TOPOLOGICAL TERMS: A SINGLE EXAMPLE

Md. Jahid

Assistant Professor(Guest) A Research Scholar

U P College, Pusa, Samastipur BRA Bihar University

(A constituent unit of LNMU Darbhanga) Muzaffarpur

Abstract

We study topology over a non-empty set and define topological terms like open set, closed set, neighborhood at a point, adherent point, accumulation point, isolated point, interior, exterior, boundary, derived set and closure of a set, dense set, connected and disconnected sets, separated sets in a topological space. Also illustrate a single example which defines all the topological terms.

Keywords

Basis, Clopen, Intersection, Isolated Point, Sub-basis, Union, Perfect set.

Introduction

A **topology** $_1$ τ over a non-empty set X is defined as a class O of subsets of X satisfying the conditions

1. X and the empty set ϕ are in O.
2. Intersection of any finite family of O is in O.
3. Union of any arbitrary family of O is in O.

Then the set X together with the topology τ is called **topological space** $_2$ denoted by (X,τ) or simply by X. The sets in the collection O are called **open sets** $_3$ of X. A subset A in of X is **closed** $_4$ if it's complement A^c is open in X. A subset of X may be both open and closed set called clopen set. The set X itself and empty set ϕ are always **clopen** $_5$ sets in a topological space (X,τ).

A subclass β of class O is called a **basis** $_6$ of topology τ if every open set of X is a union of sets of β and a subclass γ of class O is called **sub-basis** $_7$ of topology τ if the finite intersections of γ forms a basis for the topology τ.

A set N in topological space X is called **neighbourhood** $_8$ (simply written by nhd.) of a point $x \in X$, if there exists an open set O_x containing point x such that $x \in O_x \subseteq N$. Superset of a nhd. of x is also a nhd. of x.

Theorem-1 A set A in a topological space (X,) is open iff A is a nhd. Of each of it's points.

Proof: A is a set in a topological space (X,τ).

Necessity: let A is open in X, then for every point x of A, $x \in A \subseteq A$, therefore A is nhd. of x but x is arbitrary so A is a nhd. of each of its points.

Sufficiency: let A is a nhd. of each of it's points then for every point $x \in A$, ∃ open set O_x containing point x such that $x \in O_x \subseteq A$ which implies that $A \subseteq \bigcup O_x \subseteq A \Rightarrow A = \bigcup_{x \in A} O_x$ but condition (3) of topology, $\bigcup_{x \in A} O_x$ is open in X hence A is open in X. Which proves the theorem.

For a set A in X, a point $x \in X$ is called an **adherent point** $_9$ of set A if every nhd. of x contains at least one point of A. The collection of all adherent points of set A is called **adherence or closure** $_{10}$ of set A denoted by $\overline{A}$. Closure of set A can also be defined as the intersection of all closed sets contains the set A i.e. $\overline{A}$ is the smallest closed set contains the set A. If A contains at least a single element in every nhd. of x other than the point x, then x is called a **limit point or accumulation point** $_{11}$ of set A. The collection of all limit points of A is called **derived set** $_{12}$ of A and denoted by A'. Thus for $x \in X$ and every nhd. N_x of x ; $x \in \overline{A}$ iff $N_x \bigcap A \neq \phi$ and $x \in$A' iff $\{N_x - x\} \bigcap A \neq \phi$.

Theorem-2 For any set A in a topological space (X,), $\overline{A} = A \bigcup A'$ **.**

Proof: From def. (9) It is obvious that $A \subseteq \overline{A}$ and also $x \in$A'$\Rightarrow x \in \overline{A}$, $\therefore A' \subset \overline{A}$ and hence $A \bigcup A' \subseteq \overline{A}$ ..... (1)

Aim to prove A∪A' is closed, let p∈ $(AUA')^c$ ⇒ p∉(AUA') ⇒ p∉A and p∉A' ⇒p∉A and ∃ an open set P containing point p such that [P- {p}]∩A = ϕ ⇒ P∩A=ϕ ⇒ P⊆A^c ⇒p∈P⊆$(AUA')^c$ ⇒$(AUA')^c$ is a nhd. of point p but p is arbitrary therefore $(AUA')^c$ is nhd. of each of it's points hence $(AUA')^c$.

Now $(AUA')^c$ is open implies that AUA' is closed thus from def. (4) $\overline{A} \subseteq A \bigcup A'$ ……….(2)

From result (1) and result (2) we get the desired $\overline{A} = A \bigcup A'$.

For a set A in X, a point $x \in A$ is called an **interior point** $_{13}$ of set A if A is a nhd. of point x i.e. there exists an open set O_x containing point x such that $x \in O_x \subseteq A$. The collection of all interior points of set A is called **interior** $_{14}$ of set A and denoted by A^o. It is obvious that A^o is the union of all open sets contained in set A hence A^o is the largest open set contained in set A. A point x∈X is called **exterior point** $_{15}$ of set A if x is an interior point of A^c and it's collection is called **exterior** $_{16}$ of set A, denoted by ext.A . Thus ext.A = $(A^c)^o$. A point x∈X is called **frontier point** $_{17}$ of A if x is neither an interior point nor an exterior point and its collection is called **frontier** $_{18}$ denoted by front.A . A point x∈A is called **boundary point** $_{19}$ of set A if x∈front.A whose collection is called **boundary** $_{20}$ of set A.

A subset A of X is said to be **dense** $_{21}$ in X if $\overline{A} = X$ and called **nowhere dense** $_{22}$ if int($\overline{A}$) =ϕ i.e. $(\overline{A})^c$ is dense in X.The set A is said to be **perfect** $_{23}$ if A'=A and a point x∈X is called **isolated point** $_{24}$ of set A if x∈ $\overline{A}$ but x∉A' therefore A subset A of X is called perfect if A is closed and has no isolated point (by theorem-2).

Two sets A and B in topological space X are said to be **separated** $_{25}$ if A and B are disjoint and also the derived set of any one is disjoint with the other i.e. A and B satisfy i) A∩B=ϕ . ii) $\overline{A} \bigcap B$ =ϕ . iii) $A \bigcap \overline{B}$ =ϕ .

A subset A of X is said to be **disconnected** $_{26}$ if there exist two open sets G and H in X such that A⊂GUH and the sets A∩G and A∩H are disjoint, then GUH is called disconnection of set A. The set A is said to be **connected** $_{27}$ if it is not disconnected .

Example-1 Let X={a,b,c,d} and τ is a class of subsets of X given as τ={ϕ, X, {a}, {a,b}, {c,d}, {a,c,d}}.Show that τ is a topology on X and then define all terms of topology by taking suitable subset of X.

Solution: In the given class τ we observe that the all three conditions of def.(1) are satisfied hence τ ia a topology on set X and with this topology , (X,τ) is a topological space.

i) Open sets in X are ϕ,X, {a}, {a,b}, {c,d} and {a,c,d}.
ii) Closed sets in X are ϕ^c ,X^c ,$\{a\}^c$,$\{a,b\}^c$,$\{c,d\}^c$ and $\{a,c,d\}^c$ i.e. X, ϕ, {b,c,d}, {c,d} and {b}.
iii) Let a set A={a,b,c} in X, obviously a,b abd c are adherent points. Now open sets containing d are X, {c,d} and {a,c,d} , none of them is disjoint from A, therefore d is also an adherent point of A . hence by def.(10) $\overline{A}$ ={a,b,c,d}=X so by def.(21) A is dense in X.

iv) Since a belongs to open set {a} and ({a}-{a})∩A=ϕ , and c belonfs to open set {c,d} and ({c,d}-{c})∩A=ϕ therefore a and c are not accumulation points and b and d are accumulation points of A therefore A'={b,d}. Since a,c∈ $\overline{A}$ but a,c∉A' so a and c are isolated points by def.(24) and hence by def.(23) A is not a perfect set.

v) By def.(13) A^o ={a,b}. Since A^c ={d} which has no interior points so ext.A=ϕ and hence by def.(18) front.A={ c,d} and c∈A, so by def.(19) c is only the boyndary point of A.

vi) Let P and Q are two open sets given as P={a,b} and Q={c,d} . Now we found that $\overline{P}=\{c,d\}$ and $\overline{Q}=\{a,b\}$ then i) P∩Q=ϕ, ii) $\overline{P}\bigcap Q=$ ϕ, iii) $P\bigcap\overline{Q}$ =ϕ . so by def.(25) the sets P and Q are separable in X hence the set PUQ=X is disconnected by def.(26).

vii) A∩P={a,b} and A∩Q={c} then (A∩P)∩(A∩Q)=ϕ and A⊂PUQ, hence by def.(26) A is disconnected and PUQ=X is disconnection of set A.

Result

All topological terms can be defined by a single example which shows that all topological terms are related in which open sets make important and independent role to define others.

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