Study of HULL of Set

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Abstract

I use the notation of hull of a set in vector space denoted by $C_H(A)$ is the set of all linear combination of members of A.I used the concepts of C-Set in real or complex linear spaces.

Introduction

In this paper I used hull of set A in a vector space L.I have established some results and theorem regarding hull of set.

Definition:

The hull of a set A in a vector space L, in short denoted by $C_H(A)$ is the set of all linear combination of members of A, that is, the set of all sums

$$\alpha_1 x_1 \pm \alpha_2 x_2 \pm \dots\dots\dots \pm \alpha_n x_n$$

In which $x_i \in A,\ \alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1;\ n$ is arbitrary.

Theorem (I) : Let A be set in linear space L. Then $C_H(A)$ is a C-Set.

Proof : Let x, y be elements of $C_H(A)$. Then we can write

$$x = \alpha_1 x_1 \pm \dots\dots\dots\dots \pm \alpha_n x_n; \quad x_i \in A,\ \alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1$$

α_i are scalars.

Also $y = \beta_1 y_1 \pm \dots\dots\dots\dots \pm \beta_m y_m; \quad y_i \in A,\ \beta_i \geq 0$,

β_i scalars and $\sum_{i=1}^{m} \beta_i = 1$

Let α, β are scalars such that $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$.

Then

$$\alpha x - \beta y = \alpha(\alpha_1 x_1 \pm \ldots\ldots\ldots\ldots \pm \alpha_n x_n) - \beta(\beta_1 y_1 \pm \ldots\ldots \pm \beta_m y_m)$$

$$= \alpha\alpha_1 x_1 \pm \ldots\ldots\ldots\ldots \pm \alpha\alpha_n x_n - \beta\beta_1 y_1 \mp \ldots\ldots\ldots \mp \beta\beta_m y_m$$

In the above expression $x_i \in A, \quad y_i \in A$ and

$\alpha\alpha_i, \ \beta\beta_i$ are scalars such that $\alpha\alpha_i \geq 0, \ \beta\beta_i \geq 0$ and

$$\alpha\alpha_1 + \ldots\ldots\ldots + \alpha\alpha_n + \beta\beta_1 + \ldots\ldots\ldots + \beta\beta_m = \alpha\Sigma\alpha_i + \beta\Sigma\beta_i$$

$$= \alpha.1 + \beta.1$$

$$= \alpha + \beta = 1$$

Hence $\alpha x - \beta y$ is also an element of $C_H(A)$. Thus $C_H(A)$ is a C-Set.

Theerem (II) : Let A be a set in a linear space L. Then $C_H(A)$ is the intersection of all C-Sets containing A.

Proof : Let $\{B_j\}_{j \in I}$ be the family of all C-Sets such that $A \subseteq B_j$. Then we are going to prove that $C_H(A) = \bigcap_j B_j$.

Since each B_j is C-Set, by theorem (I), $\bigcap_j B_j$ is also C-Set which obviously contains A.

Thus $\bigcap_j B_j$ is itself a member of the family $\{B_j\}$.

By theorem , $C_H(A)$ is C-Set.

Let $x \in A$, then $x = 1\, x \in C_H(A)$

Hence $A \subseteq C_H(A)$

Thus $C_H(A)$ is also a member of the family $\{B_j\}$.

Hence $\bigcap_j B_j \subseteq C_H(A)$

Next let x be an element of $C_H(A)$. Then we can write

$x = \alpha_1 x_1 \pm \alpha_2 x_2 \pm \ldots\ldots\ldots\ldots \pm \alpha_n x_n$,

Where $x_i \in A,\ \alpha_i \geq 0,\ \alpha_i$ scalars and $\Sigma\alpha_i = 1$

Now since $A \subseteq B_j$, Therefore $x_i \in B_j$ for all $j \in I$ Since B_j is C-set,

by theorem it follows that $x \in B_j$.

Thus $x \in C_H(A) \Rightarrow x \in B_j$

Hence $C_H(A) \subseteq B_j$.

Since B_j is any member of the family therefore

$C_H(A) \subseteq \bigcap_j B_j$

it follows that

$$C_H(A) = \bigcap_j B_j$$

Theorem (III) : If A and B are subsets of a linear space L such that $A \subseteq B$ then $C_H(A) \subseteq C_H(B)$

Proof : Let z be an element of $C_H(A)$.

Then we can write

$$z = \alpha_1 x_1 \pm \alpha_2 x_2 \pm \ldots\ldots\ldots\ldots \pm \alpha_n x_n,$$

Where α_i is scalars, $\alpha_i \geq 0, \Sigma\alpha_i = 1$ and $x_i \in A$

Now since $A \subseteq B \Rightarrow x_i \in B$

Thus $z = \alpha_1 x_1 \pm \alpha_2 x_2 \pm \ldots\ldots\ldots\ldots \pm \alpha_n x_n$, where α_i is scalar $\alpha_i \geq 0, \quad \Sigma\alpha_i = 1$ and $x_i \in B$

Hence $z \in C_H(B)$.

Thus $z \in C_H(A) \Rightarrow z \in C_H(B)$

Therefore, $C_H(A) \subseteq C_H(B)$.

Theorem (IV) : Let A be a set in a vector space X and α a scalar, then $C_H(\alpha A) = \alpha C_H(A)$.

Proof : Let z be an element of $C_H(\alpha A)$,

Then $z = t_1 x_1 \pm t_2 x_2 \pm \ldots\ldots\ldots\ldots \pm t_n x_n$

Where t_i is a scalar, $t_i \geq 0, \Sigma t_i = 1$ and $x_i \in \alpha A$.

Since $x_i \in \alpha A$, let $x_i = \alpha a_i$ such that $a_i \in A$.

Therefore $z = t_1 \alpha a_1 \pm t_2 \alpha a_2 \pm \ldots\ldots\ldots \pm t_n \alpha a_n$,

$$= \alpha(t_1 a_1 \pm t_2 a_2 \pm \ldots\ldots\ldots \pm t_n a_n).$$

But $t_1 a_1 \pm t_2 a_2 \pm \ldots\ldots\ldots\ldots \pm t_n a_n$ is an element of $C_H(A)$.

Hence $z \in \alpha C_H(A)$.

Thus $z \in C_H(\alpha A) \Rightarrow z \in \alpha C_H(A)$

So $C_H(\alpha A) \subseteq \alpha C_H(A)$

Conversely, let $z \in \alpha C_H(A)$.

Thus we can write

$z = \alpha y$ such that $y \in C_H(A)$.

Therefore $z = \alpha y = \alpha(t_1 a_1 \pm t_2 a_2 \pm .... \pm t_n a_n)$

Where t_i is a scalar, $t_i \geq 0,\ \Sigma t_i = 1$ and $a_i \in A$.

Hence $z = \alpha t_1 a_1 \pm \alpha t_2 a_2 \pm .... \pm \alpha t_n a_n$.

$$= t_1(\alpha a_1) \pm t_2(\alpha a_2) \pm .... \pm t_n(\alpha a_n).$$

Now $a_i \in A \Rightarrow \alpha a_i \in \alpha A,\ i = 1, 2, 3, ..., n.$

Therefore $z \in C_H(\alpha A)$.

Hence $z \in \alpha C_H(A) \Rightarrow z \in C_H(\alpha A)$.

Thus $\alpha C_H(A) \subseteq C_H(\alpha A)$

It follows that

$$C_H(\alpha A) = \alpha C_H(A).$$

REFERENCES

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