The Fundamental Theorem of Line Integrals and Conservative Vector Field - A Review

***Basavaraj, Assistant Professor of Mathematics, Tunga Mahavidyalaya, Thirthahalli.**

Abstract

This paper attempts to study how **vector fields** in more than two dimensions all arguments are similar, also if the field is a gradient field, can compute more integrals with **Fundamental Theorem of Line Integrals.** Line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used; contour integral is used as well, although that is typically reserved for line integrals in the complex plane. In qualitative terms, a line integral in vector calculus can be thought of as a measure of the total effect of a given tensor field along a given curve.

The line integral over a scalar field (rank 0 tensor) can be interpreted as the area under the field carved out by a particular curve. This can be visualized as the surface created by z = f(x,y) and a curve C in the xy plane. The line integral of f would be the area of the "curtain" created—when the points of the surface that are directly over C are carved out. The path integral formulation of quantum mechanics actually refers not to path integrals in this sense but to functional integrals, that is, integrals over a space of paths, of a function of a possible path. However, path integrals in the sense of this article are important in quantum mechanics; for example, complex contour integration is often used in evaluating probability amplitudes in quantum scattering theory. This definition is not very useful by itself for finding exact line integrals. If data is provided, then we can use it as a guide for an approximate answer. Fortunately, there is an easier way to find the line integral when the curve is given parametrically or as a vector valued function. We will explain how this is done for curves in R2R2; the case for R3R3 is similar. If a vector field FF is the gradient of a function, F=∇fF=∇f, we say that FF is a **conservative vector field**. If FF is a conservative force field, then the integral for work, ∫CF·dr∫CF·dr, is in the form required by the Fundamental Theorem of Line Integrals. This means that in a conservative force field, the amount of work required to move an object from point aa to point bb depends only on those points, not on the path taken between them.

Key words: Fundamental Theorem of Line Integrals, Vector fields, conservative force field.

Introduction

A homogeneous vector field is a vector field that has the same value at every point (cf. Figure 26.22). Because the interpretation of a vector field is usually dynamic and involves motion, it is not considered appropriate to call these vector fields "constant" : To define

$$F(t) = \int_a^t f(x)dx$$

where f(x) is a continuous function. (This assumption can be weakened.) In other words, F(t) is simply the area under the f(x) curve from a to t. The Fundamental Theorem of Calculus states

$$F'(t) = f(t)$$

There is an analogous result for indefinite integrals. Let

$$F(t) = \int f(t)dt$$

Then

$$F'(t) = f(t)$$

The second version of the Fundamental Theorem of Calculus states that

$$F(t) = F(a) + \int_a^t F'(x)dx$$

This last formula can also be expressed in terms of an indefinite integral:

$$F(t) = \int F'(t)dt + C$$

where C is a constant.

Objective:

This paper intends to explore and analyze the **Fundamental Theorem for Line Integrals**; the line integral of the gradient of a function f gives the total change in the value of f from the start of the curve to its end.

Fundamental Theorem for Line Integrals

The following result for line integrals is analogous to the Fundamental Theorem of Calculus.

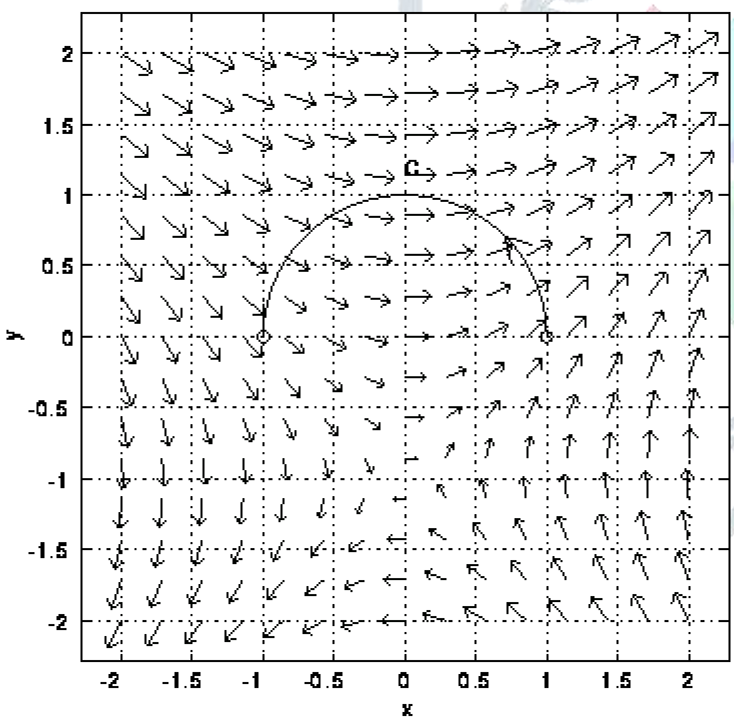
Let C be a curve in the xyz space parameterized by the vector function **r**(t)=<x(t),y(t),z(t)> for a<=t<=b. Suppose that f(x,y,z) is a differentiable function whose gradient grad f=<f_x,f_y,f_z> is continuous on C. Then

$$\int_C \text{grad} f \cdot d\mathbf{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

The above result states that the line integral of a vector field derived from a gradient depends only on the function f(x,y,z) and on the initial point (x(a),y(a),z(a)) and final point (x(b),y(b),z(b)) and not on the particular curve C. Hence, the integral is path independent. (Compare this with an example of a path dependent line integral.)

We have given the result for a function f(x,y,z) of three variables and a curve in 3 dimensional space. The above result is valid for functions of any number of variables. To verify the Fundamental Theorem for line integrals for the case that C is the top half of the circle x^2+y^2=1 traversed in the counter clockwise direction and

$$f(x, y) = xy + x.$$



A plot of the vector field and C is given above. The initial point is (1,0) and the final point (-1,0). It follows that the the value of the integral is

$$f(-1, 0) - f(1, 0) = -2.$$

This is an example where f is a function of two variables so we are dealing with a vector field in the xy plane. The vector field is

$$\text{grad} f = \langle f_x, f_y \rangle = \langle y+1, x \rangle$$

We can parameterize the curve C by vector function **r(t)**=<cos(t),sin(t)> for 0<=t<=pi. On this curve the vector field is <sin(t)+1,cos(t)> and **r**'(t)=<-sin(t),cos(t)>. It follows that

$$\int_C \text{grad} f \cdot d\mathbf{r} = \int_0^{\pi} \langle \sin t + 1, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

Cleaning this up, we have

$$\int_0^{\pi} (-\sin t - \sin^2 t + \cos^2 t) dt = \int_0^{\pi} (-\sin t + \cos(2t)) dt = -2.$$

Here we have used the identity cos^2(t)-sin^2(t)=cos(2t).

In this example the gradient function <y+1,x> is continuous at all points in the xy plane. The line integral of <y+1,x> from (1,0) to (-1,0) is equal to -2 for *any* curve joining these two points.

Conservative Vector Fields

Recall that a vector field **F** is conservative if there is a function f such that **F**=grad f. If we know that a vector field is conservative, then we can apply the Fundamental Theorem. The following result gives a test for determining if a vector field is conservative.

If **F** is a vector field defined in all of xyz space whose component functions have continuous partial derivatives and curl **F**=**0**, then **F** is a conservative vector field.

The above criterion doesn't say how to find the function f. It just says that a function f exists.

Here is an example: determine if **F**=<z,2yz,x+y^2> is conservative.

Notice that each component of **F** has continuous partial derivatives with respect to x, y, and z. If **F**=<P,Q,R>, then the definition of curl is

$$\text{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

A quick calculation shows that indeed curl **F**=**0**.

The above shows that there exists a function f(x,y,z) such that grad f=**F**=<P,Q,R>. How do we find the function f?

We know that

$$f_x = P(x,y,z) = z$$

$$f_y = Q(x,y,z) = 2yz$$

$$f_z = R(x,y,z) = x + y^2$$

We integrate to determine f. By the Fundamental Theorem of Calculus

$$f(x,y,z) = \int f_x(x,y,z)dx + G(y,z)$$

In the above integral y and z are treated as constants. We must add a function of G(y,z) because G_x(y,z)=0. Doing this calculation, we find

$$f(x,y,z) = \int z dx + G(y,z) = xz + G(y,z)$$

To determine G(y,z) we use the information that f_y=2yz and f_z=x+y^2. We have

$$f_y = 2yz = \frac{\partial}{\partial y}(xz + G(y,z)) = G_y(y,z)$$

This last equation implies

$$G_y(y,z) = 2yz$$

To determine G(y,z) we integrate with respect to y and add a constant function of z:

$$G(y,z) = \int G_y(y,z)dy + H(z) = \int 2yz dy + H(z) = y^2 z + H(z)$$

This means

$$f(x,y,z) = xz + y^2 z + H(z)$$

To find H(z) we differentiate with respect to z:

$$f_z = x + y^2 = \frac{\partial}{\partial z}(xz + y^2 z + H(z)) = x + y^2 + H'(z)$$

This last equation implies H'(z)=0. Hence, H(z)=constant. The final result is

$$f(x,y,z) = xz + y^2 z + \text{constant}$$

Line Integrals with Respect to Arc Length

Consider the following problem: a piece of string, corresponding to a curve C, lies in the xy-plane. The mass per unit length of the string is f(x,y). What is the total mass of the string?

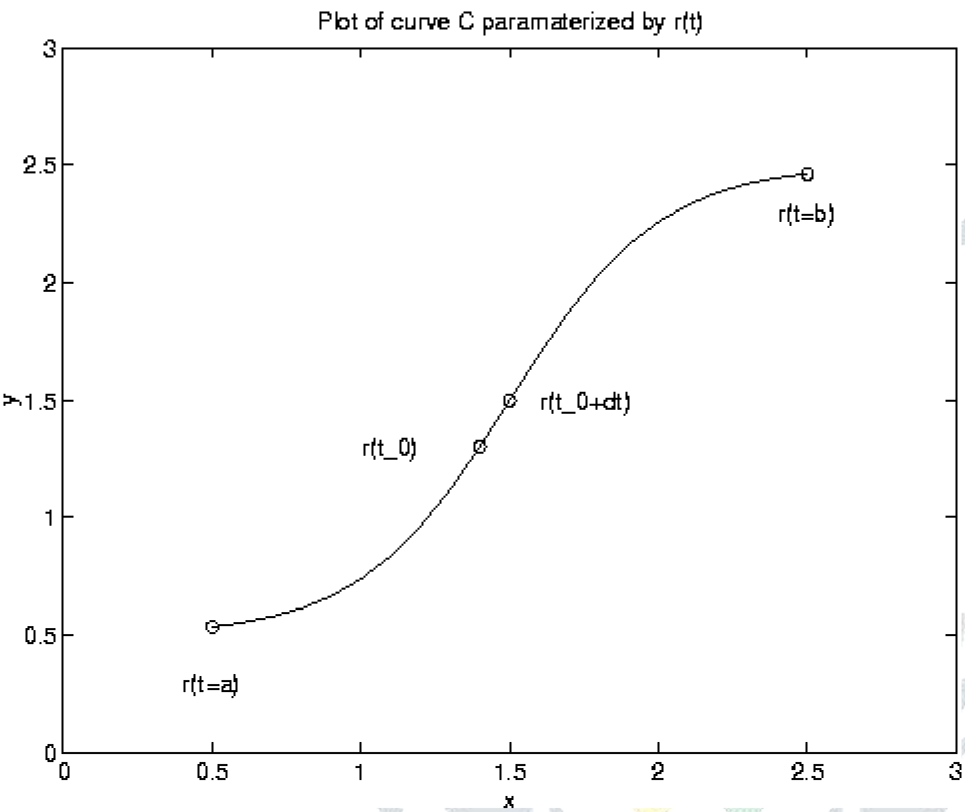
The formula for the mass is

$$\text{Mass} = \int_C f(x,y)ds.$$

The integral above is called a line integral of f(x,y) along C. It is also called a line integral with respect to arc length.

Question: how do we actually evaluate the above integral? The strategy is: (0) parameterize the curve C, (1) cut up the curve C into infinitesimal pieces, (2) determine the mass of each infinitesimal piece, (3) integrate to determine the total mass. It is assumed that C is piecewise smooth. That is, it is a union of finite number of smooth curves.

Suppose that we can describe the curve by the vector function **r**(t)=<x(t),y(t)> where a<=t<=b. Consider a portion of the curve corresponding to the infinitesimal interval t_0<=t<=t_0+dt.



The arc length of the curve, ds, on this interval is

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$$

Hence, the mass of the piece (density times length) is

$$f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt$$

The total mass of the string is the sum of all the masses of all infinitesimal pieces

$$\text{Mass} = \int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

The integral on the right is an integral of one variable.

Line integrals are not restricted to curves in the xy plane. If C is a curve in three dimensions parameterized by **r**(t)=<x(t),y(t),z(t)> with a<=t<=b, then

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}dt$$

Example

Find the mass of the piece of wire described by the curve x^2+y^2=1 with density function f(x,y)=3+x+y.

The circle of radius 1 can be parameterized by the vector function **r**(t)=<cos(t),sin(t)> with 0<=t<=2*pi. We have x(t)=cos(t) and y(t)=sin(t), so x'(t)=-sin(t) and y'(t)=cos(t). The mass is given by the formula

$$\int_0^{2\pi}(3+\cos t+\sin t)\sqrt{\sin^2 t+\cos^2 t}dt$$

The term in the square root is 1, hence we have

$$\int_0^{2\pi}(3+\cos t+\sin t)dt = 6\pi$$

Line Integrals with Respect to x, y, and z

In some applications, such as line integrals of vector fields, the following line integral with respect to x arises:

$$\int_C f(x,y,z)dx$$

This is an integral over some curve C in xyz space. It can be converted to integral in one variable. Suppose that C can be parameterized by **r**(t)=<x(t),y(t),z(t)> with a<=t<=b. Then,

$$\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))x'(t)dt$$

There are analogous formulas for integrals with respect to y and z.

In some applications, integrals with respect to x, y, and z occur in a sum:

$$\int_C (P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz)$$

If C is a curve in the xy plane and R=0, it might be possible to evaluate the line integral using Green's theorem.

Using the standard parameterization for C, this last integral becomes

$$\int_a^b (P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t))dt$$

Conclusion

Fundamental Theorem of Line Integrals implies to compute the integral of a derivative f′f′ we need to compute the values of ff at the endpoints. Something similar is true for line integrals of a certain form. In the vector setting is still "force times distance", except that "times" means "dot product". If the force varies from point to point, it is represented by a vector field FF; the displacement vector vv may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function r(t)r(t); at any point along the path, the (small) tangent vector r′Δtr′Δt gives an approximation to its motion over a short time

References

1. DeBaggis, Henry F.; Miller, Kenneth S. (1966). Foundations of the Calculus. Philadelphia: Saunders. OCLC 527896.
2. Boyer, Carl B. (1959). The History of the Calculus and its Conceptual Development. New York: Dover. OCLC 643872.
3. Bardi, Jason Socrates (2006). The Calculus Wars : Newton, Leibniz, and the Greatest Mathematical Clash of All Time. New York: Thunder's Mouth Press. ISBN 1-56025-706-7.
4. Hoffmann, Laurence D.; Bradley, Gerald L. (2004). Calculus for Business, Economics, and the Social and Life Sciences (8th ed.). Boston: McGraw Hill. ISBN 0-07-242432-X.
5. "history - Were metered taxis busy roaming Imperial Rome?". Skeptics Stack Exchange. Retrieved 29 March 2018.
6. Morris Kline, Mathematical thought from ancient to modern times, Vol. I
7. Archimedes, Method, in The Works of Archimedes ISBN 978-0-521-66160-7
8. Dun, Liu; Fan, Dainian; Cohen, Robert Sonné (1966). A comparison of Archimdes' and Liu Hui's studies of circles. Chinese studies in the history and philosophy of science and technology. 130. Springer. p. 279. ISBN 978-0-7923-3463-7.,pp. 279ff
9. Katz, Victor J. (2008). A history of mathematics (3rd ed.). Boston, MA: Addison-Wesley. p. 203. ISBN 978-0-321-38700-4.
10. Zill, Dennis G.; Wright, Scott; Wright, Warren S. (2009). Calculus: Early Transcendentals (3 ed.). Jones & Bartlett Learning. p. xxvii. ISBN 978-0-7637-5995-7. Extract of page 27
11. Katz, V.J. 1995. "Ideas of Calculus in Islam and India." Mathematics Magazine (Mathematical Association of America), 68(3):163–174.
12. "Indian mathematics".
13. von Neumann, J., "The Mathematician", in Heywood, R.B., ed., The Works of the Mind, University of Chicago Press, 1947, pp. 180–196. Reprinted in Bródy, F., Vámos, T., eds., The

Neumann Compendium, World Scientific Publishing Co. Pte. Ltd., 1995, ISBN 981-02-2201-7, pp. 618–626.

14. "Johannes Kepler: His Life, His Laws and Times". NASA. 24 September 2016. Retrieved 10 June 2018.
15. Chisholm, Hugh, ed. (1911). "Infinitesimal Calculus § History" . Encyclopædia Britannica. 14 (11th ed.). Cambridge University Press. p. 537.
16. André Weil: Number theory: An approach through History from Hammurapi to Legendre. Boston: Birkhauser Boston, 1984, ISBN 0-8176-4565-9, p. 28.
17. Blank, Brian E.; Krantz, Steven George (2006). Calculus: Single Variable, Volume 1 (Illustrated ed.). Springer Science & Business Media. p. 248. ISBN 978-1-931914-59-8.
18. Ferraro, Giovanni (2007). The Rise and Development of the Theory of Series up to the Early 1820s (Illustrated ed.). Springer Science & Business Media. p. 87. ISBN 978-0-387-73468-2.
19. Leibniz, Gottfried Wilhelm. The Early Mathematical Manuscripts of Leibniz. Cosimo, Inc., 2008. p. 228. Copy
20. Allaire, Patricia R. (2007). Foreword. A Biography of Maria Gaetana Agnesi, an Eighteenth-century Woman Mathematician. By Cupillari, Antonella. Edwin Mellen Press. p. iii. ISBN 978-0-7734-5226-8.
21. Unlu, Elif (April 1995). "Maria Gaetana Agnesi". Agnes Scott College.
22. Boyer, Carl Benjamin (1949). The History of the Calculus and its Conceptual Development. Hafner. Dover edition 1959, ISBN 0-486-60509-4
23. Courant, Richard ISBN 978-3-540-65058-4 Introduction to calculus and analysis 1.
24. Edmund Landau. ISBN 0-8218-2830-4 Differential and Integral Calculus, American Mathematical Society.
25. Robert A. Adams. (1999). ISBN 978-0-201-39607-2 Calculus: A complete course.
26. Albers, Donald J.; Richard D. Anderson and Don O. Loftsgaarden, ed. (1986) Undergraduate Programs in the Mathematics and Computer Sciences: The 1985–1986 Survey, Mathematical Association of America No. 7.
27. John Lane Bell: A Primer of Infinitesimal Analysis, Cambridge University Press, 1998. ISBN 978-0-521-62401-5. Uses synthetic differential geometry and nilpotent infinitesimals.
28. Florian Cajori, "The History of Notations of the Calculus." Annals of Mathematics, 2nd Ser., Vol. 25, No. 1 (Sep. 1923), pp. 1–46.
29. Leonid P. Lebedev and Michael J. Cloud: "Approximating Perfection: a Mathematician's Journey into the World of Mechanics, Ch. 1: The Tools of Calculus", Princeton Univ. Press, 2004.
30. Cliff Pickover. (2003). ISBN 978-0-471-26987-8 Calculus and Pizza: A Math Cookbook for the Hungry Mind.

31. Michael Spivak. (September 1994). ISBN 978-0-914098-89-8 Calculus. Publish or Perish publishing.
32. Tom M. Apostol. (1967). ISBN 978-0-471-00005-1 Calculus, Volume 1, One-Variable Calculus with an Introduction to Linear Algebra. Wiley.
33. Tom M. Apostol. (1969). ISBN 978-0-471-00007-5 Calculus, Volume 2, Multi-Variable Calculus and Linear Algebra with Applications. Wiley.
34. Silvanus P. Thompson and Martin Gardner. (1998). ISBN 978-0-312-18548-0 Calculus Made Easy.
35. Mathematical Association of America. (1988). Calculus for a New Century; A Pump, Not a Filter, The Association, Stony Brook, NY. ED 300 252.
36. Thomas/Finney. (1996). ISBN 978-0-201-53174-9 Calculus and Analytic geometry 9th, Addison Wesley.
37. Weisstein, Eric W. "Second Fundamental Theorem of Calculus." From MathWorld—A Wolfram Web Resource.