A STUDY ON "KIRCCHOFF TYPE VARIABLE INEQUALITY"

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Abstract

In this paper, we derive a sharp observability inequality for Kirchhoff plate equations with lower order terms. More precisely, for any $T > 0$ and suitable boundary observation domains (satisfying the geometric conditions that the multiplier method imposes), we prove an estimate with an explicit observability constant for Kirchhoff plate systems with an arbitrary finite number of components and in any space dimension with lower order bounded potentials.

Key words: Kirchhoff plate system, observability constant, Carleman inequalities, potential, Meshkov's construction.

1 Introduction

Let $n \geq 1$ and $N \geq 1$ be two integers. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^4 boundary Γ, Γ_0 be a nonempty open subset of Γ, and $T > 0$ be given and sufficiently large. Put $Q \triangleq (0,T)\times \Omega$, $\Sigma \triangleq (0,T)\times\Gamma$ and $\Sigma_0 \triangleq (0,T)\times\Gamma_0$. For simplicity, we will use the notation $y_i = \frac{\partial y}{\partial x_i}$, where x_i is the i-th coordinate of a generic point $x = (x_1,\ldots,x_n)$ in $\mathbb{R}^n$. Throughout this paper, we will use $C = C(T, \Omega,\Gamma_0)$ and $C^* = C^*(\Omega,\Gamma_0)$ to denote generic positive constants depending on their arguments which may vary from line to line.

Set

$$Y \triangleq \left\{ y \in H^3(\Omega) \,\middle|\, y|_\Gamma = \Delta y|_\Gamma = 0 \right\}.$$

We consider the following $\mathbb{R}^N$-valued plate system with a potential $a \in L^\infty(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n/3,\infty]$:

$$\begin{cases} y_{tt} + \Delta^2 y - \Delta y_{tt} + ay = 0 & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad y_t(0) = y^1 & \text{in } \Omega, \end{cases} \tag{1}$$

where $y = (y_1,\dots,y_N)^T$, and the initial datum (y^0,y^1) is supposed to belong to $\mathcal{H} \triangleq \{\varphi \in H^3(\Omega) \mid \varphi|_\Gamma = \Delta\varphi|_\Gamma = 0\}^N \times (H^2(\Omega)\cap H_0^1(\Omega))^N$, the state space of system (1). It is easy to show that system (1) admits one and only one weak solution $y \in C([0,T];\mathcal{H})$.

In what follows, we shall denote by $|\cdot|$, $||\cdot||_p$ and $|||\cdot|||_p$ the (canonical) norms on $\mathbb{R}^N$, $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$ and $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^{N\times N}))$, respectively.

We shall study the observability constant $K(a)$ of system (1), defined as the smallest (possibly infinite) constant such that the following observability estimate for system (1) holds:

$$\begin{aligned} &||(y^0, y^1)||^2_{\mathcal{H}} \triangleq ||y^0||^2_{(H^3(\Omega))^N} + ||y^1||^2_{(H^2(\Omega))^N} \\ &\leq K(a)\int_{\Sigma_0}\Big(\Big|\frac{\partial y}{\partial \nu}\Big|^2 + \Big|\frac{\partial y_t}{\partial \nu}\Big|^2 + \Big|\frac{\partial \Delta y}{\partial \nu}\Big|^2\Big)dxdt, \qquad \forall\,(y^0, y^1) \in \mathcal{H}. \end{aligned} \tag{2}$$

This inequality, the so-called *observability inequality*, allows estimating thetotal energy of solutions in terms of the energy localized in the observation subdomain Γ_0. It is relevant for control problems. In particular, in this linear setting, this (observability) inequality is equivalent to the so-called exact controllability property, i.e., that of driving solutions to rest by means of control forces localized in Σ_0 (see [6, 11]). This type of inequality, with explicit estimates on the observability constant, is also relevant for the control of semilinear problems ([10]). Similar inequalities are also useful for solving a variety of Inverse Problems ([9]). We remark that, as for the wave equations, (2) holds for the Kirchhoff plate only if (Ω,Γ_0,T) satisfies suitable conditions, i. e. Γ_0 needs to satisfy certain geometric conditions and T needs to be large enough.

Obviously the observability constant $K(a)$ in (2) not only depends on the potential a, but also on the domains Ω and Γ_0 and on the time T. The main purpose of this paper is to analyze only its explicit and sharp dependence on the potential a.

The main tools to derive the explicit observability estimates are the so-called *Carleman inequalities*. Here we have chosen to work in the space $\mathcal{H}$ in which Carleman inequalities can be applied more naturally. But some other choices of the state space are possible. For example, one may consider similar problems in state spaces of the form $(H_0^1(\Omega))^N\times(L^2(\Omega))^N$ or $(H^2(\Omega)\cap H_0^1(\Omega))^N\times(H_0^1(\Omega))^N$ where the Kirchhoff plate system is also well posed. But the corresponding analysis on the observability constants, in turn, is technically more involved.

One of the key points to derive inequality (2) for system (1) is the possibility of decomposing the Kirchhoff plate operator $\partial_t^2 + \Delta^2 - \partial_t^2\Delta$ as follows:

$$\partial_t^2 + \Delta^2 - \partial_t^2\Delta = (\partial_{tt} - \Delta)(I - \Delta) + \Delta, \tag{3}$$

where I is the identity operator. Actually, we set

$$z = y - \Delta y, \tag{4}$$

where y is the solution of (1). By the first equation of (1) and noting (3), it follows that

$$-ay = y_{tt} + \Delta^2 y - \Delta y_{tt} = (\partial_{tt} - \Delta)(y - \Delta y) + \Delta y = z_{tt} - \Delta z + y - z.$$

Therefore the Kirchhoff plate system (1) can be written equivalently as the following coupled elliptic-wave system

$$\begin{cases} \Delta y + z - y = 0 & \text{in } Q, \\ z_{tt} - \Delta z + y - z + ay = 0 & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \Delta y^0, \quad z_t(0) = y^1 - \Delta y^1 & \text{in } \Omega. \end{cases} \tag{5}$$

Consequently, in order to derive the desired observability inequality for system (1), it is natural to proceed in cascade by applying the global Carleman estimates to the second order operators in the two equations in system (5). We refer to [2, 3] for related works on Carleman inequalities for other cascade systems of partial differential equations.

Similar (boundary and/or internal) observability problems (in suitable spaces) have been considered for the heat and wave equations in [1], and for the Euler-Bernoulli plate equations in [5]. According to [1] and [5], the sharp observability constants for the heat, wave and Euler-Bernoulli plate equations with bounded potentials a (i.e., $p = \infty$) contain respectively the product of the following two terms (Recall that $C^* = C^*(\Omega,\Gamma_0)$ and $C = C(T, \Omega,\Gamma_0)$)

$$H_1(T, a) = \exp(C^* T ||a||_\infty), \qquad H_2(T, a) = \exp\left(C^* ||a||_\infty^{2/3}\right),$$

$$W_1(T, a) = \exp\left(C ||a||_\infty^{1/2}\right), \qquad W_2(T, a) = \exp\left(C ||a||_\infty^{2/3}\right),$$

and

$$P_1(T, a) = \exp(C^* T |||a|||_\infty^{1/2}), \qquad P_2(T, a) = \exp\left(C^* ||a||_\infty^{1/3}\right).$$

As explained in [1, 5], the role that each of these constants plays in the observability inequality is of different nature: $H_1(T, a)$, $W_1(T, a)$ and $P_1(T, a)$ are the constants which arise when applying Gronwall's inequality to establish the energy estimates for solutions of evolution equations; while $H_2(T, a)$, $W_2(T, a)$ and $P_2(T, a)$ appear when using global Carleman estimates to derive the observability inequality by absorbing the undesired lower order terms.

It is shown in [1, Theorems 1.1 and 1.2] and [5, Theorem 3] that the above observability constants are optimal for the heat, wave and Euler-Bernoulli plate systems ($N \geq 2$) with bounded potentials, in even dimensions $n \geq 2$. The proof of this optimality result uses the following two key ingredients:

1) For the heat and Euler-Bernoulli plate equations, because of the infinite speed of propagation, one can choose T as small as one likes and henceforth $H_1(T, a)$ and $P_1(T, a)$ can be bounded above by $H_2(T, a)$ and $P_2(T, a)$, respectively for $T = O(||a||_\infty^{-1/3})$ and $O(||a||_\infty^{-1/6})$. On the other hand, for the wave equation, although one has to take T to be large enough (because of the finite velocity of propagation), for any finite T, $W_1(T, a)$ can bounded by $W_2(T, a)$ because the power 1/2 for $||a||_\infty$ in $W_1(T, a)$, given by the modified energy estimate, is smaller than 2/3, the power for $||a||_\infty$ in $W_2(T, a)$, arising from the Carleman estimate. In this way, for any finite T large enough, one gets an upper bound on the observability constant (for the wave equation) of the order of $\exp(C||a||_\infty^{2/3})$.

2) Based on the Meshkov's construction [8] which allows finding potentials and non-trivial solutions for elliptic systems decaying at infinity in a superexponential way, one can construct a family of solutions (for the heat, wave and Euler-Bernoulli plate equations) with suitable localization properties showing that most of the energy is concentrated away from the observation domain. According to this, the observed energies grow exponentially as $\exp(-||a||_\infty^{2/3})$ for the wave and heat systems and as $\exp(-||a||_\infty^{1/3})$ for the Euler-Bernoulli plate ones.

Things are more complicated for the Kirchhoff plate systems under consideration. Indeed, on one hand, due to the finite speed of propagation, one has to choose the observability time T to be large enough. On the other hand, a modified energy estimate for the Kirchhoff plate systems (*see* (10) in Lemma 1 in Section 2) yields a power 1/2 for $||a||_\infty$ which can not be absorbed by the one, 1/3, arising from the Carleman estimate. To overcome this difficulty, the key observation in this paper is that, although T has to be taken to be large, one can manage to use the indispensable energy estimate only in a very short time interval when deriving the desired observability estimate. However, we do not know how to show the optimality of the observability constant at this moment. Indeed, when proving the optimality, the energy estimate has to be used in the whole time duration $[0,T]$ and this breaks down the concentration effect that Meshkov's construction guarantees, which is valid only for very small time durations for the Kirchhoff plate systems. Therefore, proving the optimality of the observability estimates obtained in this paper is an interesting open problem.

The rest of this paper is organized as follows. In Section 2 we give some preliminary energy estimate for Kirchhoff plate systems, and show some fundamental weighted pointwise estimates for the wave and elliptic operators. In Section 3 we present the sharp observability estimate for the Kirchhoff plate system. In Section 4 we explain more carefully the main difficulty to show the optimality of the observability constant for Kirchhoff plate systems by means of the above mentioned Meshkov's construction.

2 Preliminaries

In this section, we show some preliminary energy estimates for Kirchhoff plate systems, and weighted pointwise estimates for the wave and elliptic operators. The estimates for the Kirchhoff plate system will then be obtained by noting the equivalence between system (1) and the coupled wave-elliptic system (5).

2.1 Energy estimates for Kirchhoff plate systems

Denote the energy of system (1) by

$$E(t)=\frac{1}{2}\Big[|\Delta y_t(t,\cdot)|^2_{(L^2(\Omega))^N}+|y_t(t,\cdot)|^2_{(H_0^1(\Omega))^N}+|\Delta y(t,\cdot)|^2_{(H_0^1(\Omega))^N}\Big]. \quad (6)$$

Note that this energy is equivalent to the square of the norm in $\mathcal{H}$. For

$$s_0=\frac{n}{3p}, \quad (7)$$

consider also the modified energy function:

$$\mathcal{E}(t)=E(t)+\frac{1}{2}||a||_p^{\frac{2}{2-s_0}}|y(t,\cdot)|^2_{(L^2(\Omega))^N}. \quad (8)$$

It is clear that both energies are equivalent. Indeed,

$$E(t)\le\mathcal{E}(t)\le C\left(1+||a||_p^{\frac{2}{2-s_0}}\right)E(t). \quad (9)$$

The following estimate holds for the modified energy:

Lemma 1. *Let* $a\in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$ *for some* $p\in[n/3,\infty]$. *Then there is a constant* $C_0=C_0(\Omega,p,n)>0$, *independent of* T, *such that*

$$\mathcal{E}(t) \le C_0 e^{C_0 \|a\|_p^{\frac{1}{2-s_0}} |t-s|} \mathcal{E}(s), \qquad \forall\, t, s \in [0, T]. \tag{10}$$

Proof. For simplicity, we assume $N = 1$. The same proof applies to a system with any finite number of components N. Using (8) and noting system (1), it is easy to see that

$$\frac{d\mathcal{E}(t)}{dt} = -\int_\Omega a y \Delta y_t dx + \|a\|_p^{\frac{2}{2-s_0}} \int_\Omega y y_t dx. \tag{11}$$

Put $p_1 = \frac{2}{s_0 - 2p^{-1}}$ and $p_2 = \frac{2}{1-s_0}$. Noting that

$$\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{2} = 1 \quad \text{and} \quad \frac{1}{2s_0^{-1}} + \frac{1}{2(1-s_0)^{-1}} + \frac{1}{2} = 1,$$

by Hölder's inequality and Sobolev's embedding theorem, and recalling (7)-(8) and observing $s_0 p_1 = \frac{2s_0}{s_0 - 2p^{-1}} = \frac{2n}{n-6}$, we get

$$\begin{aligned}
&\left| -\int_\Omega a y \Delta y_t dx \right| \le \int_\Omega |a| |y|^{s_0} |y|^{1-s_0} |\Delta y_t| dx \\
&\le \|a\|_p \left\| |y(t,\cdot)|^{s_0} \right\|_{L^{p_1}(\Omega)} \left\| |y(t,\cdot)|^{1-s_0} \right\|_{L^{p_2}(\Omega)} \|\Delta y_t(t,\cdot)\|_{L^2(\Omega)} \\
&= \|a\|_p \|y(t,\cdot)\|^{s_0}_{L^{s_0 p_1}(\Omega)} \|y(t,\cdot)\|^{1-s_0}_{L^{(1-s_0)p_2}(\Omega)} \|\Delta y_t(t,\cdot)\|_{L^2(\Omega)} \\
&= \|a\|_p \|y(t,\cdot)\|^{s_0}_{L^{\frac{2n}{n-6}}(\Omega)} \|y(t,\cdot)\|^{1-s_0}_{L^2(\Omega)} \|\Delta y_t(t,\cdot)\|_{L^2(\Omega)} \qquad (12) \\
&= C \|a\|_p^{\frac{1}{2-s_0}} \underbrace{\|y(t,\cdot)\|^{s_0}_{L^{\frac{2n}{n-6}}(\Omega)}}_{\le \mathcal{E}(t)^{\frac{s_0}{2}}} \underbrace{\left(\|a\|_p^{\frac{1-s_0}{2-s_0}} \|y(t,\cdot)\|^{1-s_0}_{L^2(\Omega)} \right)}_{\le \mathcal{E}(t)^{\frac{1-s_0}{2}}} \underbrace{\|\Delta y_t(t,\cdot)\|_{L^2(\Omega)}}_{\le \mathcal{E}(t)^{1/2}} \\
&\le C \|a\|_p^{\frac{1}{2-s_0}} \mathcal{E}(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|a\|_p^{\frac{2}{2-s_0}} \left| \int_\Omega y y_t dx \right| &\le \frac{\|a\|_p^{\frac{1}{2-s_0}}}{2} \int_\Omega \left(\|a\|_p^{\frac{2}{2-s_0}} |y|^2 + |y_t|^2 \right) dx \\
&\le C \|a\|_p^{\frac{1}{2-s_0}} \mathcal{E}(t).
\end{aligned} \tag{13}$$

Now, combining (11)-(13), and applying Gronwall's inequality, we conclude the desired estimate (10).

2.2 Pointwise weighted estimates for the wave and elliptic operators

In this subsection, we present some pointwise weighted estimates for the wave and elliptic equations that will play a key role when deriving the sharp observability estimates for the Kirchhoff plate system.

First, we show a pointwise weighted estimate for the wave operator "∂_{tt}-Δ". For this, for any (large) $\lambda > 0$, any $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$, set

$$\ell(t,x) = \lambda\left[|x-x_0|^2 - c\left(t-\frac{T}{2}\right)^2\right]. \qquad (14)$$

By taking $(a^{ij})_{n\times n} = I$, the identity matrix, and $\theta = e^{\ell}$(with ℓ given by (14)) in [4, Corollary 4.1] (*see* also [7, Lemma 5.1]), one has the following pointwise weighted estimate for the wave operator.

Lemma 2. *For any $u = u(t,x) \in C^2(\mathbb{R}^{1+n})$, any $k \in \mathbb{R}$ and $v \triangleq \theta u$, it holds*

$$\begin{aligned}
&\theta^2|u_{tt}-\Delta u|^2 + 2\Big[\ell_t(v_t^2+|\nabla v|^2) - 2(\nabla\ell)\cdot(\nabla v)v_t - \Psi v v_t \\
&\qquad\qquad + (A+\Psi)\ell_t v^2\Big]_t \\
&\quad + 2\sum_{i=1}^n \Big\{2v_i(\nabla\ell)\cdot(\nabla v) - \ell_i|\nabla v|^2 + \Psi v v_i - 2\ell_t v_t v_i + \ell_i v_t^2 \\
&\qquad\qquad - (A+\Psi)\ell_i v^2\Big\}_i \\
&\quad \ge 2\lambda(1-k)v_t^2 + 2\lambda(k+3-4c)|\nabla v|^2 + Bv^2, \quad \forall\,(t,x)\in\mathbb{R}^{1+n},
\end{aligned} \qquad (15)$$

where

$$\begin{cases}
\Psi \triangleq \lambda(2n-2c-1+k), \\
A = 4\lambda^2\left[c^2(t-T/2)^2 - |x-x_0|^2\right] + \lambda(4c+1-k), \\
B = 8\lambda^3\left[(4c+5-k)|x-x_0|^2 - (8c+1-k)c^2(t-T/2)^2\right] + O(\lambda^2).
\end{cases} \qquad (16)$$

As a consequence of Lemma 2, we have the following pointwise weighted estimate for the elliptic operator.

Corollary 1. *Let $p = p(t,x) \in C^2(\mathbb{R}^{1+n})$, and set $q = \theta p$. Then*

$$\begin{aligned}
&\theta^2|\Delta p|^2 + 2\sum_{i=1}^n\Big\{2q_i(\nabla\ell)\cdot(\nabla q) - \ell_i|\nabla q|^2 + \widetilde{\Psi} q q_i - (\widetilde{A}+\widetilde{\Psi})\ell_i q^2\Big\}_i \\
&\qquad \ge 6\lambda|\nabla q|^2 + \widetilde{B}q^2, \qquad \forall\,(t,x)\in\mathbb{R}^{1+n},
\end{aligned} \qquad (17)$$

where

$$\begin{cases}
\widetilde{\Psi} \triangleq \lambda(2n-1), \quad \widetilde{A} = -4\lambda^2|x-x_0|^2 + \lambda, \\
\widetilde{B} = 40\lambda^3|x-x_0|^2 + O(\lambda^2), \quad \text{uniformly w.r.t. } t\in[0,T].
\end{cases} \qquad (18)$$

Proof. We fix an arbitrary $t \in [0,T]$ and view the corresponding function which depends on x as a function of (x,s) with s being a fictitious time parameter. We then set

$$U(s,x) \equiv p(t,x), \quad V(s,x) = \Xi(x)U(s,x) \qquad \forall(s,x)\in\mathbb{R}^{1+n},$$

where $\Xi = e^L$ and $L = \lambda|x-x_0|^2$. Choosing $c = 0$ in (14), and applying Lemma 2 (with $k = 0$) in the variable (x,s) to the above U and V, we get

$$\Xi^2|\Delta U|^2 + 2\sum_{i=1}^{n}\left\{2V_i(\nabla L)\cdot(\nabla V) - L_i|\nabla V|^2 + \widetilde{\Psi}VV_i - (\widetilde{A}+\widetilde{\Psi})L_iV^2\right\}_i \geq 6\lambda|\nabla V|^2 + \widetilde{B}V^2, \qquad (19)$$

with $\widetilde{\Psi}$, $\widetilde{A}$ and $\tilde{B}$ given by (18). Now, for any $c \in \mathbb{R}$, multiplying both sides of (19) by $e^{-2c\lambda(t-\frac{T}{2})^2}$, noting

$$\theta = \Xi e^{-c\lambda(t-\frac{T}{2})^2}, \quad \ell = L - c\lambda\left(t-\frac{T}{2}\right)^2 \quad \text{and} \quad q = e^{-c\lambda(t-\frac{T}{2})^2}V,$$

the desired inequality (17) follows.

Remark. The key point in Corollary 1 is that we choose the same weight θ in (17) as that in (15). This will play a key role in the sequel when we deduce the sharp observability estimate for Kirchhoff plates.

In the sequel, for simplicity, we assume $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ (For the general case where, possibly, $x_0 \in \overline{\Omega}$, we can modify an argument in [7, Case 2 in the proof of Theorem 5.1] to derive the same result). Hence

$$0 < R_0 \triangleq \min_{x\in\Omega}|x-x_0| < R_1 \triangleq \max_{x\in\Omega}|x-x_0|. \qquad (20)$$

Also, for any $\beta > 0$, we set

$$\Theta = \Theta(t) \triangleq \exp\left\{-\frac{\beta R_1}{t} - \frac{\beta R_1}{T-t}\right\}, \qquad 0 < t < T. \qquad (21)$$

It is easy to see that $\Theta(t)$ decays rapidly to 0 as $t \to 0$ or $t \to T$. The desired pointwise Carleman-type estimate with singular weight Θ for the wave operator reads as follows:

Theorem 1. *Let* $u \in C^2([0,T]\times\overline{\Omega})$ *and* $v = \theta u$. *Then there exist four constants* $T_0 > 0$, $\lambda_0 > 0$, $\beta_0 > 0$ *and* $c_0 > 0$, *independent of u, such that for all* $T \geq T_0$, $\beta \in (0,\beta_0)$ *and* $\lambda \geq \lambda_0$ it holds

$$\begin{aligned}
&\theta^2\Theta|u_{tt} - \Delta u|^2 + 2\left\{\Theta\left[\ell_t(v_t^2 + |\nabla v|^2) - 2(\nabla\ell)\cdot(\nabla v)v_t - \Psi v v_t + (A+\Psi)\ell_t v^2\right]\right\}_t \\
&\quad + 2\Theta\sum_{i=1}^{n}\left\{2v_i(\nabla\ell)\cdot(\nabla v) - \ell_i|\nabla v|^2 + \Psi v v_i - 2\ell_t v_t v_i + \ell_i v_t^2 - (A+\Psi)\ell_i v^2\right\}_i \qquad (22)\\
&\geq c_0\lambda\theta^2\Theta(u_t^2 + |\nabla u|^2 + \lambda^2u^2),
\end{aligned}$$

with A and Ψ *given by* (16).

Remark. The main difference between the pointwise estimates (15) and (22) is that we introduce a singular ''pointwise'' weight in (22). As we will see later, this point plays a crucial role in the proof of Theorem 3 in the next section. Another difference between (15) and (22) is that T is arbitrary in the former estimate; while for the later one needs to take T_0, and hence T, to be large enough.

Proof of Theorem 1. The proof is divided into several steps.

Step 1. We multiply both sides of inequality (15) by Θ. Obviously, we have (recall (16) for A and Ψ)

$$\begin{aligned}
&\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t+(A+\Psi)\ell_t v^2\Big]_t\\
&\quad=\Big\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t+(A+\Psi)\ell_t v^2\Big]\Big\}_t\\
&\qquad-\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)\\
&\qquad-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t+(A+\Psi)\ell_t v^2\Big].
\end{aligned}\tag{23}$$

Note that

$$\begin{aligned}
&\Big|-\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\Theta\Big[-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t\Big]\Big|\\
&\leq\beta TR_1t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[2|(\nabla\ell)\cdot(\nabla v)v_t|+|\Psi v_tv|\Big]\\
&\leq\beta TR_1t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[(|\nabla\ell|+1)v_t^2+|\nabla\ell||\nabla v|^2+\frac{1}{4}\Psi^2v^2\Big].
\end{aligned}\tag{24}$$

Thus by (15), and using (23)-(24), we get

$$\begin{aligned}
&\theta^2\Theta|u_{tt}-\Delta u|^2+2\Big\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t\\
&\qquad+(A+\Psi)\ell_t v^2\Big]\Big\}_t\\
&\quad+2\Theta\sum_{i=1}^{n}\Big\{2v_i(\nabla\ell)\cdot(\nabla v)-\ell_i|\nabla v|^2+\Psi vv_i-2\ell_tv_tv_i+\ell_iv_t^2\\
&\qquad-(A+\Psi)\ell_iv^2\Big\}_i\\
&\geq 2\Theta\lambda(1-k)v_t^2+2\Theta\lambda(k+3-4c)|\nabla v|^2\\
&\quad+2\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\ell_t\Theta(v_t^2+|\nabla v|^2)\\
&\quad-2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[(|\nabla\ell|+1)v_t^2+|\nabla\ell||\nabla v|^2\Big]\\
&\quad+\Theta\Big[B+2\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\ell_t(A+\Psi)\\
&\quad-\beta TR_1t^{-2}(T-t)^{-2}\frac{|T-2t|}{2}\Psi^2\Big]v^2,
\end{aligned}\tag{25}$$

where B is given by (16).

Step 2. Recalling that ℓ and Ψ are given respectively by (14) and (16), we get

$$\begin{aligned}
&2\Theta\lambda(1-k)v_t^2 + 2\Theta\lambda(k+3-4c)|\nabla v|^2 \\
&\quad + 2\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)\ell_t\Theta(v_t^2+|\nabla v|^2) \\
&\quad - 2\beta T R_1 t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[(|\nabla \ell|+1)v_t^2 + |\nabla\ell||\nabla v|^2\Big] \\
&\quad + \Theta\Big[B + 2\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)\ell_t(A+\Psi) \\
&\quad - \beta T R_1 t^{-2}(T-t)^{-2}\frac{|T-2t|}{2}\Psi^2\Big]v^2 \\
&= \lambda\Theta(F_1 v_t^2 + F_2|\nabla v|^2) + \lambda^3\Theta G v^2,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
F_1 \stackrel{\triangle}{=}\;& 2(1-k) + 2c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2 \\
& -2\beta T R_1 t^{-2}(T-t)^{-2}|T-2t|(2|x-x_0|+\lambda^{-1}),
\end{aligned} \tag{27}$$

$$\begin{aligned}
F_2 \stackrel{\triangle}{=}\;& 2(k+3-4c) + 2c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2 \\
& -4\beta T R_1 t^{-2}(T-t)^{-2}|T-2t||x-x_0|
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
G \stackrel{\triangle}{=}\;& 8\left[(4c+5-k)|x-x_0|^2 - (8c+1-k)c^2(t-T/2)^2\right] \\
& + O(\lambda^{-1}) \\
& + 8c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2\Big[c^2(t-T/2)^2 \\
& -|x-x_0|^2 + O(\lambda^{-1})\Big] \\
& - \beta(2n-2c-1+k)^2 T R_1 t^{-2}(T-t)^{-2}|t-T/2|\lambda^{-1}.
\end{aligned} \tag{29}$$

Thus, by (25) and (26), we have

$$\begin{aligned}
&\theta^2\Theta|u_{tt}-\Delta u|^2 + 2\left\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2) - 2(\nabla\ell)\cdot(\nabla v)v_t - \Psi v v_t \right. \\
&\qquad\qquad \left. +(A+\Psi)\ell_t v^2\Big]\right\}_t \\
&\quad + 2\Theta\sum_{i=1}^{n}\Big\{2v_i(\nabla\ell)\cdot(\nabla v) - \ell_i|\nabla v|^2 + \Psi v v_i - 2\ell_t v_t v_i + \ell_i v_t^2 \\
&\qquad\qquad -(A+\Psi)\ell_i v^2\Big\}_i \\
&\geq \lambda\Theta(F_1 v_t^2 + F_2|\nabla v|^2) + \lambda^3\Theta G v^2.
\end{aligned} \tag{30}$$

Step 3. Let us show that F_1, F_2 and G are positive when λ is large enough. For this purpose, we choose $c \in (0,1)$ sufficiently small so that

$$\frac{(4+5c)R_0^2}{9c} > R_1^2, \tag{31}$$

and $T(> 2R_1)$ sufficiently large such that

$$\frac{4(4+5c)R_0^2}{9c} > c^2T^2 > 4R_1^2. \tag{32}$$

Also, we choose

$$k = 1 - c. \tag{33}$$

By (33) and recalling that $c \in (0,4/5)$, it is easy to see that the nonsingular part $F_1^0 \triangleq 2(1-k)$ of F_1 (*resp.* $F_2^0 \triangleq 2(k+3-4c)$ of F_1) is positive. Using (33) again, the nonsingular part of G reads

$$\begin{aligned} G^0 \triangleq\ & 8\left[(4c+5-k)|x-x_0|^2 - (8c+1-k)c^2(t-T/2)^2\right] + O(\lambda^{-1}) \\ & \geq 2\left[4(4+5c)R_0^2 - 9c^3T^2\right] + O(\lambda^{-1}), \end{aligned}$$

which, via the first inequality in (32), is positive provided that λ is sufficiently large.

When t is near 0 and T, i.e., $t \in I_0 \triangleq (0,\delta_0)\cup(T-\delta_0,T)$ for some sufficiently small $\delta_0 \in (0,T/2)$, the dominant terms in F_i (i = 1,2) and G are the singular ones. For $t \in I_0$, the singular part of F_1 reads

$$\begin{aligned} F_1^1 \triangleq\ & 2c\beta TR_1t^{-2}(T-t)^{-2}(T-2t)^2 \\ & - 2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|(2|x-x_0| + \lambda^{-1}) \\ \geq\ & 2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|[c(T-2\delta_0) - 2R_1 - \lambda^{-1})] \\ =\ & 2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|(cT - 2R_1 - 2c\delta_0 - \lambda^{-1}), \end{aligned}$$

which, via the second inequality in (32), is positive provided that both δ_0 and λ^{-1} are sufficiently small. Similarly, for $t \in I_0$, the singular part of F_2,

$$\begin{aligned} F_2^1 \triangleq\ & 2c\beta TR_1t^{-2}(T-t)^{-2}(T-2t)^2 \\ & - 4\beta TR_1t^{-2}(T-t)^{-2}|T-2t||x-x_0| \end{aligned}$$

is positive provided that δ_0 is sufficiently small. Also, for $t \in I_0$, the singular part of G reads

$$\begin{aligned} G^1 \triangleq\ & 8\, c\beta TR_1t^{-2}(T-t)^{-2}(T-2t)^2\Big[c^2(t-T/2)^2 - |x-x_0|^2 \\ & + O(\lambda^{-1})\Big] - \beta(2n-2c-1+k)^2TR_1t^{-2}(T-t)^{-2}|t-T/2|\lambda^{-1}. \end{aligned}$$

It is easy to see that, for $t \in I_0$, it holds

$$\begin{aligned} G^1 = &\ \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ &\left\{8c|T-2t|\left[c^2(t-T/2)^2 - |x-x_0|^2 + O(\lambda^{-1})\right]\right. \\ &\left. -(2n-2c-1+k)^2(2\lambda)^{-1}\right\} \\ = &\ \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ &\left\{8c|T-2t|\left[c^2(t-T/2)^2 - |x-x_0|^2\right] + O(\lambda^{-1})\right\} \\ \geq &\ \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ &\left\{8c|T-2\delta_0|\left[c^2(\delta_0-T/2)^2 - R_1^2\right] + O(\lambda^{-1})\right\} \\ \geq &\ \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ &\left\{8c|T-2\delta_0|\left[c^2T^2/4 - R_1^2 + c^2\delta_0(\delta_0-T)\right] + O(\lambda^{-1})\right\}, \end{aligned}$$

which, via the second inequality in (32), is positive provided that both δ_0 and λ^{-1} are sufficiently small.

By (27)-(29), we see that $F_1 = F_1^0 + F_1^1$, $F_2 = F_2^0 + F_2^1$ and $G = G^0+G^1$. Since F_1^0, F_2^0 and G^0 are positive, by the above argument, we see that F_1, F_2 and G are positive for $t \in I_0$. For $t \in (0,T)\backslash I_0$, noting again the positivity of F_1^0, F_2^0 and G^0, one can choose $\beta > 0$ sufficiently small such that F_1^1, F_2^1 and G^1 are very small, hence so that F_1, F_2 and G are positive. Hence (30) yields the desired (22). This completes the proof of Theorem 1.

Similar to Theorem 1, by multiplying both sides of (17) by Θ, we have

Theorem 2. *Let* $p = p(t,x) \in C^2([0,T]\times\overline{\Omega})$, *and set* $q = \theta p$. *Then there exist two constants* $\lambda_0 > 0$ *and* $c_0 > 0$, *independent of p, such that for all* $T > 0$, $\beta > 0$ *and* $\lambda \geq \lambda_0$ *it holds*

$$\begin{aligned} \theta^2\Theta|\Delta p|^2 + 2\Theta\sum_{i=1}^{n}\left\{2q_i(\nabla\ell)\cdot(\nabla q) - \ell_i|\nabla q|^2 + \widetilde{\Psi}qq_i - (\widetilde{A}+\widetilde{\Psi})\ell_i q^2\right\}_i \\ \geq c_0\lambda\theta^2\Theta\left(|\nabla p|^2 + \lambda^2 p^2\right), \end{aligned} \tag{34}$$

with $\widetilde{A}$ *and* $\widetilde{\Psi}$ *given by* (18).

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