II-NORMAL SPACES IN TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to introduce and study a new class of spaces, called ii-normal spaces. The relationships among β*g-normal, s-normal, α-normal, γ-normal and ii-normal spaces are investigated. Moreover, we introduce the forms of generalized ii-closed (briefly gii-closed) and ii-generalized closed (briefly iig-closed) functions. We obtain characterizations of ii-normal spaces, properties of the forms of generalized ii-closed functions and preservation theorems.

Key Words: ii-open, gii-closed and iig-closed sets; ii-normal, s-normal, α-normal and β*g-normal spaces; ii-closed and ii-gii-closed functions.

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1. Introduction

Normality is an important topological property and hence it is of significance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. In 1937, Stone [**21**] introduced the concept of regular-open sets. In 1963, Levine [**13**] introduced the notion of semi-open sets and obtained their properties. In 1965, Njastad [**19**] introduced the notion of α-open sets and obtained their properties. In 1970, Levine [**14**] initiated the investigation of g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by a large number of topologists. In 1978, Maheshwari [**15**] introduced the notion of s-normal spaces and obtained their characterizations. In 1983, Abd El-Monsef [**1**] introduced the notion of β-open sets. In 1987, Bhattacharyya and Lahiri [**6**] introduced the concepts of sg-closed sets. In 1990, Arya and Nour [**3**] introduced the concepts of gs-closed sets. In 1994, Maki et al [**16**] introduced the concepts of gα-closed and αg-closed sets. In 2007, Ekicii [**8**] introduced the notion of γ-normal spaces and obtained their characterizations. In 2008, Maki et al. [**17**] introduced the concepts s*g-closed sets and s*-normal spaces, and obtained their characterizations. In 2009, Benchalli [**5**] introduced the notion of α-normal spaces and obtained their characterizations. In 2015, Sharma and Hamant [**20**] introduced the concepts of β*g-closed sets and β*-normal spaces, and obtained their characterizations. In 2019, Hamant [**11**] introduced the concept of β*g-normal spaces, and obtained their characterizations. In 2019, Mohammed and Abdullah [**2**] introduced the concepts of ii-open sets and obtained their properties. In 2019, Hamant [**10**] introduced the concepts of ii-separation axioms and ii-closed functions.

2. Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \Im) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space $(X, \Im)$ into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

2.1 Definition. A subset A of a space X is said to be:

(1) **regular open** [**21**] if A = int(cl(A)).

(2) **semi-open** [**13**] if $A \subset$ cl(int(A)).

(3) **α-open** [**19**] if $A \subset$ int(cl(int(A))).

(4) **β-open** [**1**] if A ⊂ cl(int(cl(A))).

(5) **γ-open** [**9**] if A ⊂ int(cl(A)) ∪ cl(int(A)).

(6) **ii-open** [**2**] set if there exists an open set G ∈ ℑ, such that

(i) G ≠ ϕ, X

(ii) A ⊂ cl(A ∩ G)

(iii) int(A) = G.

The complement of a regular open (resp. semi-open, α-open, β-open, γ-open, ii-open,) set is called **regular closed** (resp. **semi-closed**, **α-closed**, **β-closed**, **γ-closed**, **ii-closed**).

The intersection of all α-closed (resp. β-closed, γ-closed, ii-closed, semi-closed) sets containing A is called the **α-closure** (resp. **β-closure**, **γ-closure**, **ii-closure**, **semi-closure**) of A and is denoted by **α-cl(A)** (resp. **β-cl(A)**, **γ-cl(A)**, **ii-cl(A)**, **s-cl(A)**). Dually, the **α-interior** (resp. **β-interior**, **γ-interior**, **ii-interior**, **semi-interior**) of A, denoted by **α-int(A)** (resp. **β-int(A)**, **γ-int(A)**, **ii-int(A)**, **s-int(A)**) is defined to be the union of all α-open (resp. β-open, γ-open, ii-open, semi-open) sets contained in A.

The family of all ii-open (resp. ii-closed, regular open, regular closed, semi-open, semi-closed, α-open, α-closed, β-open, β-closed, γ-open, γ-closed) sets of a space X is denoted by ii-O(X) (resp. ii-C(X), R-O(X), R-C(X), S-O(X), S-C(X), α-O(X), α-C(X), β-O(X), β-C(X), γ-O(X), γ-C(X)).

2.2 Definition. A subset A of a space (X, ℑ) is said to be

(1) **g-closed** [**14**], if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(2) **s*g-closed** [**17**] if cl(A) ⊂ U whenever A ⊂ U and U ∈ S-O(X).

(3) **β*g-closed** [**20**] if cl(A) ⊂ U whenever A ⊂ U and U ∈ β-O(X).

(4) **gs-closed** [**3**] if s-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(5) **sg-closed** [**6**] if s-cl(A) ⊂ U whenever A ⊂ U and U ∈ S-O(X).

(6) **αg-closed** [**16**] if α-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(7) **gα-closed** [**16**] if α-cl(A) ⊂ U whenever A ⊂ U and U ∈ α-O(X).

(8) **gγ-closed** [**8**] if γ-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(9) **γg-closed** [**8**] if γ-cl(A) ⊂ U whenever A ⊂ U and U ∈ γ-O(X).

(10) **generalized ii-closed** (briefly **gii-closed**) if ii-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(11) **ii-generalized closed** (briefly **iig-closed**) if ii-cl(A) ⊂ U whenever A ⊂ U and U ∈ ii-O(X).

The complement of ii-closed (resp. gii-closed, iig-closed, g-closed, αg-closed, gα-closed, gγ-closed, γg-closed, gs-closed, sg-closed) set is said to be ii-open (resp. gii-open, iig-open, g-open, αg-open, gα-open, gγ-open, γg-open, gs-open, sg-open,).

2.3 Remark. We have the following implications for the properties of subsets:

closed	→	β^*g-closed	→	s*g-closed	→	g-closed
		↓		↓		↓
		α-closed	→	gα-closed	→	αg-closed
		↓			↓	
		s-closed	→	sg-closed	→	gs-closed
		↓				↓
		ii-closed	→	iig-closed	→	gii-closed
		↓				↓
		γ-closed	→	γg-closed	→	gγ-closed

Where none of the implications is reversible as can be seen from the following examples:

2.4 Example. Let X = {a, b, c} and ℑ = {ϕ, {a}, {b}, {a, b}, X}. Then

(1) closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(2) g-closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(3) β^*g-closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(4) s*g-closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(5) s-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {b, c}.

(6) gs-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {b, c}.

(7) sg-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {b, c}.

(8) α-closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(9) αg-closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(10) gα-closed sets in (X, ℑ) are ϕ, X, {c}, {a, c}, {b, c}.

(11) ii-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, c}, {b, c}.

(12) gii-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, c}, {b, c}.

(13) iig-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, c}, {b, c}.

(14) γ-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, c}, {b, c}.

(15) gγ-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, c}, {b, c}.

(16) γg-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, c}, {b, c}.

2.5 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b, c, d}, X}. Then

(1) closed sets in (X, $\Im$) are ϕ, X, {a}, {b, c, d}.

(2) g-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(3) β^*g-closed sets in (X, $\Im$) are ϕ, X, {a}, {b, c, d}.

(4) s^*g-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(5) s-closed sets in (X, $\Im$) are ϕ, X, {a}, {b, c, d}.

(6) gs-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(7) sg-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(8) α-closed sets in (X, $\Im$) are ϕ, X, {a}, {b, c, d}.

(9) αg-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(10) gα-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(11) ii-closed sets in (X, $\Im$) are ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(12) gii-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(13) iig-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(14) γ-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(15) gγ-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(16) γg-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

2.6 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X}. Then

(1) closed sets in (X, $\Im$) are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(2) g-closed sets in (X, $\Im$) are ϕ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(3) β^*g-closed sets in (X, $\Im$) are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(4) s^*g-closed sets in (X, ℑ) are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(5) s-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(6) gs-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(7) sg-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(8) α-closed sets in (X, ℑ) are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(9) αg-closed sets in (X, ℑ) are ϕ, X, {c}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(10) gα-closed sets in (X, ℑ) are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(11) ii-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(12) gii-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(13) iig-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(14) γ-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(15) gγ-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(16) γg-closed sets in (X, ℑ) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

3. ii-normal spaces

3.1 Definition. A space X is said to be **ii-normal** if for any pair of disjoint closed sets A and B, there exist disjoint ii-open sets U and V such that A ⊂ U and B ⊂ V.

3.2 Definition. A space X is said to be **s-normal** [**15**] (resp. **α-normal** [**5**], **γ-normal** [**8**], **β*g-normal** [**11**]) if for any pair of disjoint closed sets A and B, there exist disjoint s-open (resp. α-open, γ-open, β*g-open) sets U and V such that A ⊂ U and B ⊂ V.

3.3 Remark. The following diagram holds for a topological space (X, ℑ) :

normal → β*g-normal

↓ ↓

α-normal → s-normal → ii-normal → γ-normal

None of these implications is reversible as shown by the following examples.

3.4 Example. Let X = {a, b, c, d} and ℑ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}}. Let A = {c} and b = {d} be closed sets, there exist disjoint s-open sets U = {a, c} and V = {b, d} such that A ⊂ U and B ⊂ V. Then the space (X, ℑ) is s-normal as well as ii-normal spaces also γ-normal spaces, since every s-open set is ii-open as well as γ-open. But it is not α-normal, since U and V are not α-open sets.

3.5 Example. Let X = {a, b, c, d} and $\Im$ = {X, ϕ, {b, d}, {a, b, d}, {b, c, d}}. Then the space (X, $\Im$) is γ-normal. But it is neither ii-normal nor s-normal.

3.6 Example. Let X = {a, b, c} and $\Im$ = {X, ϕ, {a}, {b}, {a, b}, {b, c}, X}. Then the space (X, $\Im$) is normal as well as ii-normal.

3.7 Example. Let X = {a, b, c, d} and $\Im$ = {X, ϕ, {a}, {c}, {a, c}, {b, c, d}}. Then the space (X, $\Im$) is normal as well as ii-normal.

3.8 Theorem. For a space X the following are equivalent:

(1) X is ii-normal,

(2) For every pair of open sets U and V whose union is X, there exist ii-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,

(3) For every closed set H and every open set K containing H, there exists an ii-open set U such that $H \subset U \subset$ ii-cl(U) $\subset K$.

Proof: (1) $\Rightarrow$ (2) : Let U and V be a pair of open sets in an ii-normal space X such that $X = U \cup V$. Then $X - U$ and $X - V$ are disjoint closed sets. Since X is ii-normal, there exist disjoint ii-open sets U_1 and V_1 such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $A = X - U_1$, $B = X - V_1$. Then A and B are ii-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) $\Rightarrow$ (3) : Let H be a closed set and K be an open set containing H. Then $X - H$ and K are open sets whose union is X. Then by (2), there exist ii-closed sets M_1 and M_2 such that $M_1 \subset X - H$ and $M_2 \subset K$ and $M_1 \cup M_2 = X$. Then $H \subset X - M_1$, $X - K \subset X - M_2$ and $(X - M_1) \cap (X - M_2) = \phi$. Let $U = X - M_1$ and $V = X - M_2$. Then U and V are disjoint ii-open sets such that $H \subset U \subset X - V \subset K$. As $X - V$ is ii-closed set, we have ii-cl(U) $\subset X - V$ and $H \subset U \subset$ ii-cl(U) $\subset K$.

(3) $\Rightarrow$ (1) : Let H_1 and H_2 be any two disjoint closed sets of X. Put $K = X - H_2$, then $H_2 \cap K = \phi$. $H_1 \subset K$, where K is an open set. Then by (3), there exists an ii-open set U of X such that $H_1 \subset U \subset$ ii-cl(U) $\subset K$. It follows that $H_2 \subset X -$ ii-cl(U) = V , say, then V is ii-open and $U \cap V = \phi$. Hence H_1 and H_2 are separated by ii-open sets U and V. Therefore X is ii-normal.

4. Some related functions with ii-normal spaces

4.1 Definition . Let X be a topological space. A subset $N \subset X$ is called an **ii-neighbourhood** (briefly **ii-nhd**) [**10**] of a point $x \in X$ if there exist an ii-open set G such that $x \in G \subset N$.

4.2 Definition . A function $f : X \to Y$ is called

(1) **R-map** [7] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y ,

(2) **completely continuous** [**4**] if $f^{-1}(V)$ is regular open in X for every open set V of Y ,

(3) **rc-continuous** [**12**] if for each regular closed set F in Y, $f^{-1}(F)$ is regular closed in X.

4.3 Definition . A function $f : X \to Y$ is called

(1) **pre ii-open** if $f(U) \in$ ii-O(Y) for each $U \in$ ii-O(X),

(2) **pre ii-closed** if $f(U) \in$ ii-C(Y) for each $U \in$ ii-C(X),

(3) **almost ii-irresolute** if for each x in X and each ii-neighbourhood V of f(x), ii-cl(f^{-1}(V)) is an ii-neighbourhood of x.

4.4 Theorem . A function f : X → Y is pre ii-closed if and only if for each subset A in Y and for each ii-open set U in X containing $f^{-1}(A)$, there exists an ii-open set V containing A such that $f^{-1}(V) \subset U$.

Proof: (⇒) : Suppose that f is pre ii-closed. Let A be a subset of Y and U ∈ ii-O(X) containing $f^{-1}(A)$. Put V = Y − f(X − U), then V is an ii-open set of Y such that A ⊂ V and $f^{-1}(V) \subset U$.

(⇐) : Let K be any ii-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and X − K ∈ ii-O(X). There exists an ii-open set V of Y such that Y − f(K) ⊂ V and $f^{-1}(V) \subset X - K$. Therefore, we have f(K) ⊃ Y − V and $K \subset f^{-1}(Y - V)$. Hence, we obtain f(K) = Y − V and f(K) is ii-closed in Y . This shows that f is pre ii-closed.

4.5 Lemma. For a function f : X → Y , the following are equivalent:

(1) f is almost ii-irresolute,

(2) $f^{-1}(V) \subset$ ii-int(ii-cl(f^{-1} (V))) for every V ∈ ii-O(Y).

4.6 Theorem. A function f : X → Y is almost ii-irresolute if and only if f(ii-cl(U)) ⊂ ii-cl(f(U)) for every U ∈ ii-O(X).

Proof: (⇒) : Let U ∈ ii-O(X). Suppose y ∉ ii-cl(f(U)). Then there exists V ∈ ii-O(Y) such that V ∩ f(U) = ϕ. Hence, $f^{-1}(V) \cap U = \phi$. Since U ∈ ii-O(X), we have ii-int(ii-cl(f^{-1}(V))) ∩ ii-cl(U) = ϕ. Then by **Lemma 4.5**, f^{-1} (V) ∩ ii-cl(U) = ϕ and hence V ∩ f(ii-cl(U)) = ϕ. This implies that y ∉ f(ii-cl(U)).

(⇐) : If V ∈ ii-O(Y), then M = X − ii-cl(f^{-1}(V)) ∈ ii-O(X). By hypothesis, f(ii-cl(M)) ⊂ ii-cl(f(M)) and hence X − ii-int(ii-cl(f^{-1}(V))) = ii-cl(M) ⊂ f^{-1}(ii-cl(f(M))) ⊂ f^{-1}(ii-cl(f(X − f^{-1}(V)))) ⊂ f^{-1}(ii-cl(Y − V)) = f^{-1} (Y − V) = X − f^{-1}(V). Therefore, $f^{-1}(V) \subset$ ii-int(ii-cl(f^{-1}(V))). By **Lemma 4.5**, f is almost ii-irresolute.

4.7 Theorem. If f : X → Y is a pre ii-open continuous almost ii-irresolute function from an ii-normal space X onto a space Y , then Y is ii-normal.

Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is ii-normal, there exists an ii-open set U in X such that $f^{-1}(A) \subset U \subset$ ii-cl(U) $\subset f^{-1}(B)$ by **Theorem 3.8**. Then, f(f^{-1}(A)) ⊂ f(U) ⊂ f(ii-cl(U)) ⊂ f(f^{-1}(B)). Since f is pre ii-open almost ii-irresolute surjection, we obtain A ⊂ f(U) ⊂ ii-cl(f(U)) ⊂ B. Then again by **Theorem 3.8,** the space Y is ii-normal.

4.8 Theorem. If f : X → Y is a pre ii-closed continuous function from an ii-normal space X onto a space Y , then Y is ii-normal.

Proof: Let M_1 and M_2 be disjoint closed sets. Then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are closed sets. Since X is ii-normal, then there exist disjoint ii-open sets U and V such that $f^{-1}(M_1) \subset U$ and $f^{-1}(M_2) \subset V$. By **Theorem 4.4**, there exist ii-open sets A and B such that $M_1 \subset A$, $M_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is ii-normal.

4.9 Definition. A topological space (X, ℑ) is called an α-space [**12**] if ℑ = α-O(X).

4.9 Definition. A function f : X → Y is called α-closed [**18**] if for each closed set in X, f(U) is α-closed set in Y.

4.8 Theorem. If f : X → Y is an α-closed continuous surjection and X is normal, then Y is ii-normal.

Proof: Let A and B be disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X by continuity of f. Since X is normal, there exist disjoint open sets U and V in X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By **Theorem 4.3 in [18]**, there exist disjoint α-open sets G and H in Y such that A ⊂ G and B ⊂ H. Since every α-open set is ii-open, G and H are disjoint ii-open sets containing A and B, respectively. Therefore, Y is ii-normal.

5. Generalized ii-closed functions

5.1 Definition. A function f : X → Y is said to be

(1) **ii-closed** [**10**] if f(A) is ii-closed in Y for each closed set A of X,

(2) **iig-closed** if f(A) is iig-closed in Y for each closed set A of X,

(3) **gii-closed** if f(A) is gii-closed in Y for each closed set A of X.

5.2 Definition. A function f : X → Y is said to be

(1) **quasi ii-closed** if f(A) is closed in Y for each A ∈ ii-C(X),

(2) **ii-iig-closed** if f(A) is iig-closed in Y for each A ∈ ii-C(X),

(3) **ii-gii-closed** if f(A) is gii-closed in Y for each A ∈ ii-C(X),

(4) **almost gii-closed** if f(A) is gii-closed in Y for each A ∈ R-C(X).

5.3 Definition. A function f : X → Y is said to be **ii-gii-continuous** if $f^{-1}(K)$ is gii-closed in X for every K ∈ ii-C(Y).

5.4 Definition. A function f : X → Y is said to be **ii-irresolute** [**10**] if $f^{-1}(V)$ ∈ ii-O(X) for every V ∈ ii-O(Y).

5.5 Theorem. Let f : X → Y and g : Y → Z be functions. Then

(1) the composition gof : X → Z is ii-gii-closed if f is ii-gii-closed and g is continuous ii-gii-closed.

(2) the composition gof : X → Z is ii-gii-closed if f is pre ii-closed and g is ii-gii-closed.

(3) the composition gof : X → Z is ii-gii-closed if f is quasi ii-closed and g is gii-closed.

5.6 Theorem. Let f : X → Y and g : Y → Z be functions and let the composition gof : X → Z be ii-gii-closed. If f is an ii-irresolute surjection, then g is ii-gii-closed.

Proof: Let K ∈ ii-C(Y). Since f is ii-irresolute and surjective, $f^{-1}(K)$ ∈ ii-C(X) and $(gof)(f^{-1}(K)) = g(K)$. Hence, g(K) is gii-closed in Z and hence g is ii-gii-closed.

5.7 Lemma. A function f : X → Y is ii-gii-closed if and only if for each subset B of Y and each U ∈ ii-O(X) containing $f^{-1}(B)$, there exists a gii-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof: (⇒) : Suppose that f is ii-gii-closed. Let B be a subset of Y and U ∈ ii-O(X) containing $f^{-1}(B)$. Put V = Y − f(X − U), then V is a gii-open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

(⇐) : Let K be any ii-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and X − K ∈ ii-O(X). There exists a gii-open set V of Y such that $Y - f(K) \subset V$ and $f^{-1}(V) \subset X - K$. Therefore, we have $f(K) \supset Y - V$ and $K \subset f^{-1}(Y - V)$. Hence, we obtain f(K) = Y − V and f(K) is gii-closed in Y . This shows that f is ii-gii-closed.

5.8 Theorem. If f : X → Y is continuous ii-gii-closed, then f(H) is gii-closed in Y for each gii-closed set H of X.

Proof: Let H be any gii-closed set of X and V an open set of Y containing f(H). Since $f^{-1}(V)$ is an open set of X containing H, $ii\text{-}cl(H) \subset f^{-1}(V)$ and hence $f(ii\text{-}cl(H)) \subset V$. Since f is ii-gii-closed and ii-cl(H) ∈ ii-C(X), we have $ii\text{-}cl(f(H)) \subset ii\text{-}cl(f(ii\text{-}cl(H))) \subset V$. Therefore, f(H) is gii-closed in Y .

5.9 Remark. Every ii-irresolute function is ii-gii-continuous but not conversely.

5.10 Theorem. A function f : X → Y is ii-gii-continuous if and only if $f^{-1}(V)$ is gii-open in X for every V ∈ ii-O(Y).

5.11 Theorem. If f : X → Y is closed ii-gii-continuous, then $f^{-1}(K)$ is gii-closed in X for each gii-closed set K of Y .

Proof: Let K be a gii-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put V = Y − f(X − U), then V is open in Y , K ⊂ V , and $f^{-1}(V)$ ⊂ U. Therefore, we have ii-cl(K) ⊂ V and hence $f^{-1}(K) \subset f^{-1}$(ii-cl(K)) ⊂ $f^{-1}(V)$ ⊂ U. Since f is ii-gii-continuous, f^{-1}(ii-cl(K)) is gii-closed in X and hence ii-cl($f^{-1}(K)$) ⊂ ii-cl(f^{-1}(ii-cl(K))) ⊂ U. This shows that $f^{-1}(K)$ is gii-closed in X.

5.12 Corollary. If f : X → Y is closed ii-irresolute, then $f^{-1}(K)$ is gii-closed in X for each gii-closed set K of Y.

5.13 Theorem. If f : X → Y is an open ii-gii-continuous bijection, then $f^{-1}(K)$ is gii-closed in X for every gii-closed set K of Y .

Proof: Let K be a gii-closed set of Y and U an open set of X containing $f^{-1}(K)$. Since f is an open surjective, K = f(f^{-1} (K)) ⊂ f(U) and f(U) is open. Therefore, ii-cl(K) ⊂ f(U). Since f is injective, $f^{-1}(K) \subset f^{-1}$(ii-cl(K)) ⊂ f^{-1}(f(U)) = U. Since f is ii-gii-continuous, f^{-1}(ii-cl(K)) is gii-closed in X and hence ii-cl($f^{-1}(K)$) ⊂ ii-cl(f^{-1}(ii-cl(K))) ⊂ U. This shows that $f^{-1}(K)$ is gii-closed in X.

5.14 Theorem. Let f : X → Y and g : Y → Z be functions and let the composition gof : X → Z be ii-gii-closed. If g is an open ii-gii-continuous bijection, then f is ii-gii-closed.

Proof: Let H ∈ ii-C(X). Then (gof)(H) is gii-closed in Z and g^{-1}((gof)(H)) = f(H). By **Theorem 5.13**, f(H) is g ii-closed in Y and hence f is ii-gii-closed.

5.15 Theorem. Let f : X → Y and g : Y → Z be functions and let the composition gof : X → Z be ii-gii-closed. If g is a closed ii-gii-continuous injection, then f is ii-gii-closed.

Proof: Let H ∈ ii-C(X). Then (gof)(H) is gii-closed in Z and g^{-1}((gof)(H)) = f(H). By **Theorem 5.11**, f(H) is gii-closed in Y and hence f is ii-gii-closed.

6. Preservation theorems and other characterizations of ii-normal spaces

6.1 Theorem. For a topological space X, the following are equivalent :

(a) X is ii-normal,

(b) for any pair of disjoint closed sets A and B of X, there exist disjoint gii-open sets U and V of X such that A ⊂ U and B ⊂ V ,

(c) for each closed set A and each open set B containing A, there exists a gii-open set U such that cl(A) ⊂ U ⊂ ii-cl(U) ⊂ B,

(d) for each closed A and each g-open set B containing A, there exists an ii-open set U such that A ⊂ U ⊂ ii-cl(U) ⊂ int(B),

(e) for each closed A and each g-open set B containing A, there exists a gii-open set G such that A ⊂ G ⊂ ii-cl(G) ⊂ int(B),

(f) for each g-closed set A and each open set B containing A, there exists an ii-open set U such that cl(A) ⊂ U ⊂ ii-cl(U) ⊂ B,

(g) for each g-closed set A and each open set B containing A, there exists a gii-open set G such that cl(A) ⊂ G ⊂ ii-cl(G) ⊂ B.

Proof: (a) ⇔ (b) ⇔ (c) : Since every ii-open set is gii-open, it is obvious.

(d) ⇒ (e) ⇒ (c) and (f) ⇒ (g) ⇒ (c) : Since every closed (resp. open) set is g-closed (resp. g-open), it is obvious.

(c) ⇒ (e) : Let A be a closed subset of X and B be a g-open set such that A ⊂ B. Since B is g-open and A is closed, A ⊂ int(A). Then, there exists a gii-open set U such that A ⊂ U ⊂ ii-cl(U) ⊂ int(B).

(e) ⇒ (d) : Let A be any closed subset of X and B be a g-open set containing A. Then there exists a gii-open set G such that A ⊂ G ⊂ ii-cl(G) ⊂ int(B). Since G is gii-open, A ⊂ ii-int(G). Put U = ii-int(G), then U is ii-open and A ⊂ U ⊂ ii-cl(U) ⊂ int(B).

(c) ⇒ (g) : Let A be any g-closed subset of X and B be an open set such that A ⊂ B. Then cl(A) ⊂ B. Therefore, there exists a gii-open set U such that cl(A) ⊂ U ⊂ ii-cl(U) ⊂ B.

(g) ⇒ (f) : Let A be any g-closed subset of X and B be an open set containing A. Then there exists a gii-open set G such that cl(A) ⊂ G ⊂ ii-cl(G) ⊂ B. Since G is gii-open and cl(A) ⊂ G, we have cl(A) ⊂ ii-int(G), put U = ii-int(G), then U is ii-open and cl(A) ⊂ U ⊂ ii-cl(U) ⊂ B.

6.2 Theorem. If f : X → Y is a continuous quasi ii-closed surjection and X is ii-normal, then Y is normal.

Proof: Let M_1 and M_2 be any disjoint closed sets of Y . Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. Since X is ii-normal, there exist disjoint U_1, $U_2 \in$ ii-O(X) such that $f^{-1}(M_i) \subset U_i$ for i = 1, 2. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y, $M_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for i = 1, 2. Since $U_1 \cap U_2 = \phi$ and f is surjective; we have $V_1 \cap V_2 = \phi$. This shows that Y is normal.

6.3 Lemma. A subset A of a space X is gii-open if and only if F ⊂ ii-int(A) whenever F is closed and F ⊂ A.

6.4 Theorem. Let f : X → Y be a closed ii-gii-continuous injection. If Y is ii-normal, then X is ii-normal.

Proof: Let N_1 and N_2 be disjoint closed sets of X, Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y . By the ii-normality of Y , there exist disjoint V_1, $V_2 \in$ ii-O(Y) such that $f(N_i) \subset V_i$ for i = 1, 2. Since f is ii-gii-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint gii-open sets of X and $N_i \subset f^{-1}(V_i)$ for i = 1, 2. Now, put U_i = ii-int($f^{-1}(V_i)$) for i = 1, 2. Then, $U_i \in$ ii-O(X), $N_i \subset U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is ii-normal.

6.5 Corollary. If f : X → Y is a closed ii-irresolute injection and Y is ii-normal, then X is ii-normal.

Proof: This is an immediate consequence since every ii-irresolute function is ii-gii-continuous.

6.6 Lemma. A function f : X → Y is almost gii-closed if and only if for each subset B of Y and each U ∈ R-O(X) containing $f^{-1}(B)$, there exists a gii-open set V of Y such that B ⊂ V and $f^{-1}(V) \subset U$.

6.7 Lemma. If f : X → Y is almost gii-closed, then for each closed set M of Y and each U ∈ R-O(X) containing $f^{-1}(M)$, there exists V ∈ ii-O(Y) such that M ⊂ V and $f^{-1}(V) \subset U$.

6.8 Theorem. Let f : X → Y be a continuous almost gii-closed surjection. If X is normal, then Y is ii-normal.

Proof: Let M_1 and M_2 be any disjoint, closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subset U_i$, where i = 1, 2. Now, put G_i = int(cl(U_i)) for i = 1, 2, then $G_i \in$ R-O(X), $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \phi$. By **Lemma 6.7**, there exists $V_i \in$ ii-O(Y) such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where i = 1, 2. Since $G_1 \cap G_2 = \phi$ and f is surjective, we have $V_1 \cap V_2 = \phi$. This shows that Y is ii-normal.

6.9 Corollary. If f : X → Y is a continuous ii-closed surjection and X is normal, then Y is ii-normal.

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