Numerical solution of the PDE describing unsteady-state heating of the tapered rod by Forward, Backward, Central difference methods and comparison with Analytical solution

[1] Ravi Borana, [2] Vihan Gemawat

[1]Associate Professor, [2]Associate Researcher

[1]Department of Mathematics, [2]Senior year of High School,

[1] Bhavan's R.A. Sc. College, Ahmedabad, [2]Ahmedabad International School

Gujarat, India.

1.1 Introduction

The mathematicians, scientists and engineers, generally, describe the real world problem by the partial differential equation, as a result of its mathematical modeling. The mathematical modeling of most problems in science involving rates of change with respect to two or more independent variables, usually representing time, length or angle, leads either to a Partial Differential Equation (PDE) or to a set of such equations. The partial differential equations (PDEs) are obtained by the engineers and scientists in almost all the fields to describe a large number of real world problems. Many of the PDEs which result from engineering problems cannot be readily solved by analytical methods. Only a few of them have analytical solutions. Consequently, knowledge of the methods for obtaining numerical solutions of PDEs is important to the modern engineers. A numerical solution is obtained for the differential equation with specific boundary conditions. These, of course, describe some physical problem. For solving differential equations, the numerical approximation methods such as Finite Difference Methods (FDMs) are frequently used and more universally applicable than any other. The FDMs are most simple, easy and efficient to apply on partial differential equations among all the numerical methods.

The basic numerical solution schemes for the PDEs are the Finite Difference Methods. The FDMs are basic numerical solution schemes, obtained by replacing the derivatives in the given Partial Differential Equation (PDE) by the appropriate numerical differentiation formulae. Numerical methods generally provide adequate numerical solutions for the PDEs more simply and efficiently. This is certainly so with the FDMs for solving partial differential equations. FDMs generally give solutions that are either as accurate as the data warrant or as accurate as is necessary for the technical purposes for which the solutions are prepared. In both cases a finite-difference solution is as satisfactory as one calculated from an analytical formula [1, 5].

Numerical methods generally provide adequate numerical solutions more simply and efficiently. This is certainly so with finite-difference methods for solving PDEs. In these methods (Figure 1.1), the area S bounded by the closed curve C, is overlapped by the system of rectangular meshes formed by two sets of equally spaced lines, one set parallel to Ox and the other parallel to Oy, and an approximate solution to the differential equation is found at the points of intersection $P_{1,1}, P_{1,2}, \ldots, P_{i,j}, \ldots$ of the parallel lines, which points are called mesh points.(other terms in common use are grid, nodal, pivotal or lattice points) .

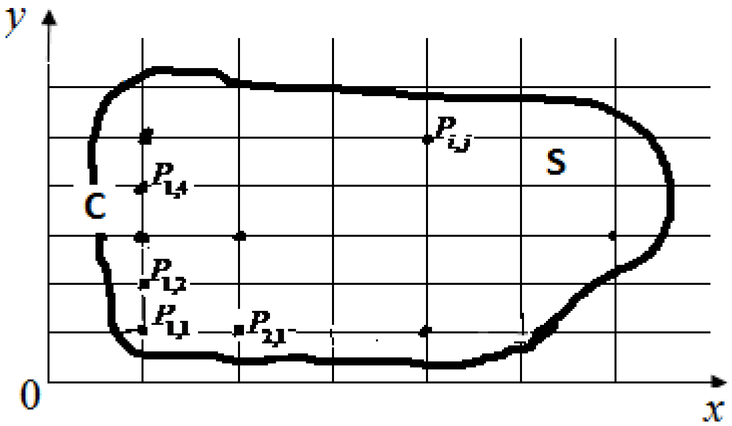

Figure 1.1 Rectangular mesh

This solution is obtained by approximating the partial differential equation over the area S by n algebraic equations involving the values of ϕ at n grid points internal to C. The approximation consists of replacing each derivative of the partial differential equation at the point $P_{i,j}$ (say) by a finite-difference approximation in terms of the values of ϕ at $P_{i,j}$ and at neighbouring grid points and boundary points and in writing down for each of the n internal mesh points the algebraic equation approximating the differential equation. This process clearly gives n algebraic equations for the n unknowns $\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{i,j}, \ldots$.[9]

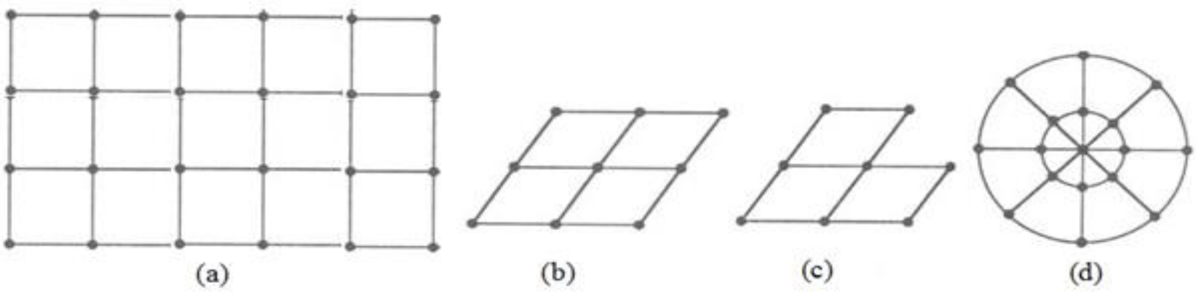

Figure 1.2 Common two dimensional grid patterns

1.2 Notation for Discrete variables

1D: $\Omega = (0, X)$, $u_i \approx u(x_i)$, $i = 0, 1, \ldots, N$

grid points $x_i = i\Delta x$ mesh size $\Delta x = \frac{X}{N}$

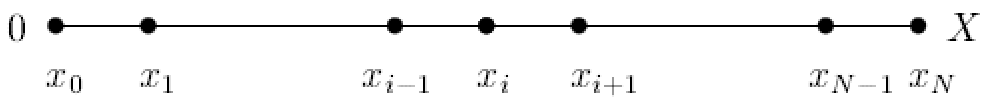

Figure 1.3 Discrete and continuous variables

There is useful relations between values of the independent variable at adjacent points are

$$x_{i+1} = x_i + \Delta x \qquad \& \qquad x_{i-1} = x_i - \Delta x \tag{1.1}$$

In the nomenclature given above, the Taylor series appears as

$$u_{i+1} = u_i + \left(\frac{du}{dx}\right)_i \Delta x + \left(\frac{d^2u}{dx^2}\right)_i \frac{(\Delta x)^2}{2!} + \left(\frac{d^3u}{dx^3}\right)_i \frac{(\Delta x)^3}{3!} + \left(\frac{d^4u}{dx^4}\right)_i \frac{(\Delta x)^4}{4!} + O(\Delta x)^5 \tag{1.2}$$

In a similar manner, the value of u at x_{i+1} is

$$u_{i-1} = u_i - \left(\frac{du}{dx}\right)_i \Delta x + \left(\frac{d^2u}{dx^2}\right)_i \frac{(\Delta x)^2}{2!} - \left(\frac{d^3u}{dx^3}\right)_i \frac{(\Delta x)^3}{3!} + \left(\frac{d^4u}{dx^4}\right)_i \frac{(\Delta x)^4}{4!} + O(\Delta x)^5 \tag{1.3}$$

Here, $O(\Delta x)^5$ is the error introduced by truncating the series. Finite difference analogs to the first and second derivatives can be obtained from the equations (1.2) and (1.3).

1.3 Finite Difference Quotients for the First and Second Derivatives

When equation (1.2) is solved for the first derivative, it takes the form

$$\left(\frac{du}{dx}\right)_i = \frac{u_{i+1} - u_i}{\Delta x} - \left(\frac{d^2u}{dx^2}\right)_i \frac{\Delta x}{2!} - \left(\frac{d^3u}{dx^3}\right)_i \frac{(\Delta x)^2}{3!} - \left(\frac{d^4u}{dx^4}\right)_i \frac{(\Delta x)^3}{4!} - \dots \tag{1.4}$$

In the equation (1.4) the term $\left(\frac{du}{dx}\right)_i = \frac{u_{i+1} - u_i}{\Delta x}$ is the first difference quotient to the first derivative. Adding the equations (1.2) and (1.3), the series is obtained as

$$u_{i+1} + u_{i-1} = 2u_i + \left(\frac{d^2u}{dx^2}\right)_i (\Delta x)^2 + 2\left(\frac{d^4u}{dx^4}\right)_i \frac{(\Delta x)^4}{4!} + \dots \tag{1.5}$$

Writing this equation explicitly for the second derivative, it becomes

$$\left(\frac{d^2u}{dx^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - \left(\frac{d^4u}{dx^4}\right)_i \frac{(\Delta x)^2}{12} - \dots \tag{1.6}$$

A finite difference quotient of the second derivative is $\left(\frac{d^2u}{dx^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$ since, first term dropped contains $(\Delta x)^2$, it is a second order correct analog. Such an analog for the first derivative can be obtained by subtracting equation (1.3) from equation (1.2).

$$\left(\frac{du}{dx}\right)_i = \frac{u_{i+1}-u_{i-1}}{2(\Delta x)} - \left(\frac{d^3u}{dx^3}\right)_i \frac{(\Delta x)^2}{6} - \dots \qquad (1.7)$$

The desired second-order-correct finite difference quotient to the first derivative, is

$\left(\frac{du}{dx}\right)_i = \frac{u_{i+1}-u_{i-1}}{2(\Delta x)}$, observe that the first term to be truncated contains $(\Delta x)^2$ [3, 7].

1.4 Finite Difference Scheme for Solving Linear Parabolic Equations

Parabolic partial differential equations arise from unsteady-state problems in which transport by conduction or diffusion is important. A general equation of this type describes the unsteady-state heating of the tapered rod. Consider the simplest parabolic equation, which describes conduction in a uniform, insulated rod:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (1.8)$$

with the simplest boundary conditions

$$\begin{aligned} u(0,\ t) &= 0 \\ u(1,\ t) &= 1 \end{aligned} \quad ;\ \text{for all } t \qquad (1.9)$$

The initial conditions to be used is

$$u(x,0) = 0; \qquad x < 1 \qquad (1.10)$$

The length variable, x, varies between 0 and 1, and the time variable, t, increases without limit from zero. The region between 0 and 1 along the x-axis is divided into R equal increments of size $\Delta x = h$, with grid points on each boundary. The time axis is divided into increments of size $\Delta t = k$ (may not be constant). $x_i = i\Delta x$; $x_{i+1} = x_i + \Delta x$ and $x_{i-1} = x_i - \Delta x$. The values of the time increment Δt may not be constant. To specify the value of u at a given point, two subscripts are used; $u(x_i, t_n) = u_{i,n}$.

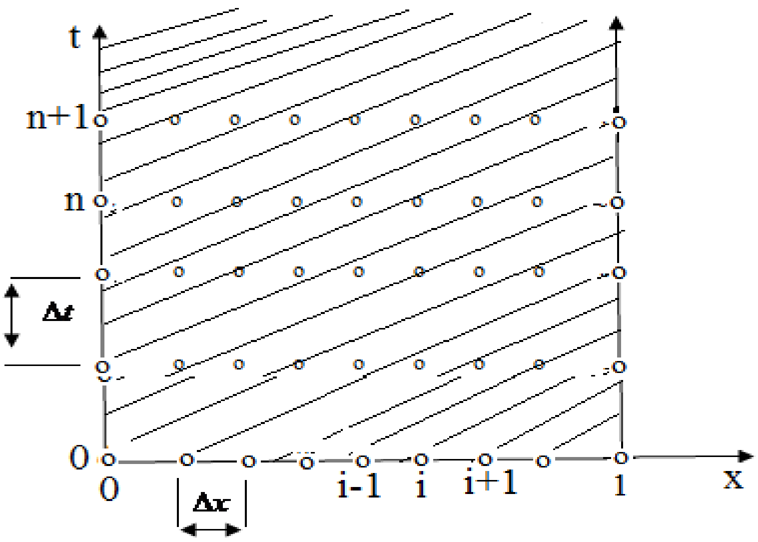

Figure 1.4 Grid points for unsteady-state problem [7]

1.5 Forward (Explicit) Difference Equation

Write the relation at the known time level which is indexed by n. This relation is

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,n} \approx \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{(\Delta x)^2} \tag{1.11}$$

It is second-order-correct in the variable x. Take the first-order-correct analog to the time derivative obtained from a Taylor series in time about the point x_i, t_n as

$$\left(\frac{\partial u}{\partial t}\right)_{i,n} = \frac{u_{i,n+1} - u_{i,n}}{\Delta t} - \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,n} \frac{\Delta t}{2!} \cdots \tag{1.12}$$

Substituting the analogs of the equations (1.11) and (1.12) into equation (1.8), the finite difference equation becomes,

$$u_{i,n+1} = r\, u_{i-1,n} + (1-2r) u_{i,n} + r\, u_{i+1,n} \quad \text{; where} \quad r = \frac{\Delta t}{(\Delta x)^2} \tag{1.13}$$

This equation is referred to as a Forward (explicit) difference equation. The

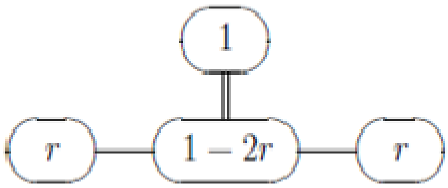

Figure 1.5 Stencil for Heat Equation for r (Forward Finite difference Scheme)

The solution of equation (1.13) approaches to that of equation (1.8), only if $r \le 0.5$.

1.6 Backward (Implicit) Difference Equation

Write the finite difference quotient at the new time level indexed by $n+1$ [7, 1].

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,n+1} = \frac{u_{i+1,n+1} - 2u_{i,n+1} + u_{i-1,n+1}}{(\Delta x)^2} \tag{1.14}$$

The analog to the time derivative obtained from Taylor series in time about the point x_i, t_{n+1}

$$\left(\frac{\partial u}{\partial t}\right)_{i,n+1} = \frac{u_{i,n+1} - u_{i,n}}{\Delta t} - \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,n+1} \frac{\Delta t}{2!} \cdots \tag{1.15}$$

Substituting the analogs of the equations (1.14) and (1.15)) into equation (1.8), we get,

$$-ru_{i-1,n+1} + (1+2r) u_{i,n+1} - ru_{i+1,n+1} = u_{i,n} \qquad \text{where, } r = \Delta t / (\Delta x)^2 \tag{1.16}$$

The equation (1.16) is a Backward (implicit) difference scheme at the unknown time level.

For $i = 1$ and $i = R-1$ for the boundary conditions of equation (1.9) are given as

$$\left(-2-\frac{(\Delta x)^2}{\Delta t}\right)u_{1,n+1}+u_{2,n+1}=\left(-\frac{(\Delta x)^2}{\Delta t}\right)u_{1,n} \tag{1.17}$$

$$u_{R-2,n+1}+\left(-2-\frac{(\Delta x)^2}{\Delta t}\right)u_{R-1,n+1}=\left(-\frac{(\Delta x)^2}{\Delta t}\right)u_{R-1,n}-1 \tag{1.18}$$

The resulting set of equations is of the form of equations (1.19); i.e. the coefficient matrix is tridiagonal [7, 8, 11]. The set of equations which has been obtained is of the form

$$\begin{aligned}
&b_1u_1+c_1u_2+0+\ldots+0=d_1\\
&a_2u_1+b_2u_2+c_2u_3+0+\ldots+0=d_2\\
&0+a_3u_2+b_3u_3+c_3u_4+0+\ldots+0=d_3\\
&\qquad\qquad\ldots\\
&0+\ldots+a_iu_{i-1}+b_iu_i+c_iu_{i+1}+0+\ldots+0=d_i\\
&\qquad\qquad\ldots\\
&0+\ldots+0+a_{R-2}u_{R-3}+b_{R-2}u_{R-2}+c_{R-2}u_{R-1}=d_{R-2}\\
&0+\ldots+0+\ldots\ \ldots0+0+a_{R-1}u_{R-2}+b_{R-1}u_{R-1}=d_{R-1}
\end{aligned} \tag{1.19}$$

Equation (1.19) represents a linear system of $R-1$ equations having $R-1$ unknowns $u_1, u_2, u_3, \ldots, u_{R-1}$, in the form of $Au=D$ where, A is a tridiagonal matrix of order $R-1$.

1.7 Cranks-Nicolson Equation (Central difference equation)

For the desired second-order-correct equation, called the Crank-Nicolson equation; all the finite differences are written about the point $x_i, t_{n+1/2}$, which is halfway between the known and the unknown time levels. In Figure 1.6, this point is shown as a cross.

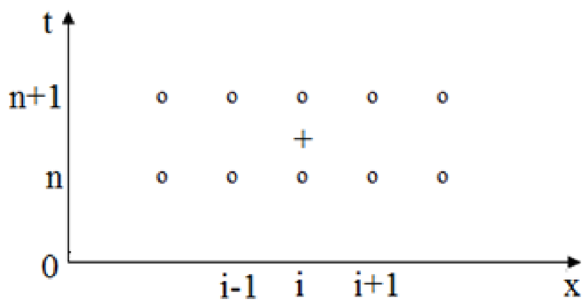

Figure 1.6 Center of analog for Crank-Nicolson equation

The values of the dependent variable, u, are computed only at the points designated by circles. The second-order-correct analog of the time derivative at the point $x_i,\ t_{n+1/2}$ is

$$\left(\frac{\partial u}{\partial t}\right)_{i,n+1/2}=\frac{u_{i,n+1}-u_{i,n}}{\Delta t}-\left(\frac{\partial^3 u}{\partial t^3}\right)_{i,n+1/2}\frac{(\Delta t)^2}{24}-\ldots- \tag{1.20}$$

The real key to the Crank-Nicolson equation is the manner of approximating $\frac{\partial^2 u}{\partial x^2}$, by the arithmetic average of its finite difference analogs at the points x_i, t_n and x_i, t_{n+1} [7, 4, 6, 2].

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,n+1} \approx \frac{1}{2}\left[\frac{u_{i+1,n+1}-2u_{i,n+1}+u_{i-1,n+1}}{(\Delta x)^2}+\frac{u_{i+1,n}-2u_{i,n}+u_{i-1,n}}{(\Delta x)^2}\right] \tag{1.21}$$

Substituting the analogs of the equations (1.20) and (1.21) into equation (2.1), we get

$$u_{i-1,n+1}+\left[-2-\frac{2(\Delta x)^2}{\Delta t}\right]u_{i,n+1}+u_{i+1,n+1}=-u_{i-1,n}+\left[2-\frac{2(\Delta x)^2}{\Delta t}\right]u_{i,n}-u_{i+1,n} \tag{1.22}$$

Substituting $r=\Delta t/(\Delta x)^2$ the Crank-Nicolson finite difference equation is obtained as,

$$-ru_{i-1,n+1}+2(1+r)u_{i,n+1}-r\,u_{i+1,n+1}=r\,u_{i-1,n}+2(1-r)u_{i,n}+r\,u_{i+1,n} \tag{1.23}$$

The boundary equations can be obtained from equation (1.23) by setting $u_{0,n+1}=u_{0,n}=0$ in the equation for $i=1$, and $u_{R,n+1}=u_{R,n}=1$ in the equation for $i=R-1$.

1.8 Numerical and Analytical Solutions of the PDE

The analytical solution of the PDE given in the equation (1.8) subject to the boundary and initial conditions of the equations (1.9) and (1.10) given in the following form [10]

$$u(x,t)=u_S(x)+u_t(x,t)=x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x)\,e^{-n^2\pi^2 t} \tag{1.24}$$

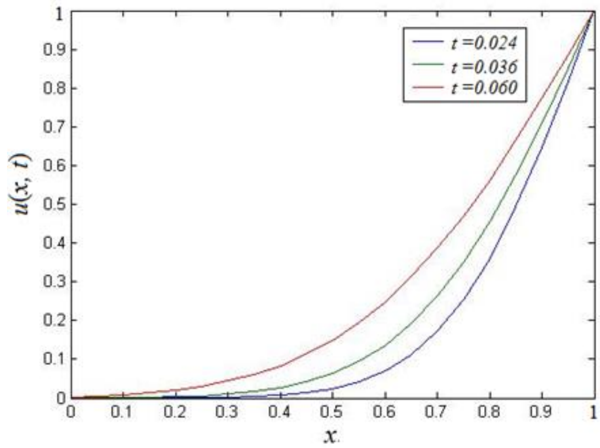

Figure 1.7 Graphical representation of the Analytical solution

Table 1.1 Numerical solution by FDMs and Analytical solution for $r = \Delta t/(\Delta x)^2 = 0.48$

	Time $t = 0.024$				**Time** $t = 0.084$			
x	**Forward**	**Backwrd**	**Central**	**Analyticl**	**Forward**	**Backwrd**	**Central**	**Analyticl**
0	0	0	0	0	0	0	0	0
0.1	0.000011	0.000145	0.000067	0.000039	0.020905	0.021049	0.020973	0.020827
0.2	0.000138	0.000626	0.000367	0.000261	0.047791	0.047822	0.047796	0.047547
0.3	0.001045	0.002375	0.001711	0.001398	0.086702	0.086198	0.086434	0.086153
0.4	0.005558	0.008184	0.006878	0.006170	0.143576	0.142119	0.142829	0.142599
0.5	0.022189	0.025369	0.023662	0.022479	0.223698	0.221082	0.222376	0.222260
0.6	0.069453	0.069901	0.069237	0.067889	0.330799	0.327205	0.328999	0.329019
0.7	0.175812	0.169052	0.171736	0.170904	0.466028	0.462065	0.464056	0.464181
0.8	0.368691	0.354831	0.361241	0.361310	0.627116	0.623686	0.625417	0.625574
0.9	0.653851	0.641661	0.647619	0.648077	0.808127	0.806131	0.807141	0.807247
1	1	1	1	1	1	1	1	1

Table 1.2 Numerical solution by FDMs and Analytical solution for Heat equation

	For $r = 0.52$ **at Time** $t = 0.065$				**For** $r = 0.56$ **at Time** $t = 0.112$			
x	**Forward**	**Backwrd**	**Central**	**Analytic**	**Forward**	**Backwrd**	**Central**	**Analytc**
0	0	0	0	0	0	0	0	0
0.1	0.023773	0.010814	0.010484	0.010273	0.776866	0.046284	0.046578	0.046572
0.2	0.055725	0.026504	0.026008	0.025626	1.510239	0.097461	0.098063	0.098062
0.3	0.10337	0.052709	0.052363	0.051892	2.152292	0.158082	0.159003	0.159023
0.4	0.172582	0.096225	0.096436	0.095989	2.650154	0.232002	0.23322	0.233277
0.5	0.266426	0.164681	0.165793	0.165486	2.950927	0.321996	0.323438	0.323537
0.6	0.384364	0.265256	0.267345	0.267248	3.012078	0.429412	0.430938	0.431068
0.7	0.522485	0.402565	0.405279	0.405378	2.814291	0.553892	0.555309	0.555449
0.8	0.674903	0.576318	0.578907	0.579099	2.372237	0.69327	0.694369	0.694487
0.9	0.835657	0.779777	0.781365	0.781511	1.73879	0.843694	0.844295	0.844363
1	1	1	1	1	1	1	1	1

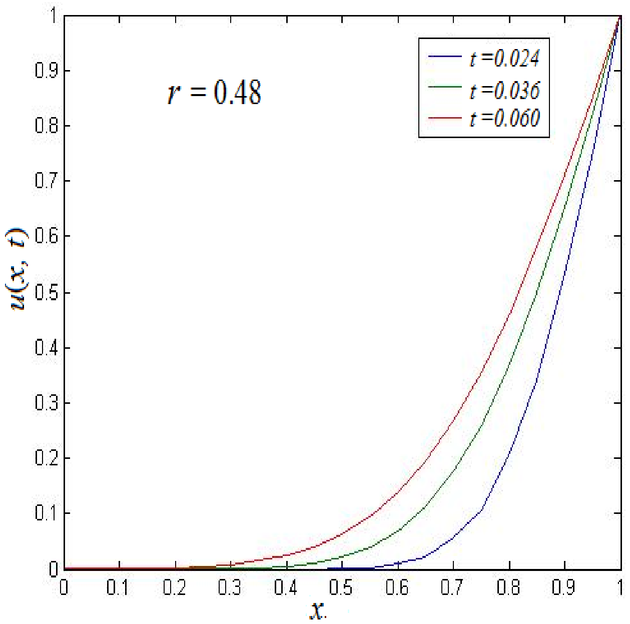

Figure 1.8 Graph of Forward difference equation

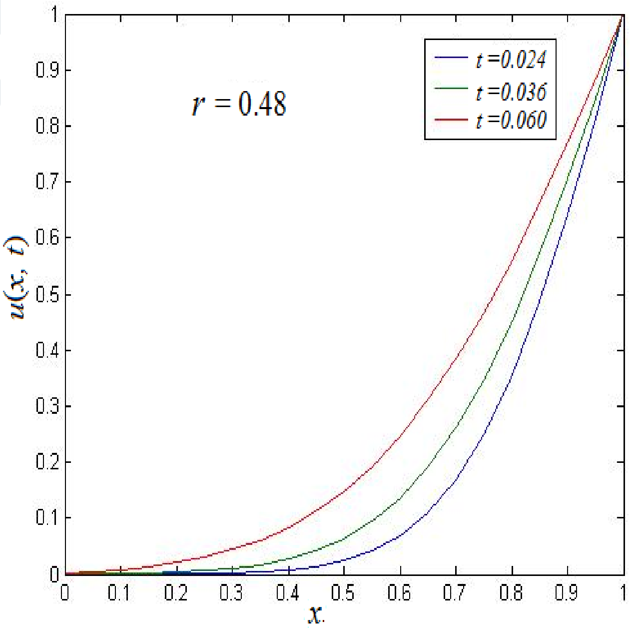

Figure 1.9 Graph of Backward difference eqn

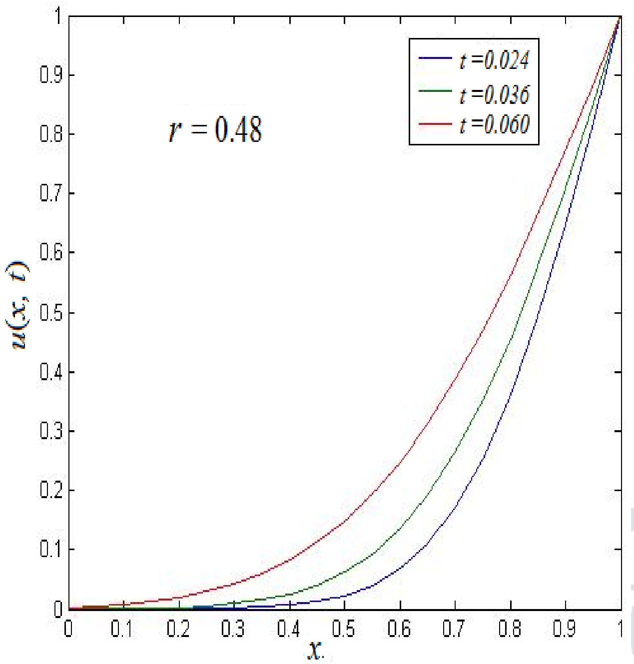

Figure 1.10 Graph of Central difference equation

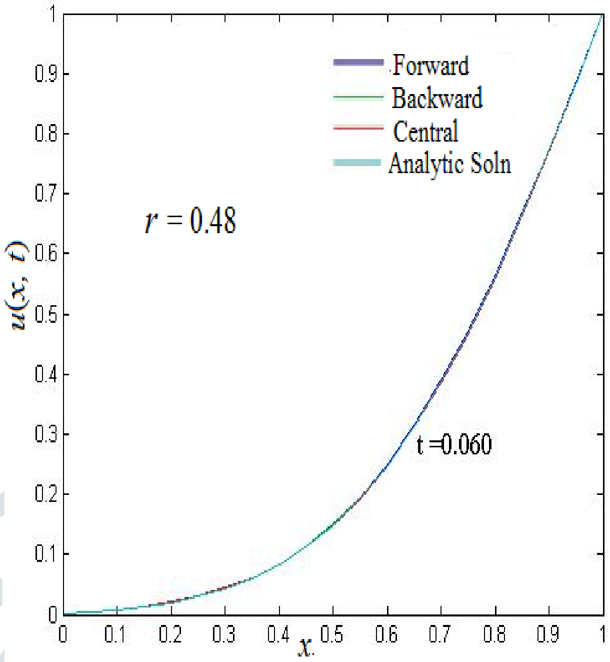

Figure 1.11 Graph of Forward, Backward, Central equation and Analytic solution

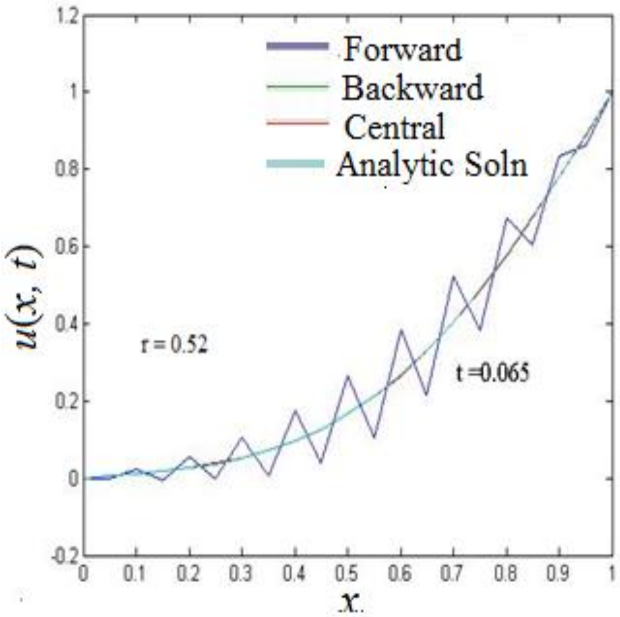

Figure 1.12 Graph: Forward, Backward, Central difference schemes and Analytical Solution for $r = 0.52$

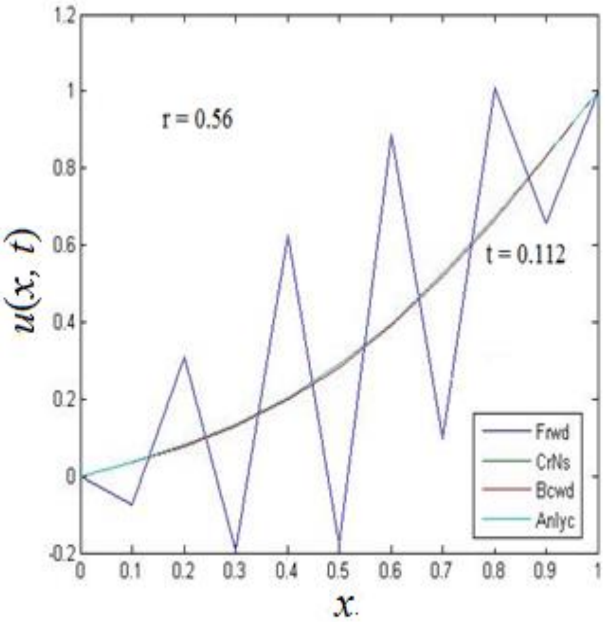

Figure 1.13 Graph: Forward, Backward, Central difference schemes and Analytical Solution for $r = 0.56$

Results, discussion and conclusion: In the present paper, first and second order derivatives have been approximated from the Taylor's series expansions. Second order correct ratio for the derivative gives better approximation than that of the first order correct. Parabolic partial differential equation describes unsteady-state problem in which transport by conduction or diffusion is important. A general equation of this type represents the unsteady-state heating of the tapered rod. The simplest parabolic PDE, which describes conduction in a uniform, insulated rod, has been considered with the realistic initial and boundary conditions. The analytical solution and numerical solutions for the PDE, equation (1.8) subject to the initial and boundary conditions given in the equations (1.9) and (1.10) respectively have been obtained by applying Forward difference equation, Backward difference equation and Crank-Nicolson (Central) difference equation, for the different values of the ratio r, $r = \Delta t/(\Delta x)^2 = 0.48 < \frac{1}{2}$, $r = 0.52 > 0.5$ and for $r = 0.56 > 0.5$. The graphical and numerical solutions given by the figures, Figure 1.8 to Figure 1.13, Table 1.1 and Table 1.2, clearly show that the forward difference equation of FDMs gives very closed numerical solution for the governing equation (1.8) subject to the given initial and boundary conditions as long as the value of the ratio $r = \Delta t/(\Delta x)^2 \leq 0.5$. It is also observed that the backward difference equation and Crank-Nicolson (central) difference equation give a numerical solution for the PDE (1.8) that matches well with the analytical solution without having any restriction on the ratio r.

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