A Technique for Fractional Programs with Restriction on Variables

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Abstract : This research study deals with very special class of single ratio fractional programs.The objective function considered in tjhis study is ratio of non linear functions each separable in nature..Numerator as well as denominator both are transformed to linear function by incorporating/using.the piecewise linear approximation method via grid point and thereby a linear fractional progam is determined which is converted/formulated into a linear program by making use of Charne's and Cooper transformation method. In the course of action, the variables associated being assumed to be bounded i.e. are finite valued. The procedure is explained stepwise by an algorithm and is explained with help of an illustrative example..

Index Terms: Piecewise linear approximation,;Optimization.Grid points search method.;Separability.;Affine Functios,;Linear programming.

I. INTRODUCTION

Consider the separable fractional programming problem

$$\text{(SFPP)} \quad \min \frac{U(x)}{V(x)} = \min \frac{\sum_{j=1}^{n} u_j(x_j) + \alpha}{\sum_{j=1}^{n} v_j(x_j) + \beta}.$$

$$= \min \frac{\sum_{j=1}^{n} p_j x_j^2 + \sum_{j=1}^{n} q_j x_j + \alpha}{\sum_{j=1}^{n} r_j x_j^2 + \sum_{j=1}^{n} s_j x_j + \beta} \qquad (1)$$

$$\text{subject to} \qquad \sum_{j=1}^{n} g_{ij}(x_j) \le b_i \quad \text{for } i = 1, 2, \ldots, m$$

where $u(x)$ and $v(x)$ are separable non-linear functions of x with the condition that either p_j and r_j are both zero or non-zero; g_{ij} are linear functions of x $(1 \le i \le m,\ 1 \le j \le n)$, $b_i \in \mathbb{R}^n$, $v(x)$ is positive on the constraints set

S = $\{ x_i \in \mathbb{R}^m \ : \ g_{ij}(x_j) \le b_i, 1 \le i \le m, 1 \le j \le n \}$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

S is assumed to be a non-empty convex polyhedron.

Fractional Programminghas been dealt by many authors [1.2.4.5.6.7] Charne's et al. reformulated linear fractional programming problem into a linear programming problem by applying. the transformation. Fractional programs arise in various circumstances/cases: in management science as well as other areas. Maximization of productivity, maximization of return on investment, maximization of cost/time give rise a fractional program.

Nonlinear programming problems have gained great importance since they arie in many fields like financial analysis of firms,selection problem;cutting stock problem; stochastic processing.Single ratio fractional programs generally appeared in the literature in 1960s. Much work has been carried out on theory,classification and applications in this regard. Single ratio fractional programs have been stated in the monographs by Craven[6].A model has been presented by Chang[4] in which auxilliary constraints have been used to linearize the mixed 0–1 fractional programming problem. Here we use approximation technique for finding the solution so as to overcome the complexities/infeasibility of the problem. For solving large scale problems, approximation techniques are employed.The problem presented in this paper is non-concave fractional program.A number of methods of concave programming are available for finding solution by transforming a non-concave fractional program into concave program. Portfolio selection problems and stochastic decision making problems [3.8,9,10] are non-concave fractional programs.

2. METHODOLOGY AND THEORETICAL FRAMEWORK

The following definitions are employed:

Definition 1: **"**Non-linear programs where the objective functions and the constraint functions can be expressed as the sum of functions, each involving only one variable, are called Separable Programs."

Definition 2:" Let f be a real valued function defined on a convex set S in $\mathbb{R}^m$. The function f is said to be strictly convex on S if

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall\ x_1, x_2 \in S \text{"}$$

and for each $\lambda \in (0, 1)$.

$v = 1, \ldots, p_j,$

While carrying the solution process, the property of linear functions that it is both concave and convex in nature is used.

3.Reformulation into Linear Program:

Consider the problem

$$\text{(SFPP)} \quad \min \frac{\sum_{j=1}^{n} u_j(x_j)+\alpha}{\sum_{j=1}^{n} v_j(x_j)+\beta} = \min \frac{\sum_{j=1}^{n} p_j x_j^2 + \sum_{j=1}^{n} q_j x_j + \alpha}{\sum_{j=1}^{n} r_j x_j^2 + \sum_{j=1}^{n} s_j x_j + \beta} . \qquad (1)$$

$$\text{subject to } \sum_{j=1}^{n} g_{ij}(x_j) \le b_i \quad \text{for } i = 1, 2, \ldots, m;$$

$x_j \geq 0$ and bounded for $1 \le j \le n$ where either both "p_j" and "r_j "are zero or non-zero.

Let $T = \{j : u_j \text{ and } v_j \text{ are linear}\}$

For $j \notin T$, let "u_j" and "v_j". be strictly convex and g_{ij} linear for i = 1, 2, ...,m.

Assuming for each $j \notin T$, u_j, v_j .g_{ij}. For i = 1, ..,m are replaced by their piecewise linear approximation via grid points

x_{vj} for v = 1, ..., p_j, obtain the fractional program

$$\text{(L/FP)} \quad \min \frac{\sum_{j\in t}^{n} u_j(x_j) + \sum_{j\notin T} \sum_{v=1}^{b_j} u_j(x_{vj}) + \alpha}{\sum_{j\in T} v_j(x_j) + \sum_{j\notin T} \sum_{v=1}^{b_j} v_j(x_{vj}) + \beta}$$

$$\text{subject to } \sum_{j=1}^{n} g_{ij}(x_j) + \sum_{j\notin T} \sum_{v=1}^{p_j} \lambda_{vj} g_{ij}(x_{ij}) \le b_i , \quad \text{for } i = 1,2, \ldots, -m$$

$$\lambda_{vj} \ge 0 \quad \text{for } v = 1, 2, \ldots, p_j \quad j \notin T$$

$$x_j \ge 0 \quad \text{and bounded for} \quad j \in T$$

with atmost two adjacent λ_{vj}'s are positive for $j \notin T$.

The above problem is a linear fractional program with the exception that almost two adjacent λ_{vj}'s are positive for $j \notin T$

By Charne'sCooper transformation[5],

$$t = \frac{1}{\sum_{j\in T} v_j(x_j) + \sum_{j\notin T} \sum_{v=1}^{p_j} \lambda_{vj} v_j(x_{vj}) + \beta}$$

$$y_j = t\ x_j, \quad j \in L ,$$

this fractional program reduces to the linear programming problem

$$\text{(L/PP)} \quad \sum_{j\in T} u_j(y_j) + \sum_{j\notin T} \sum_{v=1}^{p_j} \lambda'_{vj} u_j(x_{vj}) + \alpha t$$

subject to

$$\sum_{j\in T} g_{ij}(y_j) + \sum_{j\notin T} \sum_{v=1}^{p_j} \lambda'_{vj} g_{ij}(x_{vj}) - t b_i \le 0, \quad i = 1, \ldots. m$$

$$\sum_{v=1}^{p_j} \lambda'_{vj} - t = 0$$

$$\sum_{j\in T} v_j(y_j) + \sum_{j\notin T} \sum_{v=1}^{p_j} \lambda'_{vj} v_j(x_{vj}) + \beta t = 1$$

$$y_j \ge 0 \quad \text{for} \quad j \in T$$

$$\lambda'_{vj} \ge 0 \quad \text{for} \quad v = 1, \ldots, p_j, j \notin T$$

Let the solution to (LPP) be $\hat{x}_j$ *for* $j \in T$ *and* $\hat{\lambda}_{vj}$ for V = 1, ,.., p_j, $j \notin T$.

We shall now prove the following:

Theorem 1: Let atmost two λ_{vj}'s be positive for $j \notin T$ .Then they must be adjacent.

Proof: If $\hat{\lambda}_{ij}$ and $\hat{\lambda}_{\rho_j}$ are positive for each $j \notin T$, it only remains to show that x_{ij} and x_{ρ_j} must be adjacent.

Let x_{ij} and x_{ρ_j} be not adjacent for $\hat{\lambda}_{ij} > 0$.

Therefore, there exists a grid point $x_{\rho_j} \in (x_{ij}, x_{\rho_r})$ which is expressible as

$x_{\delta_j} = \alpha_1 x_{ij} + \alpha_2 x_{\rho_j}$

where $\alpha_1, \alpha_2 > 0$ s.t. $\alpha_1 + \alpha_2 = 1$

Consider the optimal solution of (LPP).

Let for each $j \notin T, m_j$ be the optimum Lagrangian multiplier associated with the constraint $\sum_{v=1}^{p_j} \lambda'_{vj} = 1$ $l_i \geq 0$ for $i = 1, \ldots, m$ be the optimum, Lagrangian multipliers associated with the first on constraints and nj be the optimum Lagrangian multiplier associated with the constraint

$$\sum_{j \in T} v_j(y_j) + \sum_{j \notin T} \sum_{v=1}^{p_j} \lambda'_{vj} v_j(x_{vj}) + \beta t = 1 \qquad (4)$$

Hence,a subset of the Kuhn-Tucker necessary conditions are satisfied

i.e.

$$\sum_{j \notin T} u_j(x_{ij}) + \sum_{i=1}^{m} l_i g_{ij}(x_{ij}) + m_j + \sum_{j \notin T}^{p_j} n_j v_j(x_{ij}) = 0 \qquad (a)$$

$$\sum_{j \notin T} u_j(x_{\rho_i}) + \sum_{i=1}^{m} l_i g_{\rho j}(m_j) + n_j + \sum_{j \notin T} n_j v_j(x_{\rho j}) = 0 \qquad (b)$$

$$\sum_{j \notin T} u_j(x_{vj}) + \sum_{i=1}^{m} l_i g_{ij}(x_{ij}) + \sum_{j \notin T} n_j v_j(x_{vj}) \geq 0 \qquad (c)$$

Since u_j and v_j are strictly convex in nature, it is observed that condition (c) is not satisfied for ν = γ.

Conditions "(a) and (b)" imply that

$$\sum_{j \in T} u_j(x_{ij}) + \sum_{i=1}^{m} l_i g_{ij}(x_{vj}) + m_j + \sum_{j \notin T}^{p_j} n_j v_j(x_{vj}) <$$

$$\sum_{j \notin T} \alpha_i u_j(x_{ij}) + \alpha_2 u_j(x_{\rho_j}) + \sum_{i=1}^{m} l_i(\alpha_1 g_{ij}(x_{ij}) + \alpha_2 g_{ij}(x_{\rho j})$$

$$+ \sum_{j \notin T} n_j(\alpha_1 v_j(x_{ij}) + \alpha_2 v_j(x_{p_j}) + m_j = 0 \qquad (5)$$

which is a contradiction to condition (c) for $\nu = \gamma$. So, our assumption that x_{ij} and $x_{\rho j}$ are not adjacent stands wrong.

Hence, x_{ij} and $x_{\rho j}$ are adjacent and the theorem stands proved.

4. ALGORITHMIC DEVELOPMENT

Step 0 : Consider (SFPP) defined above

Step 1 : Consider the set T.

Step 2 : Reduce the given (SFPP) to (LFP) by replacing u_j, v_j and g_{ij} with their piecewise linear approximations for i = 1, ...,m and $j \notin T$.

Step 3 : By using Charne's and cooper transformation, solve (LFP) by finding optimal solution of the corresponding (LPP) problem. The solution so obtained is optimal to (SFPP).

5. ILLUSTRATION

Consider the following (FP)

(FP) .. min $\dfrac{x_1^2 + 2x_1^2 + x_4^2 - 3x_1 + 2x_3 + x_4 + 2}{x_1^2 + x_2^2 + x_4^2 + x_1 + 2x_2 + 1}$;;.

subject to .

$4x_1 + x_2 + x_3 + x_4 \leq 5.2$;

$$2.2x_1 + x_2 - x_3 \le 3.2;$$
$$3 - x_1 + x_2 + x_3 \le 1.5;$$
$$6.1x_1 - x_3 \le 2.1;$$
$$4.1x_1 - x_2 + x_3 + 2x_4 \le 4.1;$$
$$0 \le x_1 \le 4.6;$$
$$0 \le x_2 \le 3.4;$$
$$0 \le x_3 \le 3.5;$$
$$0 \le x_4 \le 4.6;$$

Here "p_j and r_j" are both non-zero; g_{ij} are linear functions of x $(1 \le i \le m,\ 1 \le j \le n)$ ..

Since non-linear terms appear involving x_3, So $T = \{3\}$ and no grid points need to considered for x_3.;

Also, x_1, x_2 and x_4 lie in interval [0, 4].

Take the grid points 0, 2 and 4.0 for the variables x_1, x_2 and x_4

$$.\ x_{11} = 0 \quad x_{21} = 2 \quad x_{41} = 4.0;$$
$$.\ x_{12} = 0 \quad x_{22} = 2 \quad x_{42} = 4.0;$$
$$.\ x_{14} = 0 \quad x_{24} = 2 \quad x_{44} = 4.0;$$

Hence., $x_1 = 0\lambda_{11} + 2\lambda_{21} + 4\lambda_{41} = 2\lambda_{21} + 4\lambda_{41};$

$$x_2 = 0\lambda_{12} + 2\lambda_{22} + 4\lambda_{42} = 2\lambda_{22} + 4\lambda_{42};$$
$$x_4 = 0\lambda_{14} + 2\lambda_{24} + 4\lambda_{44} = 2\lambda_{24} + 4\lambda_{44};$$
$$\lambda_{11} + \lambda_{21} + \lambda_{41} = 1.0;$$
$$\lambda_{12} + \lambda_{22} + \lambda_{42} = 1.0;$$
$$\lambda_{14} + \lambda_{24} + \lambda_{44} = 1.0;$$
$$\lambda_{\nu 1},\ \lambda_{\nu 2},\ \lambda_{\nu 4} \ge 0, \text{ for } \nu = 1,2,4$$

So. piecewise linear approximation of

$$u(x) = (-2\lambda_{21} + 4\lambda_{41}) + (8\lambda_{22} + 32\lambda_{42}) + (6\lambda_{24} + 20\lambda_{44}) - 2x_3 + 2.0; \quad (7)$$

and the piecewise linear approximation of

$$v(x) = (6\lambda_{21} + 20\lambda_{41}) + (8\lambda_{22} + 24\lambda_{42}) + (4\lambda_{24} + 16\lambda_{44}) + 1.0; \quad (8)$$

Therefore, (FP) gets reduced to the following Linear Fractional Program

$$\min \frac{(-2\lambda_{21} + 4\lambda_{41}) + (8\lambda_{22} + 32\lambda_{42}) + (6\lambda_{24} + 20\lambda_{44}) - 2x_3 + 2}{(6\lambda_{21} + 20\lambda_{41}) + (8\lambda_{22} + 24\lambda_{42}) + (4\lambda_{24} + 16\lambda_{44}) + 1}.$$

subject to

$$(2\lambda_{21} + 4\lambda_{41}) + (2\lambda_{22} + 4\lambda_{42}) + x_3 + (2\lambda_{24} + 4\lambda_{44}) \le 5.0;$$
$$(2\lambda_{21} + 4\lambda_{41}) + (2\lambda_{22} + 4\lambda_{44}) - x_3 \le 3.0;$$
$$-(2\lambda_{21} + 4\lambda_{41}) + (2\lambda_{22} + 4\lambda_{44}) + x_3 \le 1.0;$$
$$(2\lambda_{21} + 4\lambda_{41}) - x_3 \le 1.0;$$
$$(2\lambda_{21} + 4\lambda_{41}) - (2\lambda_{22} + 4\lambda_{42}) + x_3 + (4\lambda_{24} + 8\lambda_{44}) \le 4.0;$$
$$\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{41} = 1;$$
$$\lambda_{12} + \lambda_{22} + \lambda_{32} + \lambda_{42} = 1;$$
$$\lambda_{14} + \lambda_{24} + \lambda_{34} + \lambda_{44} = 1;$$
$$\lambda_{\nu j} \ge 0 \quad \nu = 1,2,3,4; \quad j = 1,2,4; \quad 0 \le x_3 \le 3;$$

Atmost two λ_{vj}'s are positive : $j \notin T$. The condition two λ_{vj}'s are positive for $j \notin T$ is relaxed, above problem get reduced to linear fractional program..

By using Charne's Coopper transformation

$$t = \frac{1}{(6\lambda_{21} + 20\lambda_{41}) + (8\lambda_{22} + 24\lambda_{42}) + (4\lambda_{24} + 16\lambda_{44}) + 1},$$

(LPP) $\min(-2\lambda'_{21} + 4\lambda'_{41}) + (3\lambda'_{22} + 32\lambda'_{42}) + (6\lambda'_{24} + 20\lambda'_{44}) - 2y_3 + 2t$

subject to

$$(2\lambda'_{21}+4\lambda'_{41})+(4\lambda'_{22}+4\lambda'_{42})+y_3+(2\lambda'_{24}+4\lambda'_{44})-5t\le 0;$$

$$(2\lambda'_{21}+4\lambda'_{41})+(2\lambda'_{22}+4\lambda'_{44})-y_3-3t\le 0; \quad (10)$$

$$-(2\lambda'_{21}+4\lambda'_{41})+(2\lambda'_{22}+4\lambda'_{44})+y_3-t\le 0;$$

$$(2\lambda'_{21}+4\lambda'_{41})=y_3-2t\le 0;$$

$$(2\lambda'_{21}+4\lambda'_{41})-(2\lambda'_{22}+4\lambda'_{42})+y_3+(4\lambda'_{24}+8\lambda'_{44})-4t\le 0;$$

$$\lambda'_{11}+\lambda'_{21}+\lambda'_{31}+\lambda'_{41}-t=0;$$

$$\lambda'_{12}+\lambda'_{22}+\lambda'_{32}+\lambda'_{42}-t=0;$$

$$\lambda'_{14}+\lambda'_{24}+\lambda'_{34}+\lambda'_{44}-t=0;$$

$$\lambda'_{vj}\ge 0 \quad \text{for } v=1,2,3,4; \;\; j=1,2,4;$$

$$y_3-3t\le 0;$$

$$y_3\ge 0;$$

The optimal solution to the problem stated above is: $(\lambda'_{11}, \lambda'_{21}, \lambda'_{31}, \lambda'_{41},\lambda'_{12}, \lambda'_{22}, \lambda'_{32}, \lambda'_{42}\;\; \lambda'_{14}, \lambda'_{24}, \lambda'_{34}, \lambda'_{44}, t)=1$

(0.05, 0,0,0.14,0.18,0,0,0,0.45,0,0,0,0,0.18)

INFERENCES: While developingmethodology for solving problem, a fractional programming problem is considered in which the numerator & denominator are separable functions which in turn are replaced by their corresponding piecewise linear approximations by making use of grid punts. The parent problem is finally reduced to linear fractional program excluding the constraint at most, two λ_{vj} 's are positive for $j \notin T$. The Linear Fractional Program is then reduced to a linear program by using the Charne's Cooper transformation.

7. REFERENCES

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