FIXED POINT RESULTS WITH $(\psi, \varphi) - (\alpha, \beta, \gamma)$ WEAK CONTRACTIVE MAPPINGS

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Abstract: In fixed point theory, the study of common fixed points under various types of contractive mappings is the centre for vigorous research activity. In this direction, J.R.Morales, E.M.Rojas and R.K.Bisht have introduced the new class of contractive conditions having functions as contractive parameters. The main objective of this paper is to generalize this class of contractive mappings to a new type of weakly contractive mappings and subsequently obtain some fixed point results in metric space which generalizes many recent fixed point results.

Index Terms: **Coincidence point, fixed point, (E.A) property, CLRg property, $(\psi, \varphi) - (\alpha, \beta, \gamma)$ weakly contractive mappings, absorbing maps.**

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1. INTRODUCTION AND PRELIMINARIES

The existence of fixed point was established through Banach-contraction principle, which is the fundamental result of metric fixed point theory. This principle was generalized by Jungck [1] by considering two commutative maps. Since then, many results on the existence of fixed points of various types of contractive mappings under the weaker forms of commutativity like compatibility, R-weak commutativity, weak compatibility etc. have come through by various authors. In this direction R.K.Bisht et al.[5] have introduced the class of $\psi - (\alpha, \beta, \gamma)$ contractive mappings with functions as contractive parameters satisfying the stability condition at zero and obtained results under weak compatibility.

In this paper, the weaker form of $\psi - (\alpha, \beta, \gamma)$ contractive mappings called $(\psi, \varphi) - (\alpha, \beta, \gamma)$ weakly contractive mappings will be introduced and some fixed point theorems will be proved by employing this new form of contractive condition and the notion of absorbing maps[8], which are neither a class of compatible maps nor a subclass of non compatible maps, as they do not enforce commutativity at its coincidence points unlike weakly compatible maps. Thus these results extend and generalizes the result of R.K.Bisht et al.[5] and many more results in the literature.

Definition 1.1. [2] Let f and g be two self maps defined on a metric space (X,d). Then the pair (f,g) is said to satisfy the (E.A) property, if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} \mathrm{g x_n} = t \text{ for some } t \in X.$$

Definition 1.2. [3] Let f and g be two self maps defined on a metric space (X,d). Then the pair (f,g) is said to satisfy the Common Limit in the Range of g (CLRg) property, if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \mathrm{gx_n} = gt \text{ for some } t \in X.$$

Definition 1.3. [8] A pair of self mappings *(f,g)* of a metric space (X, d) is called g-absorbing, if there exist some real number $R > 0$ such that $d(gx, gfx) \le Rd(fx, gx)$ for all x in X .

Definition 1.4. Let (f,g) and (S,T) be two pairs of self mappings of a metric space (X,d). Then we say that (f,g) and (S,T) satisfies the common (E.A) property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \mathrm{gx_n} = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} \mathrm{Ty_n} = t \text{ for some } t \in X.$$

Definition 1.5. [5] Let (X, d) be a metric space and f, g be self maps on X. The pair (f,g) is called $\psi - (\alpha, \beta, \gamma)$ contractive, if $\forall\, x,y \in X$

$$\psi(d(fx, fy)) \le \alpha(d(gx,gy))\psi(d(gx,gy)) + \beta(d(gx,gy))\psi(d(fx,gx)) + \gamma(d(gx,gy))\psi(d(fy, gy)),$$

where $\alpha,\beta,\gamma : [0,\infty) \to [0,1)$ satisfying $\alpha(t)+ \beta(t) + \gamma(t) < 1$ for all $t \in [0,\infty)$ and

$$\limsup_{x\to t^+} \frac{\alpha(x)+\beta(x)}{1-\gamma(x)} < 1 \text{ for all } t \in [0, \infty),$$

$\psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(t_n) \to 0$ implies that $t_n \to 0$.

2. MAIN RESULT

Now the weaker form of $\psi - (\alpha, \beta, \gamma)$ contractive mappings will be defined and by employing this definition some theorems are proved.

Definition 2.1. Let (X,d) be a metric space and f,g be self maps on X. The pair (f,g) is called $(\psi, \varphi) - (\alpha, \beta, \gamma)$ weakly contractive if

$$\psi(d(fx, fy)) \le \alpha(d(gx,gy))[\psi(d(gx,gy)) - \varphi(d(gx,gy))] + \beta(d(gx,gy))[\psi(d(fx,gx)) - \varphi(d(gx,gy))] + \gamma(d(gx,gy))[\psi(d(fy,gy)) - \varphi(d(gx,gy))] \quad \forall\, x,y \in X,$$

where $\alpha,\beta,\gamma : [0,\infty) \to [0,1)$ satisfying $\alpha(t)+ \beta(t) + \gamma(t) < 1$ for all $t \in [0,\infty)$ and

$$\limsup_{x\to t^+} \frac{\alpha(x)+\beta(x)}{1-\gamma(x)} < 1 \text{ for all } t \in [0, \infty),$$

$\psi,\varphi : [0,\infty) \to [0,\infty)$ are both continuous functions such that $\psi(t),\varphi(t) > 0$ for all $t \in (0,\infty)$ and $\psi(0) = \varphi(0) = 0$.

Theorem 2.2. *Let (X,d) be a metric space and f, g be two self mappings on X satisfying (E.A) property. Let the pair (f,g) be a* $(\psi,\varphi)-(\alpha,\beta,\gamma)$ *weakly contractive pair. If* $g(X)$ *is closed, then f and g have a unique point of coincidence. Further, if f is g-absorbing then they have a unique common fixed point.*

Proof: Given that (f,g) satisfies (E.A) property. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} gx_n = t \text{ for some } t \in X.$$

Since $g(X)$ is closed, $t \in g(X)$ which implies $t = gu$ for some u in X.

Now we prove that $t = fu$: Suppose that $t \neq fu$, then consider

$$\psi(d(fx_{n+1}, fu)) \le \alpha(d(gx_{n+1},gu))[\psi(d(gx_{n+1},gu)) - \varphi(d(gx_{n+1},gu))]$$
$$+ \beta(d(gx_{n+1},gu))[\psi(d(fx_{n+1},gx_{n+1})) - \varphi(d(gx_{n+1},gu))]$$
$$+ \gamma(d(gx_{n+1},gu))[\psi(d(fu,gu)) - \varphi(d(gx_{n+1},gu))],$$

letting $n \to \infty$ we obtain $\psi(d(t, fu)) < \psi(d(fu,t))$, a contradiction. Therefore $fu = t = gu$. Thus t is a point of coincidence of f and g.

Now f is g-absorbing, implies there exists $R > 0$ such that

$$d(gx,gfx) \le R\, d(fx,gx) \qquad \text{for all } x \in X. \text{ Since } u \in X$$

we get $fu = gfu$. i.e., fu is a fixed point of g.

Now we prove that $ffu = fu$: Suppose $ffu \neq fu$, then consider

$$\psi(d(fu, ffu)) \le \alpha(d(gu,gfu))[\psi(d(gu,gfu)) - \varphi(d(gu,gfu))]$$
$$+ \beta(d(gu,gfu))[\psi(d(fu,gu)) - \varphi(d(gu,gfu))]$$
$$+ \gamma(d(gu,gfu))[\psi(d(ffu,gfu)) - \varphi(d(gu,gfu))].$$

Thus $\psi(d(fu, ffu)) < \psi(d(ffu, fu))$, a contradiction. Therefore $ffu = fu = gfu$. Hence fu is a common fixed point of f and g.

The uniqueness of the fixed point can be proved easily.

The above theorem is illustrated by the following example.

Example: Let $X = [0,1]$ and d be the usual metric on X. Define $f,g : X \to X$ by $fx = \frac{x}{7}$ and $gx = \frac{x}{2}$ all x in X.

Let $\alpha,\beta,\gamma : [0,\infty) \to [0,1)$ defined by

$$\alpha(t) = \frac{1}{2(t+1)},\ \beta(t) = \frac{t}{10(t+1)},\ \gamma(t) = \frac{1}{10(t+1)}\ \forall\, t \in [0,\infty)$$

Let $\psi,\varphi : [0,\infty) \to [0,\infty)$ be continuous maps defined by $\psi(t) = t^2$ and $\varphi(t) = \frac{t}{2}$

Note that f and g satisfies all the conditions of Theorem 2.2 and have a unique common fixed point at $x = 0$.

Theorem 2.3. *Let (X,d) be a metric space and f, g,S,T be four self mappings on X such that*

(1) (f,g) and (S,T) satisfies common (E.A) property.

(2) gX and TX are closed and

$$\psi(d(fx,Sy)) \le \alpha(d(gx,Ty))[\psi(d(gx,Ty)) - \varphi(d(gx,Ty))]$$
$$+ \beta(d(gx,Ty))[\psi(d(fx,gx)) - \varphi(d(gx,Ty))]$$
$$+ \gamma(d(gx,Ty))[\psi(d(Sy,Ty)) - \varphi(d(gx,Ty))] \qquad \forall\, x,y \in X.$$

Then (f,g) and (S,T) have a unique point of coincidence. Further, if f is g-absorbing and S is T-absorbing, then they have a unique common fixed point.

Proof. Given that (f,g) and (S,T) satisfies common (E.A) property. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \text{ for some } t \in X.$$

Since gX and TX are closed, there exist $u,v \in X$ such that $t = gu = Tv$.

Now we prove that $fu = gu$:

Suppose that $gu \neq fu$, then consider

$$\begin{aligned}\psi(d(fu,Sy_n)) \leq\ & \alpha(d(gu,Ty_n))[\psi(d(gu,Ty_n)) - \varphi(d(gu,Ty_n))] \\ & + \beta(d(gu,Ty_n))[\psi(d(fu,gu)) - \varphi(d(gu,Ty_n))] \\ & + \gamma(d(gu,Ty_n))[\psi(d(Sy_n,Ty_n)) - \varphi(d(gu,Ty_n))],\end{aligned}$$

letting $n \to \infty$ we obtain, $\psi(d(fu, gu)) < \psi(d(fu,gu))$, a contradiction.

Therefore $fu = t = gu$. Thus t is a point of coincidence of f and g. Similarly we can prove that $Tv = Sv = t$. Thus $fu = gu = Sv = Tv = t$.

Now f is g-absorbing, implies there exists $R_1 > 0$ such that

$$d(gx,gfx) \leq R_1\, d(fx,gx) \qquad \text{for all } x \in X.$$

Since $u \in X$ we get $fu = gfu$. i.e., fu is a fixed point of g.

Now we prove that $ffu = fu$: Suppose $ffu \neq fu$, then consider

$$\begin{aligned}\psi(d(ffu,Sv)) \leq\ & \alpha(d(gfu,Tv))[\psi(d(gfu,Tv)) - \varphi(d(gfu,Tv))] \\ & + \beta(d(gfu,Tv))[\psi(d(ffu,gfu)) - \varphi(d(gfu,Tv))] \\ & + \gamma(d(gfu,Tv))[\psi(d(Sv,Tv)) - \varphi(d(gfu,Tv))].\end{aligned}$$

Thus $\psi(d(ffu,fu)) < \psi(d(ffu,fu))$, a contradiction. Therefore $ffu = fu = gfu$.

Hence fu is a common fixed point of f and g. Similarly we can prove that $SSv = TSv = Sv$. Since $fu = Sv = t$, t is the common fixed point of f, g, S and T.

The uniqueness of the fixed point can be proved easily.

Now we present an example to illustrate Theorem 2.3.

Example: Let $X = [0,1]$ and d be the usual metric on X. Define $f,g,S,T : X \to X$ by $fx = \frac{x}{5}$, $gx = x$, $Sx = \frac{x}{10}$ and $Tx = \frac{x}{3}$ all x in X. Let $\alpha,\beta,\gamma : [0,\infty) \to [0,1)$ defined by

$$\alpha(t) = \frac{9}{20},\ \beta(t) = \frac{t}{4} = \gamma(t)\ for \text{ all } t \in [0,\infty)$$

Let $\psi,\varphi : [0,\infty) \to [0,\infty)$ be continuous maps defined by $\psi(t) = t$ and $\varphi(t) = \frac{t}{3}$

Here f,g,S and T satisfies all the conditions of Theorem 2.3 and have a unique common fixed point at $x = 0$.

Theorem 2.4. *Let (X,d) be a metric space and f, g,S be three self mappings on X such that*

(1) (f,g) and (S,g) satisfies Common (E.A) Property.

(2) gX is closed and

$$\begin{aligned}\psi(d(fx,Sy)) \leq\ & \alpha(d(gx,gy))[\psi(d(gx,gy)) - \varphi(d(gx,gy))] \\ & + \beta(d(gx,gy))[\psi(d(fx,gx)) - \varphi(d(gx,gy))] \\ & + \gamma(d(gx,gy))[\psi(d(Sy,gy)) - \varphi(d(gx,gy))] \qquad \forall\, x,y \in X.\end{aligned}$$

Then (f,g) and (S,g) have a unique point of coincidence. Further, if f and S are g-absorbing, then they have a unique common fixed point.

Proof. Put $T = g$ in the proof of Theorem 2.3.

In Theorem 2.3, if we replace the property (E.A) by $CLR_{(g,T)}$ property then the closeness of the range of mappings can be relaxed.

Theorem 2.5. *Let (X,d) be a metric space and f, g, S, T be four self mappings on X satisfying* $CLR_{(g,T)}$ *property and*

$$\psi(d(fx,Sy)) \le \alpha(d(gx,Ty))[\psi(d(gx,Ty)) - \varphi(d(gx,Ty))] + \beta(d(gx,Ty))[\psi(d(fx,gx)) - \varphi(d(gx,Ty))] + \gamma(d(gx,Ty))[\psi(d(Sy,Ty)) - \varphi(d(gx,Ty))] \quad \forall x,y \in X.$$

Then (f,g) and (S,T) have a unique point of coincidence. Further, if f is g-absorbing and S is T-absorbing, then they have a unique common fixed point.

Proof. Given that f, g, S and T satisfies $CLR_{(g,T)}$ property. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \text{ where } t = gu = Tv \text{ for some } t,u,v \in X$$

The rest of the proof follows easily from the proof of Theorem 2.3.

By restricting f, g, S, T suitably, one can derive the corollaries involving two as well as three self mappings as follows:

Corollary 2.6. *Let (X,d) be a metric space and f, g,S be three self mappings on X satisfying* CLR_g *Property and*

$$\psi(d(fx,Sy)) \le \alpha(d(gx,gy))[\psi(d(gx,gy)) - \varphi(d(gx,gy))] + \beta(d(gx,gy))[\psi(d(fx,gx)) - \varphi(d(gx,gy))] + \gamma(d(gx,gy))[\psi(d(Sy,gy)) - \varphi(d(gx,gy))] \quad \forall x,y \in X.$$

Then (f,g) and (S,g) have a unique point of coincidence. Further, if f and S are g-absorbing, then they have a unique common fixed point.

Proof. Follows from Theorem 2.5 by setting $T = g$.

Corollary 2.7. *Let (X,d) be a metric space and f, g be two self mappings on X satisfying* CLR_g *Property and*

$$\psi(d(fx,fy)) \le \alpha(d(gx,gy))[\psi(d(gx,gy)) - \varphi(d(gx,gy))] + \beta(d(gx,gy))[\psi(d(fx,gx)) - \varphi(d(gx,gy))] + \gamma(d(gx,gy))[\psi(d(fy,gy)) - \varphi(d(gx,gy))] \quad \forall x,y \in X.$$

Then (f,g) has a unique point of coincidence. Further if f is g-absorbing then they have a unique common fixed point.

Proof. Follows from Theorem 2.5 by setting $S = f$ and $T = g$.

REFERENCES

[1] Jungck G., 1986, Compatible mappings and common fixed points, Int.J.Math.Math.Sci., 9, 771-779.

[2] M.Aamri, D.El.Moutawakil, 2002, Some new common fixed point theorems under strict contractive conditions, J.Math.Anal.Appl., 270, 181-188.

[3] W.Sintunavarat, P.Kumam, 2011, Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type, J. Inequalities and Appl., Article ID: 3:2011.

[4] Pant R.P., Bisht R.K., Arora D., 2011, Weak reciprocal continuity and fixed point theorems, Ann. Univ.Ferrara, 57(1), 181-190.

[5] Morales J.R., Rojas E.M., Bisht R.K., 2014, Common fixed points for pairs of mapping with variable contractive parameters, Abstract and Applied Analysis, Article ID: 209234, 1-7.

[6] Pant R.P., Bisht R.K., 2012, Occasionally weakly compatible mappings and fixed points, Bull.Belgian Math.Soc, 19(4), 665-661.

[7] Liu Z., Li X., Minkan S., 2011, Fixed point theorems for mappings satisfying contractive conditions of integral type and applications, Fixed Point Theory Appl., Article ID: 64:2011.

[8] Gopal D., Pant R.P., Ranadive A.S., 2008, Common fixed points of absorbing maps, Bull. Marathwada Math.Soc., 9(1), 43-48.