On a special type of operator called (λ,μ)-jection of third order

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Abstract
This paper deals with an operator called (λ,μ)-jection or a (λ,μ)-jection of third order and investigate forms of such an operator in R^2.

Keywords
(λ,μ)-jection, projection, trijection, λ-jection

I. Introduction

Dr. P.Chandra defined a trijection operator in his Ph.D. thesis titled "Investigation into the theory of operators and linear spaces"[1]. An operator E is defined a projection if $E^2 = E$ as given in Dunford, N. and Schwartz, J. [2], p.37 or Rudin [3], p.126.E has been defined a trijection operator if $E^3 = E$ and is generalisation of projection operator. Navin Kumar Singh [4] has defined a (3,2)-jection operator if $E^3 = E^2$ in his Ph.D. thesis of V.K.S. University Ara of 2019. I had previously defined E to be a λ-jection if $E^3 + \lambda E^2 = (1+\lambda)E, \lambda$ being a scalar.

To generalise these operators, I have defined E to be a (λ,μ)-jection if $\lambda E^3 + \mu E^2 = (\lambda+\mu)E$
λ, μ being scalars. We see that
When $\lambda = 1, \mu = 0\ then\ E^3 = E$, a trijection
When $\lambda = 0, \mu = 1\ then\ E^2 = E$, a projection
When $\lambda = 1, \mu = -1\ then\ E\ is\ a$ (3,2)-jection
When $\lambda = 1, \mu = \lambda\ then\ E\ is\ a\ \lambda$-jection

II. Main Results

Theorem 1
Let E be an operator defined on R^2 by
$E(x,y) = (ax+by, cx+dy)$ with a,b,c,d in R
We find out conditions for E to be a (λ,μ)-jection
Proof:
Let $E(x,y) = (ax+by, cx+dy)$
Then $E^2(x,y) = E(E(x,y)) = E(ax+by, cx+dy)$
$= (a(ax+by) + b(cx+dy), c(ax+by) + d(cx+dy))$
$= ((a^2+bc)x + b(a+d)y, c(a+d)x + (bc+d^2)y)$
$= (Ax+By, Cx+Dy)$
where $A = a^2+bc, B = b(a+d), C = c(a+d), D = bc+d^2$
Let $ad - bc = m\ and\ a+d = n. Then\ d = n - a$
Hence $a(n-a) - bc = m \Rightarrow an - a^2 - bc = m$
So, $A = a^2 + bc = an - m\ and\ bc = an - m - a^2$
Also $B = bn, C = cn$
$D = bc + d^2 = bc + (n-a)^2 = a^2 + bc - 2an + n^2 = an - m - 2an + n^2$
$= n^2 - an - m$

Then,
$E^3(x,y) = E(E^2(x,y)) = E(Ax+By, Cx+Dy)$

$= (a(Ax + By) + b(Cx + Dy), c(Ax + By) + d(Cx + Dy))$
$= ((aA + bC)x + (aB + bD)y, (cA + dC)x + (cB + dD)y)$
$= (A_1x + B_1y, C_1x + D_1y)$
Where $A_1 = aA + bC = a(an - m) + bcn = a^2n - am + (an - m - a^2)n$
$= an^2 - mn - am$
$B_1 = aB + bD = abn + b(n^2 - an - m) = b(n^2 - m)$
$C_1 = cA + dC = c(an - m) + (n - a)cn = c(n^2 - m)$
$D_1 = cB + dD = cbn + (n - a)(n^2 - an - m)$
$= (an - m - a^2)n + n^3 - an^2 - mn - an^2 + a^2n + am$
$= n^3 - an^2 - 2mn + am$

Hence
$\lambda E^3(x, y) + \mu E^2(x, y) = (\lambda + \mu)E(x, y)$
gives
$\lambda(A_1x + B_1y, C_1x + D_1y) + \mu(Ax + By, Cx + Dy) = (\lambda + \mu)(ax + by, cx + dy)$
We equate coefficients of x,y in both co-ordinates and get
$\lambda A_1 + \mu A = a(\lambda + \mu), \lambda B_1 + \mu B = b(\lambda + \mu)$
$\lambda C_1 + \mu C = c(\lambda + \mu), \lambda D_1 + \mu D = d(\lambda + \mu)$
Now consider
$\lambda A_1 + \mu A = a(\lambda + \mu)$
$\Rightarrow \lambda(an^2 - mn - am) + \mu(an - m) = \lambda a + \mu a$
$\Rightarrow a\lambda(n^2 - m) - \lambda mn + \mu an - \mu m - \lambda a - \mu a = 0$ ------------------------------- (1)
Consider
$\lambda B_1 + \mu B = b(\lambda + \mu)$
$\Rightarrow \lambda b(n^2 - m) + \mu bn = b(\lambda + \mu)$
Assuming $b \neq 0$,
$\lambda(n^2 - m) = \lambda + \mu - \mu n$ ----------------------------------- (2)
$n^2 - m = 1 + \frac{\mu}{\lambda} - \frac{\mu}{\lambda}n$
Using (2) in (1),
$a(\lambda + \mu - \mu n) - \lambda mn + \mu an - \mu m - \lambda a - \mu a = 0$
$\Rightarrow -\lambda mn - \mu m = 0$
$\Rightarrow m(\lambda n + \mu) = 0$
$\Rightarrow m = 0 \; or \; \lambda n + \mu = 0$ i.e. $\lambda n = -\mu$ ---------------------------- (3)
Consider $\lambda C_1 + \mu C = c(\lambda + \mu)$
$\Rightarrow \lambda c(n^2 - m) + \mu cn = c(\lambda + \mu)$
Assuming $c \neq 0$, we have
$\lambda(n^2 - m) + \mu n = \lambda + \mu$
Which is same as relation (2) above, and we get (3)
Finally consider
$\lambda D_1 + \mu D = d(\lambda + \mu)$
$\Rightarrow \lambda(n^3 - an^2 - 2mn + am) + \mu(n^2 - an - m) = (n - a)(\lambda + \mu)$
$\Rightarrow \lambda n^3 - a\lambda(n^2 - m) - 2\lambda mn + \mu(n^2 - an - m) = n(\lambda + \mu) - a(\lambda + \mu)$
$\Rightarrow \lambda n^3 - a\lambda(n^2 - m) - 2\lambda mn + \mu(n^2 - m) - \mu an = n(\lambda + \mu) - a(\lambda + \mu)$
Using (2) in above relation,
$\lambda n^3 - a(\lambda + \mu - \mu n) - 2\lambda mn + \mu(1 + \frac{\mu}{\lambda} - \frac{\mu}{\lambda}n) - a\mu n = \lambda n + \mu n - a\lambda - a\mu$
$\Rightarrow \lambda n^3 - 2\lambda mn + \mu + \frac{\mu^2}{\lambda} - \frac{\mu^2}{\lambda}n - \lambda n - \mu n = 0$ ----------------------------- (4)
From (3), let m=0 and use in (4)
Then $\lambda n^3 + \mu + \frac{\mu^2}{\lambda} - \frac{\mu^2}{\lambda}n - \lambda n - \mu n = 0$
$\Rightarrow \lambda n^3 - n(\frac{\mu^2}{\lambda} + \lambda + \mu) + \mu + \frac{\mu^2}{\lambda} = 0$
This is a cubic in n, whose roots are easily found to be

$$1, \frac{\mu}{\lambda} \; and \; -(1+\frac{\mu}{\lambda})$$

So when m=0, then n assumes the above values.

Theorem 2

Let m=0, n=1, then

$$E(x,y) = (ax+by, cx+(1-a)y), where\ bc = a-a^2$$

In this case $E^2 = E$ i.e, E is a projection.

Proof:-

From (2),

$\lambda(1-0) = \lambda + \mu - \mu = \lambda$, true for all values of $\lambda \; and \; \mu$.

$m = 0 \Rightarrow ad - bc = 0$

$n = 1 \Rightarrow a + d = 1 \Rightarrow d = 1 - a$

Hence $a(1-a) - bc = 0$

$\Rightarrow a = a^2 + bc = A \; and \; bc = a - a^2$

Also $B = bn = b, C = cn = c$

$D = n^2 - an - m = 1 - a = d$

So in this case,

$E(x,y) = (ax+by, cx+(1-a)y) \; where \; bc = a - a^2$

and $E^2(x,y) = (Ax+By, Cx+Dy) = (ax+by, cx+dy) = E(x,y)$

Thus $E^2 = E$ i.e.- E is a projection.

Theorem 3

Let $m = 0 \; and \; n = \frac{\mu}{\lambda}$. Then

$E(x,y) = (ax+by, cx+(-\frac{1}{2}-a)y) \; where \; bc = -\frac{1}{2}a - a^2$

Also $E^2 = -\frac{1}{2}E$

Proof:-

Let $m = 0 \; and \; n = \frac{\mu}{\lambda}$

Due to (2),

$\lambda(\frac{\mu^2}{\lambda^2}) = \lambda + \mu - \mu * \frac{\mu}{\lambda}$

$\Rightarrow \frac{\mu^2}{\lambda} = \lambda + \mu - \frac{\mu^2}{\lambda}$

$\Rightarrow 2\frac{\mu^2}{\lambda} = \lambda + \mu \Rightarrow 2\mu^2 - \lambda\mu - \lambda^2 = 0$

$\Rightarrow (2\mu + \lambda)(\mu - \lambda) = 0 \Rightarrow \mu = -\frac{\lambda}{2} \; or \; \lambda$

$\Rightarrow 2\mu = -\lambda \; or \; \mu = \lambda \Rightarrow \frac{\mu}{\lambda} = -\frac{1}{2} \; or \; 1$

i.e. $n = -\frac{1}{2} \; or \; 1$

We have already considered the case $m = 0, n = 1$ in Theorem 2.

So let $m = 0, n = -\frac{1}{2}$

Then $a + d = -\frac{1}{2} \Rightarrow d = -\frac{1}{2} - a$

Also $bc = an - m - a^2 = -\frac{a}{2} - a^2$

So, $a^2 + bc = -\frac{1}{2}a, b(a+d) = -\frac{b}{d}, c(a+d) = -\frac{c}{2}$

$bc + d^2 = D = n^2 - an - m = \frac{1}{4} + \frac{a}{2} = -\frac{1}{2}(-\frac{1}{2} - a) = -\frac{1}{2}d$

Hence in this case

$E(x,y) = (ax+by, cx+(-\frac{1}{2}-a)y) \; where \; bc = -\frac{1}{2}a - a^2$

and $E^2(x,y) = (-\frac{1}{2}ax - \frac{1}{2}by, -\frac{1}{2}cx - \frac{1}{2}dy) = -\frac{1}{2}E(x,y)$

Hence $E^2 = -\frac{1}{2}E$

Theorem 4

Let $m = 0 \; and \; n = -(1 + \frac{\mu}{\lambda})$

Then $E(x,y) = (ax + by, cx + (-(1 + \frac{\mu}{\lambda}) - a)y)$

where $bc = -(1 + \frac{\mu}{\lambda})a - a^2$

In this case $E^2 = -(1 + \frac{\mu}{\lambda})E$

Proof:

Let $m = 0 \; and \; n = -(1 + \frac{\mu}{\lambda})$

Using (2),

$$\lambda(1 + \frac{\mu}{\lambda})^2 = \lambda + \mu + \mu(1 + \frac{\mu}{\lambda}) = \lambda + 2\mu + \frac{\mu^2}{\lambda}$$

$$\Rightarrow \lambda(1 + \frac{\mu^2}{\lambda^2} + \frac{2\mu}{\lambda}) = \lambda + 2\mu + \frac{\mu^2}{\lambda}$$

$$\Rightarrow \lambda + \frac{\mu^2}{\lambda} + 2\mu = \lambda + 2\mu + \frac{\mu^2}{\lambda}$$

which is true for all λ, μ except when $\lambda = 0$

Hence $n = a + d = -(1 + \frac{\mu}{\lambda}) \Rightarrow d = -(1 + \frac{\mu}{\lambda}) - a$

Also $bc = ad = a[-(1 + \frac{\mu}{\lambda}) - a] = -(1 + \frac{\mu}{\lambda})a - a^2$

$$\Rightarrow A = a^2 + bc = -(1 + \frac{\mu}{\lambda})a, B = bn = -(1 + \frac{\mu}{\lambda})b$$

$$C = cn = -(1 + \frac{\mu}{\lambda})c$$

$$D = bc + d^2 = n^2 - an - m = n^2 - an = n(n - a) = nd = -(1 + \frac{\mu}{\lambda})d$$

So in this case,

$$E(x,y) = (ax + by, cx + (-(1 + \frac{\mu}{\lambda}) - a)y)$$

where $bc = -(1 + \frac{\mu}{\lambda})a - a^2$

and

$$E^2(x,y) = (-(1 + \frac{\mu}{\lambda})ax - (1 + \frac{\mu}{\lambda})by, -(1 + \frac{\mu}{\lambda})cx - (1 + \frac{\mu}{\lambda})dy)$$

$$= -(1 + \frac{\mu}{\lambda})(ax + by, cx + dy) = -(1 + \frac{\mu}{\lambda})E(x,y)$$

or $E^2 = -(1 + \frac{\mu}{\lambda})E$

An interesting case is when $\mu = -1, \lambda = 1. Then \; \lambda + \mu = 0$Then E is a
Then E is a (3,2)-jection [4] and $E^2 = 0$. Then $E^3 = 0$ also.

Theorem 5

Let $n = -\frac{\mu}{\lambda}$ and m is arbitrary.

Then $E(x,y) = (ax + by, cx - ay) \; where \; bc = 1 - a^2$

Also $E^2 = I$

Proof:

Here $n = -\frac{\mu}{\lambda}$

Using (4),

$$-\lambda\frac{\mu^3}{\lambda^3} + 2\lambda m\frac{\mu}{\lambda} + \mu + \frac{\mu^2}{\lambda} + \frac{\mu^2}{\lambda} * \frac{\mu}{\lambda} + \mu + \frac{\mu^2}{\lambda} = 0$$

$$\Rightarrow 2m\mu + 2\mu + 2\frac{\mu^2}{\lambda} = 0 \Rightarrow m\mu + \mu + \frac{\mu^2}{\lambda} = 0$$

$$\Rightarrow \mu(m + 1 + \frac{\mu}{\lambda}) = 0 \Rightarrow \mu = 0 \; or \; m = -(1 + \frac{\mu}{\lambda})$$

If we choose $\mu = 0 \; then \; n = 0 \; and \; m = -1$

Then $a + d = 0 \Rightarrow d = -a$

$bc = an - m - a^2 = 1 - a^2$

Then $A = a^2 + bc = 1, B = bn = 0, C = cn = 0$

$D = bc + d^2 = bc + a^2 = 1$

Thus in this case

$E(x, y) = (ax + by, cx - ay)\ where\ bc = 1 - a^2$

Also $E^2(x, y) = (x, y)\ i.e. E^2 = I$

Theorem 6

Let $m = -(1 + \frac{\mu}{\lambda})\ and\ n = -\frac{\mu}{\lambda}$

Then $E(x, y) = (ax + by, cx - (\frac{\mu}{\lambda} + a)y)$

where $bc = 1 + \frac{\mu}{\lambda} - \frac{a\mu}{\lambda} - a^2$

Then $E^2 = -\frac{\mu}{\lambda}E + (1 + \frac{\mu}{\lambda})I$

Proof:-

Due to (2),

$\lambda(\frac{\mu^2}{\lambda^2} + 1 + \frac{\mu}{\lambda}) = \lambda + \mu + \frac{\mu^2}{\lambda}$, true for all $\lambda, \mu\ (\lambda \neq 0)$

Here $a + d = -\frac{\mu}{\lambda} \Rightarrow d = -\frac{\mu}{\lambda} - a$

$\Rightarrow -d\lambda = \mu + a\lambda$

Now $A = an - m = -\frac{a\mu}{\lambda} + 1 + \frac{\mu}{\lambda}$

$B = bn, C = cn$

$$D = n^2 - an - m = n(n - a) - m = nd - m = -\frac{d\mu}{\lambda} + 1 + \frac{\mu}{\lambda}$$

$$bc = an - m - a^2 = -\frac{a\mu}{\lambda} + 1 + \frac{\mu}{\lambda} - a^2 = 1 + \frac{\mu}{\lambda} - \frac{a\mu}{\lambda} - a^2$$

Hence

$$E(x, y) = (ax + by, cx - (\frac{\mu}{\lambda} + a)y)$$

where $bc = 1 + \frac{\mu}{\lambda} - \frac{a\mu}{\lambda} - a^2$

and $E^2(x, y) = (Ax + By, Cx + Dy) = ((an - m)x + bny, cnx + (nd - m)y)$

$= (anx - mx + bny, cnx + dny - my)$

$= (anx + bny, cnx + dny) - m(x, y)$

$= n(ax + by, cx + dy) - m(x, y)$

$= nE(x, y) - mI(x, y)$

Hence $E^2 = nE - mI$

$$= -\frac{\mu}{\lambda}E + (1 + \frac{\mu}{\lambda})I$$

Theorem 7

Let $b = c = 0$ i.e. $E(x, y) = (ax, dy)$

In this case a and d both takes values in the set $\{0, 1, -\frac{(\lambda+\mu)}{\lambda}\}$ i.e. 9 possibilities

Proof:

In this case,

$E(x, y) = (ax + dy)\ where\ bc = 0$

Hence $E^2(x, y) = (a^2x, d^2y), E^3(x, y) = (a^3x, d^3y)$

So substituting in condition for (λ, μ)-jection,

$\lambda(a^3x, d^3y) + \mu(a^2x, d^2y) = (\lambda + \mu)(ax, dy)$

$\Rightarrow (\lambda a^3x + \mu a^2x, \lambda d^3y + \mu d^2y) = (a(\lambda + \mu)ax, d(\lambda + \mu)y)$

Comparing coefficients of x,y

$\lambda a^3 + \mu a^2 = (\lambda + \mu)a$

$\lambda d^3 + \mu d^2 = (\lambda + \mu)d$

Now $\lambda a^3 + \mu a^2 - (\lambda + \mu)a = 0$

$\Rightarrow a[\lambda a^2 + \mu a - (\lambda + \mu)] = 0$

$\Rightarrow a[\lambda(a^2 - 1) + \mu(a - 1)] = 0$

$\Rightarrow a(a-1)[\lambda(a+1)+\mu]=0$

$\Rightarrow a=0,1,-\dfrac{(\lambda+\mu)}{\lambda}$

Similarly, $d=0,1,-\frac{(\lambda+\mu)}{\lambda}$

Considering these values of a and d, forms for E(x,y) are given by

When $a=0, d=0, E(x,y)=(0,0)\ i.e. E=0, zero\ operator$

When $a=0, d=1, E(x,y)=(0,y)\ Here\ E^2=E$

When $a=0, d=-\frac{(\lambda+\mu)}{\lambda}, E(x,y)=(0,-\frac{(\lambda+\mu)}{\lambda}y)$

In this way other possibilities are

$E(x,y)=(x,0)\ Here\ E^2=E$

$E(x,y)=(x,y)\ i.e. E=I$

$E(x,y)=(x,-\dfrac{(\lambda+\mu)}{\lambda}y)$

$E(x,y)=(-\dfrac{(\lambda+\mu)}{\lambda}x,0)$

$E(x,y)=(-\dfrac{(\lambda+\mu)}{\lambda}x,y)$

$E(x,y)=(-\dfrac{(\lambda+\mu)}{\lambda}x,-\dfrac{(\lambda+\mu)}{\lambda}y)\ i.e. E=-\dfrac{(\lambda+\mu)}{\lambda}I$

III. References

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