_n) = u$ for some $u \in X$ such that

$$\lim_{n\to\infty} M(ABP(z_n), PAB(z_n), t) = 1.$$

As (P, AB) sub sequentially continuous, we have

$$\lim_{n\to\infty} P(z_n) = u \Rightarrow \lim_{n\to\infty} AB(Pz_n) = ABu \quad \text{and}$$

$$\lim_{n\to\infty} ABz_n = u \Rightarrow \lim_{n\to\infty} P(ABz_n) = Pu.$$

Since

$$\lim_{n\to\infty} M(ABPz_n, PABz_n, t) = 1$$

$$ABu = Pu.$$

Since (Q, ST) is faintly compatible and subsequently continuous then there exist sequence $\{z_n'\}$ in X

$$\lim_{n\to\infty} Q(z_n') = \lim_{n\to\infty} ST(z_n') = w \text{ for some } w \in X \text{ such that}$$

$$\lim_{n\to\infty} M(Q(STz_n'), ST(Qz_n'), t) = 1.$$

As (Q, ST) subsequently continuous, we have

$$\lim_{n\to\infty} Qz_n{}' = w \Rightarrow \lim_{n\to\infty} ST(Qz_n{}') = STw$$

$$\lim_{n\to\infty} STz_n{}' = w \Rightarrow \lim_{n\to\infty} Q(STz_n{}') = Qw.$$

Since,

$$\lim_{n\to\infty} M(QSTz_n{}', STQz_n{}', t) = 1$$

Qw = STw.

Since pairs (*P*, *AB*) and (*Q*, *ST*) are faintly compatible, we have

Pu = ABu

and PPu = PABu = ABPu = ABABu. (2)

And Qw = STw

and QQw = STQw = QSTw = STSTw . (3)

Since P(X) ⊂ST(X), there exist a point w ∈ X such that

Pu = STw. (4)

Also, Q(X) ⊂AB(X), there exist a point u ∈ X such that

Qw = ABu. (5)

Now we claim that Pu is the unique common fixed point of P and AB. First, we assert that Pu is a fixed point of P. If PPu $\neq$ Pu, then by (1), we have

d (PPu, Pu) = d (PPu, ABu)

= d (PPu, Qw)

$\leq$ a_1d (ABPu, STw) + a_2 d (PPu, ABPu) + a_3 d (Qw, STw)

+ a_4 d (ABPu, Qw) + a_5 d (PPu, STw)

= a_1d (ABPu, STw) + 0 + 0+ a_4 d (PPu, Qw) + a_5 d (PPu, Qw)

= a_1d (PPu, Pu) + a_4 d (PPu, Pu) + a_5 d (PPu, Pu)

= ($a_1 + a_4 + a_5$) d (PPu, Pu)

which is a contradiction. Hence Pu is a fixed point of P.

By (2) Pu is a common fixed point of P and AB.

Now we claim that Qw is the unique common fixed point of Q and ST and we assert that Qw is a fixed point of Q.

If QQw $\neq$ Qw, then by (1), we have

d (QQw, Qw) < d (QQw, Qw)

which is a contradiction. Hence Qw is a fixed point of Q.

By (3) Qw is a common fixed point of Q and ST.

Now by (2) and (5) we have

Pu = ABu = Qw.

Hence, Pu = Qw is a common fixed-point A, B, S, T, P and Q.

For the uniqueness, let v be another common fixed point of A, B, S, T, P and Q.

Let Pu = ABu = u and Qv = STv = v.

If u $\neq$ v, then from (6), we have

d (u, v) = d (Pu, Qv)

$\leq$ a_1d (ABu, STv) + a_2 d (Pu, ABu) + a_3 d (Qv, STv)

+ a_4 d (ABu, Qv) + a_5 d (Pu, STv)

= a_1d (u, v) + 0 + 0 + a_4 d (u, v) + a_5 d (u, v)

= ($a_1 + a_4 + a_5$)d(Pu, STV), a contradiction.

Therefore, u = v. Hence A, B, S, T, P and Q have unique common fixed point

Theorem 2.2. Let (X, d) be a complete cone metric space and P be a normal cone. Let A, B, S, T, P, Q: X→X be mappings such that

(i) P(X)⊂ST(X) and Q(X)⊂AB(X)

(ii) The pairs (P, AB) and (Q, ST) are faintly compatible and subsequently continuous, and satisfying the following condition,

d (Px, Qy) $\leq \varphi$ (g (x, y)) (6)

where g (x, y) = d (ABx, STy) + γ (d (ABx, Px) + d (STy, Qy)), for all x, y $\in$ X and φ :R+→R+ continuous.

Then A, B, S, T, P and Q have unique common fixed point.

Proof. Since the pairs (P, AB) are faintly compatible and subsequently continuous, then there exist sequence $\{z_n\}$ in X, where $\lim_{n\to\infty} P(z_n) = \lim_{n\to\infty} AB(z_n) = u$ for some $u \in X$ such that

$$\lim_{n\to\infty} M(ABP(z_n), PAB(z_n), t) = 1.$$

As (P, AB) sub sequentially continuous, we have

$$\lim_{n\to\infty} P(z_n) = u \Rightarrow \lim_{n\to\infty} AB(Pz_n) = ABu \text{ and}$$

$$\lim_{n\to\infty} ABz_n = u \Rightarrow \lim_{n\to\infty} P(ABz_n) = Pu,$$

Since,

$$\lim_{n\to\infty} M(ABPz_n, PABz_n, t) = 1$$

$$ABu = Pu.$$

(Q,ST) is faintly compatible and subsequently continuous then there exist sequence $\{z_n'\}$ in X,

$\lim_{n\to\infty} Q(z_n') = \lim_{n\to\infty} ST(z_n') = w$ for some w ∈ X such that

$$\lim_{n\to\infty} M(Q(STz_n'), ST(Qz_n'), t) = 1$$

As (B, QT) is subsequently continuous, we have

$$\lim_{n\to\infty} Qz_n' = w \Rightarrow \lim_{n\to\infty} ST(Qz_n') = STw$$

$$\lim_{n\to\infty} STz_n' = w \Rightarrow \lim_{n\to\infty} Q(STz_n') = Qw.$$

Since

$$\lim_{n\to\infty} M(QSTz_n', STQz_n', t) = 1$$

Qw = STw.

Since pairs (P, AB) and (Q, ST) are faintly compatible, we have

Pu = ABu

and PPu = PABu = ABPu = ABABu. (7)

And Qw = STw

and QQw = STQw = QSTw = STSTw. (8)

Since P(X) ⊂ST(X), there exist a point w ∈ X such that

Pu = STw. (9)

Also, Q(X) ⊂AB(X), there exist a point u ∈ X such that

Qw = ABu. (10)

Now we claim that Pu is the unique common fixed point of P and AB. First, we assert that Au is a fixedpoint of P. If PPu ≠Pu, then by (6), we have

d (PPu, Pu) = d (PPu, ABu)
= d (PPu, Qw)
≤ φ (g (Pu, u))
≤ φ (d (ABPu, STw) + γ [d (ABPu, PPu) +d (STw, Qw)])
≤ φ (d (PPu, Pu) + γ [0+0])
≤φ (d (PPu, Pu))
<d (PPu, Pu)

which is a contradiction. Hence, Pu is a fixed point of P. By (7) Pu is a common fixed point of P and B.

Now we claim that Qw is the unique common fixed point of Q and T and we assert that Qw is a fixed point of Q.

If QQu≠Qu, then by (6), we have

d(QQw, Qw)<d(QQw, Qw)

which is a contradiction.

Hence, Qw is a fixed point of Q.

By (8), Qw is a common fixed point of Q and ST.

Now by (7) and (10), we have

Pu = ABu = Qw.

Hence, Pu = Qw is a common fixed-point A, B, S, T, P and Q.

For the uniqueness, let v be another common fixed point of A, B, S, T, P and Q.

Let Pu = ABu = u and Qv = STv = v.

If $u \neq v$, then from (6), we have

$$
\begin{aligned}
d(u, v) &= d(Pu, Qv) \\
&\leq \varphi(g(u, v)) \\
&= \varphi(d(ABu, STv) + \gamma[d(ABu, Pu) + d(STv, Qv)]) \\
&= \varphi(d(Pu, Qv) + \gamma[0+0]) \\
&= \varphi(d(u, v)), \text{ a contradiction.}
\end{aligned}
$$

Therefore, $u = v$. Hence A, B, S, T, P and Q have unique common fixed point. Then A, B, S and T have unique common fixed point.

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