TOPOLOGICAL VECTOR SPACES

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Abstract

This article deals with the concept of Topological Vector Spaces with an example and the definitions required for this. Also we will study continuous mapping on topological spaces.

Keywords

Cartesian product, Induced, Metric, Norm, Set, Topology etc.

Introduction

To understand the concept of topological vector spaces, we require following concepts….

(Definition-1)**VECTOR SPACE:** A non empty set A is called a Vector space or Linear space over a Field (F,+,·) if A is an Abelian group under an operation which we denote '+'and if for every a∈F , α∈A there is defined an element aαεA subjected to

1. a(α+β) = aα+aβ

2. (a+b)α = aα+bα

3. a(bα) = (ab)α

4. 1α = α For all a,bεF and α,βεA , where 1 represents the unit element of F under multiplication.

(Definition-2)**TOPOLOGY :** A Topology τ defined over a non empty set X is the collection of subsets of set X satisfies the following axioms…

[A1] X and Φ belong to τ.

[A2] The union of any family of sets in τ belongs to τ.

[A3] The intersection of any two sets in τ belongs to τ.

The set X together with the Topology τ is called Topological Spaces and written as the pair (X,τ) or simply by X.

(Definition-3)**Metric space:** Let X be a non-empty set. A metric on X, denoted by d is defined as the mapping from the Cartesian product X×X into the set of real numbers R satisfying the following conditions…….

(M1) $d(x,y) \geq 0$ for every x,y∈X. This is called non-negative restriction.

(M2) $d(x,y)=0$ if and only if x=y for x,y∈X.

(M3) $d(x,y) = d(y,x)$ for every x,y∈X. This is called symmetric property of d. and

(M4) $d(x,y) + d(y,z) \geq d(x,z)$ for every x,y,z∈X. This is called triangle inequality of d.

The set X together with the metric d is called metric space and denoted by the pair (X,d) or simply X. There can always be defined a topology on X induced by the metric d called natural topology or usual topology on X hence a metric space is always a topological space.

(Definition-4)NORM : For a given linear space E, over a field F , a norm on E is defined as a map $x \to \|x\|$ from X into the set R^+ of non negative real numbers which satisfies the following axioms ….

(N1) $\|x\|=0$ iff $x=0$ (N2) $\|\lambda x\|=|\lambda|\cdot\|x\|$ $\forall\ \lambda \in F,\ x \in X$. (N3) $\|x+y\| \le \|x\|+\|y\|$ $\forall\ x,y \in X$.

A linear space on which a norm is defined is called normed linear space or simply normed space. There can always be defined a metric d induced by norm $\|\ \|$ given as $d(x,y)=\|x-y\|$ for every $x,y \in X$. And definition-3 implies that a topology can be induced by a metric, therefore, every norm space is a topological space.

Property-1 $\|x\|-\|y\| \le \|x-y\|$ for very $x,y \in X$.

Since $\|x\|=\|(x-y)+y\| \le \|x-y\|+\|y\|$ (by axiom N3), implies that $\|x\|-\|y\| \le \|x-y\|$.

(Definition-5)CONTINUOUS MAPPING ON TOPOLOGICAL SPACES: Let us consider X and Y are two Topological spaces and f :X→Y. The mapping f is said to be continuous at a point $x \in X$ if for each neighborhood V of y= f (x) in Y, f^{-1} (V) is a neighborhood of x in X. f is said to be continuous on X into Y if f is continuous at each $x \in X$.

Sequentially (in view of metric spaces), f is continuous if for $x \in X$ every sequence (x_n) in X converging to x , the sequence $(f(x_n))$ in Y converging to $f(x)$ i.e. $x_n \to x \Rightarrow f(x_n) \to f(x)$. i.e. if d_1 & d_2 are metrices in X and Y respectively then for ϵ>0 there exists δ>0 such that $d_1(x_n,x)<\delta \Rightarrow d_2(f(x_n),f(x))<\varepsilon$.

Theorem-1 **Norm function is a continuous function.**

Solution: Let X is a normed space with the norm $\|\ \|$ from X into R. Let the metric induced by norm in X is $d_1=\|x-y\|$ and $d_2=\|\|x\|-\|y\|\|$ be the usual metric in R. Now let (x_n) be a sequence in the normed space X such that $x_n \to x$ in X. Then for the mapping $\|\ \|$, we find that

$\|x_n\|-\|x\| \le \|x_n-x\|$ (by property-1), which implies that $\|\|x_n\|-\|x\|\| \le d_1(x_n,x) \to 0$ as $x_n \to x$ in X. Therefore, we get $d_2(\|x_n\|,\|x\|)=\|\|x_n\|-\|x\|\| \to 0$ then definition-5 implies that norm $\|\ \|$ is a continuous mapping.

(Definition-6)TOPOLOGICAL VECTOR SPACES : Let E is a vector space over a field K(real or complex) and a topology τ is defined on it. The set E is called a **topological vector space** if the maps

(I) $(x,y) \to x+y$ from $E \times E \to E$ and (II) $(\lambda,x) \to \lambda.x$ from $K \times E \to E$ are continuous and then it is abbreviated by TVS. The topology defined on $E \times E$ is the product topology $\tau \times \tau$ and the topology defined on $K \times E$ is the product topology $\mu \times \tau$ where μ is the usual topology defined on the field K.

Theorem-2 Let E is a normed vector space over a field K then the maps (i) $(x,y) \to x+y$ from E×E into E. and (ii) $(\lambda,x) \to \lambda x$ from K×E into E. are continuous mapping.

Solution: Let <(x,y)> be a sequence in the space E×E converging to a point (a,b) in E×E which implies that x→a and y→b in E for which taking $\|x-a\|<\varepsilon/2$ and $\|y-b\|<\varepsilon/2$, then we have

$\|(x+y)-(a+b)\|=\|(x-a)+(x-b)\| \le \|x-a\|+\|y-b\|<\varepsilon$ (by property N3 of definition-4)

Which implies that $(x+y) \to (a+b)$. So the mapping $(x,y) \to x+y$ from E×E into E is continuous.

Now let $(\lambda, x) \to (\alpha, a)$ in the space K×E which implies that $\lambda \to \alpha \,\&\, x \to a$ in their respective spaces. Such that

$\|x-a\| < \dfrac{\varepsilon}{2|\lambda|} \,\&\, |\lambda - \alpha| < \dfrac{\varepsilon}{2\|a\|}$, then we have

$\|\lambda x - \alpha a\| = \|\lambda x - \lambda a + \lambda a - \alpha a\| \le \|\lambda(x-a)\| + \|a(\lambda - \alpha)\| \le |\lambda|\|x-a\| + \|a\||\lambda - \alpha| < \varepsilon$ i.e. $\lambda x \to \alpha a$ so the mapping $(\lambda, x) \to \lambda . x$ from $K \times E \to E$ is continuous.

Example-1 A normed vector space equipped with the topology defined by its norm is a topological vector space.

Solution: Let E is a normed vector space over a field K and d is the metric induced by the norm $\| \;\|$ on E given as $d(x, y) = \|x - y\|$ and then a topology τ is induced by d on E. Now from theorem-2, we find that the maps

(i) $(x, y) \to x + y$ from E×E into E. and (ii) $(\lambda, x) \to \lambda x$ from K×E into E. are continuous mapping.

Therefore by definition-6, E is a topological vector space.

result

For defining the concept of a TVS, we need the concept of vector space, topology and continuous mapping in topological spaces. Example-3 is an example of a TVS for which we need the concept of norm.

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