THE FREDHOLM ALTERNATIVE AND COMPACT OPERATORS ON BANACH SPACES

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ABSTRACT

In this paper, we prove two interesting cases of the Fredholm alternative on Banach spaces, one is that for any compact operator T & identity operator I on a Banach spaces X with a non-zero complex number λ, the injectivity & surjectivity of (T- λ I) are equivalent and the other one is that either (T- λ I) is bijective or has non-trivial Kernel and nontrivial Co-Kernel of the same dimension.

Keywords: Fredholm alternative, compact operator, Banach space, Kernel &Co Kernel.

1. INTRODUCTION

The Fredholm alternative named after Ivar Fredholm is one of Fredholm's Theorems and is a result in Fredholm theory. It is as follows: If K is a compact operator on a Banach space X, then either the homogeneous equation possesses a non-trivial solution or the inhomogeneous equation (T- λ I) x = y has for every y $\in$ X, unique solution x $\in$ X for $\lambda \neq 0$ and the identity map I on X.

Fredholm (1900, 1903) treated compact operators as limiting case of finite rank operators .Riesz (1917) defined and made direct use of the compactness condition, more opt for Banach spaces.

Definition (1.1)

A continuous linear operator T: X $\rightarrow$ Y with Banach spaces X&Y, is compact if T maps bounded sets in X to pre-compact sets in Y that is sets with compact closure.

We recall that a finite rank operator is one with finite- dimensional image and is clearly compact .If T is compact then Ker(T- λ I) is finite dimensional.

Proposition (2.1)

If T is a compact operator on a Banach space X then injectivity & surjectivity of (T- λ I) are equivalent with identity operator I & non-zero $\lambda \in C$.

Proof: -

Suppose (T- λ I) is injective. Let V_n = (T- λ I)^n X. Since images of Banach spaces under (T- λ I) for compact operator T & $\lambda \neq 0$ are closed by induction these are closed subspaces of X. For x $\notin$ (T- λ I) X and any y $\in$ X,

(T- λ I)^n_x - (T- λ I) (T- λ I) $^{n+1}_y$

= (T- λ I)^n (x-(T (- λ I)$_y$)

Injectivity of (T- λ I) implies injectivity of (T- λ I)^n ,so this is not o. That is, (T- λ I)^n_x $\notin$ (T- λ I)$^{n+1}_X$

.Thus the claim of subspaces V$_n$ is strictly decreasing. Take v$_n$ $\in$ V$_n$ such that $|$ v$_n$ $|$=1and away from say by $\inf_{y \in V_{n+1}} |$ V$_n$ -y $| \geq \frac{1}{2}$

The effect of T is T$_{V_m}$ -T$_{V_{m+n}}$ $= \lambda V_m$ + (T- λ I)$_{V_m}$ -T$_{V_{m+n}}$ $\in \lambda I_{V_m} + V_{n+1}$ (integer m≥ 1& n≥ 1)

Since V $_{m+1}$ is T-stable. Thus $\left| \; T_{V_m} - T_{V_{m+n}} \; \right| \rangle |\lambda| . \frac{1}{2}$

This is impossible .Since compact T maps the bounded set $\{V_n\}$ to a pre-compact set. Thus, the claim of subspaces V $_n$ cannot be strictly decreasing, and have surjectivity (T- λ I) X = X.

On the other hand suppose (T- λ I) is surjective. Then the adjoint (T- λ I)$*$ is injective .Since adjoints of compact operators are compact, we already know that (T- λ I)* is surjective. Then (T- λ I)** is injective. The natural inclusion X $\to$ X $**$ shows that (T- λ I) is a restriction of (T- λ I)**,

So (T- λ I) is necessarily injective.

Theorem (2.1):

If K is a compact operator on a Banach Space X then dim Ker (T- λ I)=dim Co Ker(T- λ I) for identity map I & $\lambda \neq 0$

The above theorem is the Fredholm alternative for operators (T- λ I) with T compact &I the identity map with $\lambda \neq 0$: either (T- λ I) is bijective ,or has non-trivial Co-Kernal of the same dimension.

Proof:-As we know from the property of compact operators, the compactness of T implies the finite dimensionality of Ker (T- λ I) for $\lambda \neq 0$ &I , the identity map on X. For y,...,y $_n$ $\in$ X linearly independent modulo (T- λ I) X ,by Hahn-Banach theorem, there are η_1 ,..., η_n $\in$ X$*$ vanishing on the image (T- λ I)X and η_i (y $_j$)= ∂_{ij} .Such η_i are in the kernel of the adjoint (T- λ I)$*$ .We know T* is compact so Ker(T- λ I)$*$ is finite dimensional.

We have proved that injectivity& surjectivity of (T- λ I) are equivalent ,and that the Kernal & CoKernal areb finite dimensional.

Let x $_1$, ..., x $_m$ (with m ≥ 1) xpan the Kernal , and let y $_1$,...,y $_n$ (with n ≥ 1) span the Co Kernal,and show that m = n.

For m $\leq$ n , Let X$'$ be a closed complementary subspace to the kernel of (T- λ I).Let F be the finite rank operator which is o on X$'$ and Fx $_i$ = y $_i$.

The adjusted operator T$'$ =T+F is compact. For (T$'$ - λ I) x = 0 ,

(T- λ I) x = F x $\in$ (T- λ I) X $\cap$ span y $_1$,..., y $_n$ = $\{0\}$.

That is (T$'$ - λ I) is injective, so is surjective ,so m = n , In the opposite case m $\geq$ n, let F x $_i$ = y $_i$ for i $\leq$ n and F x $_i$ = y $_n$ for i $\geq$ n. With T$'$ =T+F again.In this case (T$'$ - λ I) is surjective,so is injective,and m = n.

Conclusion

Hence, T is a compact operator on a Banach space X then injectivity & surjectivity of (T- λ I) are equivalent with identity operator I & non-zero $\lambda \in C$ and K is a compact operator on a Banach Space X then dim Ker (T- λ I)=dim Co Ker(T- λ I) for identity map I & $\lambda \neq 0$

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