Divergence on Dimensional Space of Cylindrical Coordinates: A Study

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Abstract

This paper attempts to study how the divergence of a **vector field** is intuitively, vector field $\dlvf$ gives the velocity of some fluid flow,with **Divergence on dimensional space** appears that the fluid is exploding outward from the origin of cylindrical coordinates. The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Divergence is a vector operator that operates on a vector field, producing a scalar field giving the quantity of the vector field's source at each point. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. As an example, consider air as it is heated or cooled. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value. While the air is cooled and thus contracting, the divergence of the velocity has a negative value. The divergence of a vector field is often illustrated using the example of the velocity field of a fluid, a liquid or gas. A moving gas has a velocity, a speed and direction, at each point which can be represented by a vector, so the velocity of the gas forms a vector field. If a gas is heated, it will expand. This will cause a net motion of gas particles outward in all directions. Any closed surface in the gas will enclose gas which is expanding, so there will be an outward flux of gas through the surface. So the velocity field will have positive divergence everywhere. Similarly, if the gas is cooled, it will contract. There will be more room for gas particles in any volume, so the external pressure of the fluid will cause a net flow of gas volume inward through any closed surface.

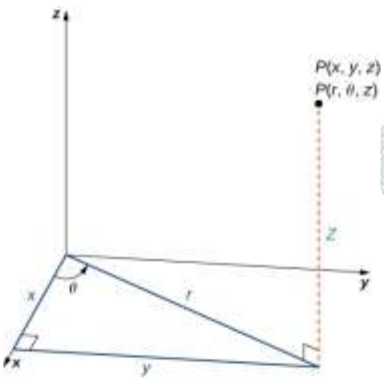
Key words: divergence, polar coordinate, vector field, integral, cylindrical coordinate.

Introduction

In the cylindrical coordinate system, a point in space ((Figure)) is represented by the ordered triple integral where

- are the polar coordinates of the point's projection in the *xy*-plane
- is the usual in the Cartesian coordinate system

The right triangle lies in the *xy*-plane. The length of the hypotenuse is and is the measure of the angle formed by the positive *x*-axis and the hypotenuse. The *z*-coordinate describes the location of the point above or below the *xy*-plane.



The divergence theorem, more commonly known especially in older literature as Gauss's theorem (e.g., Arfken 1985) and also known as the Gauss-Ostrogradsky theorem, is a theorem in vector calculus that can be stated as follows. Let V be a region in space with boundary ∂V. Then the volume integral of the divergence $\nabla \cdot \mathbf{F}$ of $\mathbf{F}$ over V and the surface integral of $\mathbf{F}$ over the boundary ∂V of V are related by

$$\int_V (\nabla \cdot \mathbf{F})\, d\, V = \int_{\partial V} \mathbf{F} \cdot d\, \mathbf{a}. \qquad (1)$$

The divergence theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.

A special case of the divergence theorem follows by specializing to the plane. Letting S be a region in the plane with boundary ∂S, equation (1) then collapses to

$$\int_S \nabla \cdot \mathbf{F}\, d\, A = \int_{\partial S} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\, s. \qquad (2)$$

If the vector field $\mathbf{F}$ satisfies certain constraints, simplified forms can be used. For example, if $\mathbf{F}(x, y, z) = v(x, y, z)\, \mathbf{c}$ where $\mathbf{c}$ is a constant vector $\neq \mathbf{0}$, then

$$\int_S \mathbf{F} \cdot d\,\mathbf{a} = \mathbf{c} \cdot \int_S v\,d\,\mathbf{a}. \quad (3)$$

But

$$\nabla \cdot (f\,\mathbf{v}) = (\nabla f) \cdot \mathbf{v} + f\,(\nabla \cdot \mathbf{v}), \quad (4)$$

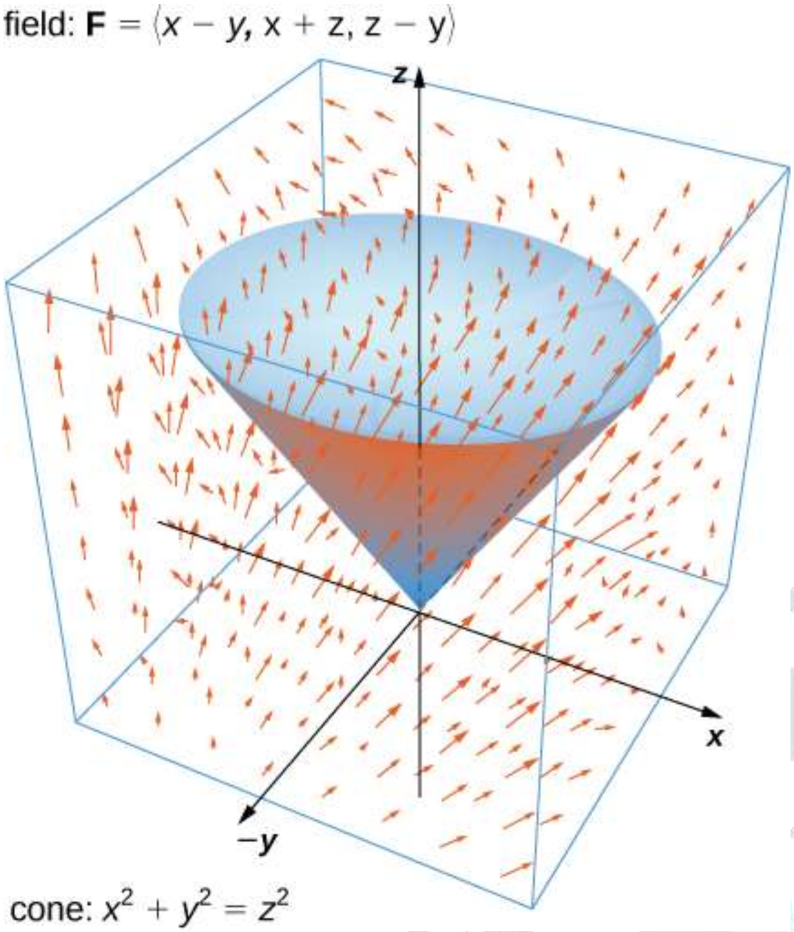
so

$$\int_V \nabla \cdot (\mathbf{c}\,v)\,d\,V = \int_V [(\nabla v) \cdot \mathbf{c} + v\,\nabla \cdot \mathbf{c}]\,d\,V \quad (5)$$

$$= \mathbf{c} \cdot \int_V \nabla v\,d\,V$$

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative divF over a solid to a flux integral of F over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field F over a closed surface *S* to a triple integral of the divergence of F over the solid enclosed by *S*. The divergence theorem translates between the flux integral of closed surface *S* and a triple integral over the solid enclosed by *S*. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

- The divergence theorem relates a surface integral across closed surface *S* to a triple integral over the solid enclosed by *S*. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.



Divergence of a flux means the excess of the outward flux over the inward flux through any closed surface per unit volume. For electric flux density *D*, the ∇ • *D* means

∫sD•n→ds=∫v∇•DdV=∫ρdV=0 if Q(the net charge)is located outside the surface S=Q if Q is inside the volume V enclosed by the surface S.

If the region enclosed by the surface *S* has a net charge *Q* that is equal to ∫ρ*dV*, then we can write

∇•D=ρ

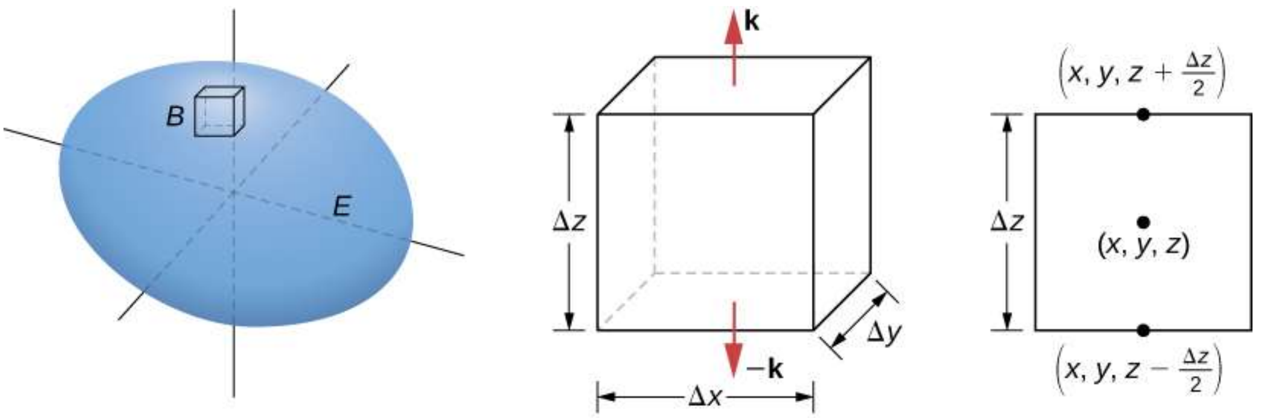
Objective:

This paper intends to explore and analyze **vector field** and the flow out of spheres are closely related. if a fluid is expanding (i.e., the flow has positive **divergence** everywhere inside the sphere), the net flow is out of a **spherical/cylindrical coordinates**

Divergence of a vector field

The divergence of a vector field $\mathbf{F}$, denoted $\text{div}\,(\mathbf{F})$ or $\nabla \cdot \mathbf{F}$ (the notation used in this work), is defined by a limit of the surface integral

$$\nabla \cdot \mathbf{F} \equiv \lim_{V \to 0} \frac{\oint_S \mathbf{F} \cdot d\,\mathbf{a}}{V} \tag{1}$$



(a) (b) (c)

where the surface integral gives the value of $\mathbf{F}$ integrated over a closed infinitesimal boundary surface $S = \partial V$ surrounding a volume element V, which is taken to size zero using a limiting process. The divergence of a vector field is therefore a scalar field. If $\nabla \cdot \mathbf{F} = 0$, then the field is said to be a divergenceless field. The symbol ∇ is variously known as "nabla" or "del."

The physical significance of the divergence of a vector field is the rate at which "density" exits a given region of space. The definition of the divergence therefore follows naturally by noting that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or out of the region. By measuring the net flux of content passing through a surface surrounding the region of space, it is therefore immediately possible to say how the density of the interior has changed. This property is fundamental in physics, where it goes by the name "principle of continuity." When stated as a formal theorem, it is called the divergence theorem, also known as Gauss's theorem. In fact, the definition in equation (1) is in effect a statement of the divergence theorem.

For example, the continuity equation of fluid mechanics states that the rate at which density ρ decreases in each infinitesimal volume element of fluid is proportional to the mass flux of fluid parcels flowing away from the element, written symbolically as

$$\nabla \cdot (\rho\, \mathbf{u}) = -\frac{\partial \rho}{\partial t}, \tag{2}$$

where $\mathbf{u}$ is the vector field of fluid velocity. In the common case that the density of the fluid is constant, this reduces to the elegant and concise statement

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

which simply says that in order for density to remain constant throughout the fluid, parcels of fluid may not "bunch up" in any place, and so the vector field of fluid parcel velocities for any physical system must be a divergenceless field.

Maxwell equations

Divergence is equally fundamental in the theory of electromagnetism, where it arises in two of the four Maxwell equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

where MKS units have been used here, $\mathbf{E}$ denotes the electric field, ρ is now the electric charge density, ϵ_0 is a constant of proportionality known as the permittivity of free space, and $\mathbf{B}$ is the magnetic field. Together with the two other of the Maxwell equations, these formulas describe virtually all classical and relativistic properties of electromagnetism.

A formula for the divergence of a vector field can immediately be written down in Cartesian coordinates by constructing a hypothetical infinitesimal cubical box oriented along the coordinate axes around an infinitesimal region of space. There are six sides to this box, and the net "content" leaving the box is therefore simply the sum of differences in the values of the vector field along the three sets of parallel sides of the box. Writing $\mathbf{F} = (F_x, F_y, F_z)$, it therefore following immediately that

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{6}$$

This formula also provides the motivation behind the adoption of the symbol $\nabla \cdot$ for the divergence. Interpreting ∇ as the gradient operator $\nabla = (\partial / \partial x, \partial / \partial y, \partial / \partial z)$, the "dot product" of this vector operator with the original vector field $\mathbf{F} = (F_x, F_y, F_z)$ is precisely equation (6).

While this derivative seems to in some way favor Cartesian coordinates, the general definition is completely free of the coordinates chosen. In fact, defining

$$\mathbf{F} \equiv F_1 \, \hat{\mathbf{u}}_1 + F_2 \, \hat{\mathbf{u}}_2 + F_3 \, \hat{\mathbf{u}}_3, \tag{7}$$

the divergence in arbitrary orthogonal curvilinear coordinates is simply given by

$$\nabla \cdot \mathbf{F} \equiv \frac{1}{h_1 \, h_2 \, h_3} \left[\frac{\partial}{\partial u_1} (h_2 \, h_3 \, F_1) + \frac{\partial}{\partial u_2} (h_3 \, h_1 \, F_2) + \frac{\partial}{\partial u_3} (h_1 \, h_2 \, F_3) \right]. \tag{8}$$

The divergence of a linear transformation of a unit vector represented by a matrix A is given by the elegant formula

$$\nabla \cdot \frac{\mathsf{A}\,\mathbf{x}}{|\mathbf{x}|} = \frac{\mathrm{Tr}(\mathsf{A})}{|\mathbf{x}|} - \frac{\mathbf{x}^{\mathrm{T}}(\mathsf{A}\,\mathbf{x})}{|\mathbf{x}|^3}, \tag{9}$$

where $\mathrm{Tr}(\mathsf{A})$ is the matrix trace and $\mathbf{x}^{\mathrm{T}}$ denotes the transpose.

The concept of divergence can be generalized to tensor fields, where it is a contraction of what is known as the covariant derivative, written

$$\nabla \cdot A \equiv A^{\alpha}_{;\alpha}.$$

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. Green's Theorem says these are equal, or roughly, that the sum of the "microscopic" swirls over the region is the same as the "macroscopic" swirl around the boundary.

Conclusion

divergence allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to electrostatic fields. An important result in this subject is Gauss' law. This law states that if *S* is a closed surface in electrostatic field **E**, then the flux of **E** across *S* is the total charge enclosed by *S* (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin. if there is a single point charge at the origin. In this case, Gauss' law says that the flux of **E** across *S* is the total charge enclosed by *S*. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if *S* is a piecewise smooth closed surface and *Q* is the total amount of charge inside of *S*, then the flux of **E** across *S* is

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