# Topical operators on the Herglotz waves functions

**Osman Abdalla Adam Osman[1], Mortada Gawhar Abdelaziz Gawhar[2], Manofali Mousa Ali Ahmed[3], Gafar Mohamed Osman Ali[4], Abdelrahman Mahmoud Mohamed Mahmoud[5], Sahar Ahmed ldris Mohammed Ali[6]**

**[1,2,3,4,5,6]Department of Mathematics.**

**[1]College of Science and Arts in Uglat Asugour– Qassim University – Saudi Arabia , [2]National Construction Training Center – Kingdom of Saudia Arabia, [3]College of Engineering and InformationTechnology _Onayzah - Onayzah College private - Kingdom of Saudia Arabia, [4]College of Engineering and IT _Onaizah College private _ Kingdom of Saudia Arabia, [5]DEANSHIP OF PREPARATORY YEAR - S U P P O R T I N G S T U D I E S Northern Borders University - Kingdom of Saudi Arabia, [6]College of industrial engineering _king Khalid University - Abha- Kingdom of Saudi Arabia.**

**Email: o.osman@qu.edu.sa[1]**

***Abstract:*** The Herglotz space wave functions in $\mathbb{R}^2$ wher is consist of all the solutions to the Helmholtz equation of Fourier's transform in $\mathbb{R}^2$ analysis of the scale of support in a circle with a density $L^2(S^1)$. This space has the construction of a Hilbert space while reproducing the kernel. In this article we study the Toeplitz operators with some nonnegative and limited radial symbols, which are compact Toeplitz operator of compressed and his belonging to Schatten classes $s_{1+\varepsilon}$.

***IndexTerm:s*** Toepliz operator, Fourier's transform, Bergmann spaces, kernel, Herglotz wave function, Helmholtz, Hilbert space, Carleson symbol, Berezin transform.

***Introduction:***

Theory of the Toepliz operators in the Bergmann spaces of the homogeneous functions it have played many important role in the operators of the theory of complex analysis for many years. We can answer natural questions such as those that describe to us a characterization of symbols that show compact Toeplitz operators or that are limited to different values, but the Toepliz operators in Bergmann spaces for Halomorphic functions and as in the spaces of harmonic functions and kernel creation. ( [1] and [2],[3],[4] ).

We needed from this paper to study those toeplitz operators by the spaces of all Herglotz wave functions in $\mathbb{R}^2$,The Herglotz wave function in $\mathbb{R}^n$ which are solutions to the Helmutz equation.

$$\Delta u + u = 0 \tag{1}$$

From $\mathbb{R}^n$ it can be represented by Fourier transform in $\mathbb{R}^n$ and the measure $\emptyset dy$, where $\emptyset \in L^2(S^{n-1})$ and $\lambda$ is the Lebesgue measure in $S^{n-1}$. And from the Herglotz wave function write as

$$u(x) = \int_{S^{n-1}} e^{ix.\omega} \emptyset(\omega) d\lambda(\omega) \tag{2}$$

It has been given a number of descriptions of the Herglotz wave functions, ([5],[6],[7]). Such as the necessary and sufficient conditions the Herglotz wave functions, which is

$$\lim_{R\to\infty} \left(\frac{1}{R}\int_{|x|<R} |u(x)|^2 dx\right)^{\frac{1}{2}} < \infty. \tag{3}$$

Referring to $W^2$ in relation to the Herglotz space functions in $\mathbb{R}^n$.

Through the intrinsic relationship of the harmonic analysis, we find that the functions of Herglotz waves have a role in the acoustic dispersion of nonheterogeneous obstacles and media in that some of the Herglotz wavelength functions determine whether some patterns of the far field are dense in $L^2(S^{n-1})$. [8], which leads to dispersal of elastic waves [9], We can say that the whole function $u$ is a solution to the equation of the Helmholtz in $\mathbb{R}^n$. The space $W^2$ from equation (3) is the Hilbert space and the map in (2) is called the expansion operator, it gives the similarity isomorphism $L^2(S^{n-1})$ onto $W^2$.

Through the dimensions $n = 2$ in [10] that is the extension operator $\emptyset \to \widehat{\emptyset d}\theta$, That is an isomorphism from $L^2(S^1)$ to space represents the complete solution to $u$ of the Helmholtz equation which is given by

$$\|u\|^2 = \int_{\mathbb{R}^2} (|u(x)|^2 + |\partial_\theta u(x)|^2)\, W(x) dx, \tag{4}$$

If $W(x) = \frac{1}{1+|x|^3}.\, \partial_\theta u = \nabla.\frac{x^\perp}{|x|}.\, x^\perp = (-x_2, x_1)$ and $d\theta$ is to be Lebesgue measure on $S^1$ [11]. So from the set of solutions of the Helmholtz equation (3) is equivalent to (4) and thus provides us with a new characterization of the functions of the Herglotz wave in $\mathbb{R}^2$, and we find that the space of all entire functions as in $u$, is $\|u\| < \infty$.

From all this we confirm that the result is in [1], which means that $W^2$ is in $\mathbb{R}^2$.

From the $W^2$ elements such as the Newmann series can be expanded

$$u(x) = \sum_{n\in\mathbb{Z}} a_n J_n(r) e^{in\theta} \tag{5}$$

when $x = re^{i\theta} = r(cos\theta + isin\theta)$ and $\sum_{n\in\mathbb{Z}}|a_n|^2 < \infty$, that where $J_n(r)$ is the Bessel function to order $n \in \mathbb{Z}$ , [6].
If we consider the $W^2$ space associated with the Hilbert space of $L^2(S^1)$ as space taking into account the kernel output mentioned in [10,11].

When $w(r) = \frac{r}{1+r^3}$ for $r > 0$, if $H^2$ it contains space that consists of all distributions $u \in D'(\mathbb{R}^2\backslash\{0\})$ such that $u, \frac{\partial u}{\partial \theta} \in L^2(w)$, then can write $L^2(w)$ for $L^2(\mathbb{R}^2\backslash\{0\}, Wdx)$ .

$H^2$ That I mentioned in the base $\|.\|$ in (4) is a Hilbert space and $W^2$ it is that element space $u \in H^2$ who extends $\mathbb{R}^2$ and extends Helmholtz equation.

If $\in W^2$ , then

$$\|u\|^2_{W^2} \sim \sum_{n\in\mathbb{Z}}|a_n|^2 \sim \|\emptyset\|^2_{L^2(S^1)},$$

when $\emptyset = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ and $u = \widehat{\emptyset d\theta}$ .

Denote $F_n(re^{i\theta}) = J_n(r)e^{in\theta}$, then $\{e_n\}_{n\in\mathbb{Z}}$ with $e_n = F_n/\|F_n\|_{W^2}$ It is the orthogonal basis to $W^2$. Let $\beta_n = \|F_n\|_{W^2}$, in [6].

$$\beta_n^2 = \int_0^\infty (n^2+1)J_n^2(r)w(r)dr \sim \int_1^\infty (n^2+1)J_n^2(r)\frac{dr}{r^2} \sim 1. \tag{6}$$

we can take $P: H^2 \to W^2$

is orthogonal projection. A reproducing kernel to $W^2$ it is a closed subspace of $H^2$ is, $K(x,y) = \sum_{n\in\mathbb{Z}} e_n(x)\overline{e_n(y)} = \sum_{n\in\mathbb{Z}} \frac{J_n(r)J_n(S)e^{in(\theta-\varphi)}}{\beta_n^2}$, when $x = re^{i\theta}$ and $y = Se^{i\varphi}$. The kernel $K(x,y)$ is real and,

$$Pu(x) = \langle u, K(x,.)\rangle_{H^2} = \int_{\mathbb{R}^2}\left(K(x,y)u(y) + \partial_\varphi K(x,y)\partial_\varphi u(y)\right)W(y)dy.$$

If that $\|K(x,.)\|^2_{W^2} = \sum_n \frac{J_n^2(r)}{\beta_n^2} \sim 1$, hence , [11].

$$\sum_{n\in\mathbb{Z}} J_n^2(r) = 1 \tag{7}$$

From studying the Topolitz operators in $W^2$ for nonnegative radial symbols, $\rho_j$, the Topolitz operators previously mentioned by $T_{\rho_j}$ and previously mentioned in the classic holorporphic numbers can be defined as

$$T_{\rho_j}(u) = P(\rho_j u).$$

We can study the continuity of Toeplitz operators is function $\rho_j$ defined on $[0,\infty)$ is a Carleson symbol if that, $W^2 \to W^2_{\rho_j}$ it be continuous if $W^2_{\rho_j}$ is to be defined by replacing the measure as $W(x)dx$ when $\rho_j(x)W(x)dx$, at definition of $W^2$.
We can prove the a nonnegative $\rho_j \in L^1([0,\infty), w(r))$, the Toeplitrz operator $T_{\rho_j}$ is bounded in $W^2$ if $\rho_j$ is a Carleson symbol this is also the boundary of the sequence,

$$\left\{(1+n^2)\int_0^\infty J_n^2(r)\rho_j(r)w(r)dr\right\}_{n\in\mathbb{Z}}.$$

We find that in [12] of weights adapted to Helmholtz equations, we want to replace the Basel functions in the above characterization and they hide the geometric meaning of the Carlison symbols, the main result gives a sufficient condition in the integration of symbols that relate to a group of one-parameter with logical weights that do not involve special functions.

**Definition 1.** We can Given a radial function $\rho_j = \rho_j(|x| \geq 0)$ and $\Omega = \mathbb{R}^2\backslash\{0\}$**,** the Toeplitz operator can be defined with

$$T_{\rho_j}u(x) = P(\rho_j u)(x) = \int_{\mathbb{R}^2}\left(K(x,y)u(y) + \partial_\varphi K(x,y)\partial_\varphi u(y)\right)\rho_j(|y|)W(y)dy,$$

that $u$ is the linear of span of $\{e_n\}_{n\in\mathbb{Z}}$.
It can be seen that all symbol $\rho_j \in L^\infty(0,\infty)$, from the definition of abounded Toeplitz operator on $W^2$, Also $T_{\rho_j}$, is to be a multiplier acting on $W^2$

$$T_{\rho_j}\left(\sum a_n e_n\right) = \sum \gamma_n a_n e_n$$

where

$$\gamma_n = \gamma_{-n} = \frac{2\pi(n^2+1)}{\beta_n^2}\alpha_n \tag{8}$$

and

$$\alpha_n = \int_0^\infty J_{\rho_j}^2(r)\rho_j(r)w(r)dr. \tag{9}$$

This can be seen in a note for $x = re^{i\theta} = r(cos\theta + isin\theta$

$$T_{\rho_j}\left(\frac{F_m}{\beta_m}\right)(x) = \int_0^\infty \int_0^{2\pi}\left(\sum_n \frac{J_n(r)J_n(s)e^{in(\theta-\varphi)}}{\beta_n^2}\right)\frac{J_m(s)e^{im\varphi}}{\beta_m} d\varphi \rho_j(s)w(s)ds$$

$$+\int_0^\infty \int_0^{2\pi} \partial_\varphi\left(\sum_n \frac{J_n(r)J_n(s)e^{in(\theta-\varphi)}}{\beta_n^2}\right)\partial_\varphi\left(\frac{J_m(s)e^{im\varphi}}{\beta_m}\right)d\varphi\rho_j(s)w(s)ds$$

$$= \left(\frac{2\pi(1+m^2)}{\beta_m^2}\int_0^\infty J_m^2(s)\rho_j(s)w(s)ds\right)\frac{F_m}{\beta_m}(x) = \gamma_m\frac{F_m}{\beta_m}(x).$$

And that also follows $T_{\rho_j}$, is limited if and only if $\{\gamma_n\}_n$ is limited

It follows immediately that $T_{\rho_j}$ is bounded if and only if $\{\gamma_n\}_n$ is bounded, that is, provided in

$$\int_0^\infty J_n^2(r)T_{\rho_j}(r)w(r)dr \le \frac{C}{n^2+1}. \tag{10}$$

And in particular by (7), we can have that $\rho_j \in L^1([0,\infty), w(r))$ if $T_{\rho_j}$ is bounded. But the converse is not true from Remark 9.

**Definition 2.** it call $\rho_j$ a Carleson symbol if it

$$\int_{\mathbb{R}^2}\left(|u(x)|^2+\left|\partial_\varphi u(x)\right|^2\right)\rho_j(x)W(x)dx \le C\int_{\mathbb{R}^2}\left(|u(x)|^2+\left|\partial_\varphi u(x)\right|^2\right)W(x)dx$$

that any $u \in W^2$, that can be, if the identity, $W^2 \to W^2_{\rho_j}$

is acontinuous, when $W^2_{\rho_j}$, where defined by replacing the measure $W(x)dx$ by $\rho_j(x)W(x)dx$ by definition of $W^2$.

From operator notice $T^{1/2}_{\rho_j}$ it can be a densely defined if, $T^{1/2}_{\rho_j}u = \sum_n \gamma^{1/2}_{\rho_j} a_n e_n$,

If $u = \sum_{n\in\mathbb{Z}} a_n e_n$ is the linear of span of $\{e_n\}_{n\in\mathbb{Z}}$.

Such that $u$,

$$\|u\|^2_{W^2_{\rho_j}} = \int_{\mathbb{R}^2}\left(|u(x)|^2+\left|\partial_\varphi u(x)\right|^2\right)\rho_j(x)W(x)dx$$

$$= \int_0^\infty\int_0^{2\pi}\left|\sum_n \frac{a_n J_n(s)e^{in\varphi}}{\beta_m}\right|^2 d\varphi w(s)\rho_j(x)ds + \int_0^\infty\int_0^{2\pi}\left|\sum_n \frac{a_n in J_n(s)e^{in\varphi}}{\beta_m}\right|^2 d\varphi w(s)\rho_j(x)ds$$

$$= 2\pi\int_0^\infty \sum_n(n^2+1)\frac{|a_n|^2 J_n^2(s)}{\beta_n^2}w(s)\rho_j(x)ds = 2\pi\sum_n \gamma_n\frac{|a_n|^2}{\beta_n^2} = C\left\|T^{1/2}_{\rho_j}u\right\|_{W^2}.$$

**Proposition3.** From the following equivalence statements:

1- $\rho_j$ this is a Carleson symbol,

**2-** $T_{\rho_j}$ this is bounded.

In order for us to complete, we must have an estimate for the Basel functions, as these estimates can be summarized in the following elements.

**Lemma4.** Let $\mu \ge 0$ , $\varepsilon \ge 1$ and $a > 1$**,** If there is a constant $C$ and it depends on $1+\varepsilon$ and $a$, like that

$$\frac{1}{C}\mu^{\frac{1}{3}-\frac{1+\varepsilon}{3}}\sum_{j=0}^{M-1}2^{j\left(1-\frac{1+\varepsilon}{4}\right)} \le \int_{\frac{\mu}{a}}^{a\mu}\left|J_\mu(r)\right|^{1+\varepsilon}dr \le C\mu^{\frac{1}{3}-\frac{1+\varepsilon}{3}}\sum_{j=0}^{M-1}2^{j\left(1-\frac{1+\varepsilon}{4}\right)},$$

when $\mu^{\frac{2}{3}} \sim 2^M$.

**Lemma5.** If $\mu \ge 1/2$**,** if there exists a universal constant $A > 0$, then:

1- for $r \ge \mu+\mu^{1/3}$, it have,

$$J_\mu(r) = \frac{1}{\sqrt{2\pi}}\frac{\cos\theta(\mu,r)}{(r^2-\mu^2)^{\frac{1}{4}}} + h(\mu,r), \tag{11}$$

and

$$\theta(\mu,r) = (r^2-\mu^2)^{\frac{1}{2}} - \mu\, ar\cos\frac{\mu}{r} - \frac{\pi}{4} \tag{12}$$

also

$$|h(\mu,r)| \le \begin{cases} A\left(\frac{\mu^2}{(r^2-\mu^2)^{\frac{7}{4}}}+\frac{1}{r}\right) & \text{if } \mu+\mu^{1/3} \le r \le 2\mu \\ \frac{A}{r} & \text{if } \quad r \ge 2\mu. \end{cases} \tag{13}$$

2- $\mu^{1/3}\left|J_\mu(r)\right| \sim A$ if $\mu-\mu^{1/3} < r < \mu+\mu^{1/3}$ .

3- if We choose a constant $\alpha_0$. independent of , $0 < \alpha_0 < 1/2$, represent this $t_0$ we can defined it by equation $\mu$ when $\alpha_0 = \mu - t_0\mu^{1/3}$**,** such as $\frac{\alpha_0^2}{2}\mu^{2/3} \le t_0 \le \alpha_0^2\mu^{2/3}$

where

(a) for $\mu$ sech $\alpha = \mu - t\mu^{1/3}$ and $1 \le t \le t_0$, we have

$$\left|J_\mu(\mu\sec h\alpha)\right| \le A\frac{e^{-\mu(\alpha-\tanh\alpha)}}{\mu^{1/3}t^{1/4}},$$

(b) for $1 \le \mu$ such $\alpha \le \mu$ $\alpha \le \mu$ such $\alpha_0$ we have

$$\left|J_\mu(\mu\sec h\alpha)\right| \le A\frac{e^{-\mu(\alpha-\tanh\alpha)}}{\mu^{1/2}}.$$

4- For $r \ge 2\mu$

$$J_\mu(r) = \sqrt{\frac{2}{\pi r}}\left(\cos\left(r - \frac{\pi\mu}{2} - \frac{\pi}{4}\right) + 0\left(\frac{1}{r}\right)\right)$$

that is auniformly in $\mu$.

**Proof**: In part 1 of Lemma (5) we get it from using fixed methods in [13,14] and part 2 and 4 its proved in [15] and part 3 in [16].

**Lemma6**. If $1 \le r \le \mu - \mu^{1/3}$, then : $\emptyset(r) = \alpha(r) - \tanh\alpha(r)$, when $\alpha(r)$ given by the equation $\mu sech\alpha(r) = r$. Such as

1- $\emptyset$ is to be a decreasing function with $\emptyset'(r) = -\frac{(\mu^2-r^2)^{1/2}}{\mu^r}$

2- $\emptyset(\mu\,\text{sech}\,\alpha_0) \coloneqq \beta_0 > 0$,

3- $\emptyset(1) \le \log(2\mu) - \frac{\sqrt{\mu^2-1}}{\mu}$

**Theorem7**. Let $\rho_j$ is nonnegative function $L^1([0,\infty), w(r)dr)$ so for each $\mu_0 > 0$,

$$\sup_{\mu>\mu_0} \int_{\mu}^{\infty} \frac{\mu^2}{\mu^{2/3}+r^{1/2}(r-\mu)^{1/2}} \rho_j(r)w(r)dr < \infty. \tag{14}$$

Which means $\rho_j$ it is called a Carleson symbol in (14) takes replacing $\mu_0$ if its takes a positive number.

The above condition is almost necessary, as was mentioned in Remark12 and Proposition13. Where the proposals is proved in Carlson symbols $\rho_j$ from the integrals mentioned in (8) for the sake of $\mu$ have been proven and it reflects the weakness of theorem7 and this is an open question that can change the record in proposition13.

In order to prove that theorems we can use sharp asymptotics of the Bessel functions $J_\mu$, With the control on the dependence in the parameter $\mu$, contained in Lemmas (5,6,7).

We can study the compactness of the Toeplitz operators. A compactness of $T_{\rho_j}$ that is that is equivalent to corresponding pressures that include $W^2 \to W^2_{\rho_j}$ for the determining that change of the type of Prizin specialized to study of factors belonging to the Schatten of that types of classes $s_{1+\varepsilon}$. That can be using a Berezin transform, we can characterize all the Hilbert-Schmidt and Trace Class Teeplitz operators.

From this paper $\rho_j$ will can fined a nannegative radial function. It can write A~B for two nonnegative quantities $A, B$, that is positive constants $C_1, C_2$ when as $C_1, A \le B \le C_2 A$ . Such as $C$ it denote a positive constsnt for any change in each occurrence.

**Proof:** From conditions in equation (14) is a independent of $\mu_0$ can provided $\rho_j \in L^1([0,\infty), w(r)dr)$. Let $\rho_j \in L^1([0,\infty), w(r)dr)$ satisfying in equation(14). Define by

$$H(r,\mu) = \frac{\mu^2}{\mu^{2/3}+r^{1/2}|r-\mu|^{1/2}}.$$

We can have $H(r,\mu) \sim \widetilde{H}(r,\mu)$, when

$$\widetilde{H}(r,\mu) = \begin{cases} \mu^{4/3} & r \in \left[\mu-\mu^{1/3}, \mu+\mu^{1/3}\right] \\ \dfrac{\mu^2}{\mu^{1/2}(r-\mu)^{1/2}} & r > \mu + \mu^{1/3} \end{cases}$$

and uniformly for $r \ge \mu - \mu^{1/3}$ and $\mu > \mu_0$.

Notce it from equation (14) it follows that there exists $\mu_1 > 0$ such as

$$\sup_{\mu>\mu_1} \int_{\mu-\mu^{1/3}}^{\infty} \widetilde{H}(r,\mu)\, \rho_j(r)w(r)dr < \infty.$$

In the fact if we put $\mu_1$ largere enough then $\mu - \mu^{1/3} \le r \le \mu$ when $\mu \ge \mu_1$, we have $\mu - \mu^{1/3} \sim r \sim \mu$ . Then so

$$\widetilde{H}(r,\mu) = \mu^{4/3} \le CH\left(r, \mu - \mu^{1/3}\right)$$

and

$$\sup_{\mu>\mu_1} \int_{\mu-\mu^{1/3}}^{\infty} \widetilde{H}(r,\mu)\, \rho_j(r)w(r)dr \le$$
$$C \sup_{\mu>\mu_1} \int_{\mu-\mu^{1/3}}^{\infty} H\left(r, \mu-\mu^{1/3}\right) \rho_j(r)w(r)dr < \infty$$

This is together with (14) implies that

$$\sup_{\mu\ge 0} \mu^2 \int_{\mu+\mu^{1/3}}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-\mu)^{1/2}} w(r)dr < \infty$$

and

$$\sup_{\mu\ge 0} \mu^{4/3} \int_{\mu-\mu^{1/3}}^{\mu+\mu^{1/3}} \rho_j(r)\, w(r)dr < \infty. \tag{15}$$

Now we can prove the sequence $\{\gamma_n\}_n$ when corresponding to $\rho_j$ is bounding. the end of this we decompose for $n \ge 1$

$$\int_0^\infty J_n^2(r)\rho_j(r)w(r)dr = \int_0^{n \operatorname{sech} \alpha_0} + \int_{n \operatorname{sech} \alpha_0}^{n-n^{1/3}} + \int_{n-n^{1/3}}^{n+n^{1/3}} + \int_{n+n^{1/3}}^{2n} + \int_{2n}^{\infty} = \sum_{i=1}^{5} J_i$$

any integration can be estimated according from Lemma6

That can be $|J_\mu(r)| \le C/n$, wher $r \in [0, n \operatorname{sech} \alpha_0)$, like

$$\mathcal{J}_1 \le \frac{C}{n^2} \int_0^\infty \rho_j(r)w(r)dr \le \frac{C}{n^2}.$$

Into $\mathcal{J}_3$, that can be $|J_n(r)| \sim n^{-1/3}$ in $[n - n^{1/3}]$, there

$$\mathcal{J}_3 \le \frac{C}{n^{2/3}} \int_{n-n^{1/3}}^{n+n^{1/3}} \rho_j(r)w(r)dr \le \frac{C}{n^2}.$$

To consider $\mathcal{J}_4$, we can know that $|J_n(r)| \le \frac{C}{r^{1/4}(r-n)^{1/4}}$ if $r \in \left[n+n^{1/3}, 2n\right]$, this means that $\mathcal{J}_4 \le \frac{C}{n^2}$.

For $\mathcal{J}_5$, that can be $|J_\mu(r)| \le Cr^{-1/2}$, then

$$\int_{2n}^{\infty} J_n^2(r)\rho_j(r)\frac{dr}{r^2} \le C\int_{2n}^{\infty} \frac{r^{1/2}(r-n)^{1/2}}{rr^{1/2}(r-n)^{1/2}} \rho_j(r)\frac{dr}{r^2} \le C\int_{2n}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-n)^{1/2}}\frac{dr}{r^2} \le \frac{C}{n^2}.$$

In the end we estimate $\mathcal{J}_2$, we split it $\left[n \operatorname{sech} \alpha_0, n - n^{1/3}\right] \subset \bigcup_0^M I_j$ with

$I_j = \{r = n - sn^{1/3} : 2^j < s \le 2^{j+1}\}$ and $M = \left[\log_2(1 - \operatorname{sech} \alpha_0) n^{2/3}\right]$ for each $I_j$

$$|J_n(r)| \le \frac{e^{-n(\alpha(r)-\tanh(r))}}{2^{j/4} n^{1/3}},$$

When $r = n \operatorname{sech} \alpha$

then:

$$\mathcal{J}_2 = \sum_j \int_{I_j} J_n^2(r)\rho_j(r)\frac{dr}{r^2} \le \sum_j \frac{C}{2^{j/2} n^{2/3}} \int_{I_j} e^{-2n(\alpha(r)-\tanh(r))} \rho_j(r)\frac{dr}{r^2} \le \sum_j \frac{C}{2^{j/2} n^{2/3}} \sup_{I_j} e^{-2n(\alpha(r)-\tanh\alpha(r))} \int_{I_j} \rho_j(r)\frac{dr}{r^2}$$

The idea is simply

$$\psi(\mu) = \mu - \mu^{1/3} \text{ and } \varphi(\mu) = \mu + \mu^{1/3} \tag{16}$$

The function $\varphi^{-1}$ is exists in $[0,\infty[$. Put $\mu_j$ such as $\varphi(\mu_j) = n - 2^{j+1}n^{1/3}$. We can have $(\mu_j)\sim\mu_j\sim n$ .

$$\int_{I_j} \rho_j(r)w(r)dr = \int_{I_j} \frac{r^{1/2}(r-\mu_j)^{1/2}}{r^{1/2}(r-\mu_j)^{1/2}}\rho_j(r)w(r)dr$$

$$\leq \int_{I_j} \frac{r^{1/2}\left(r-\varphi(\mu_j)\right)^{1/2}}{r^{1/2}(r-\mu_j)^{1/2}}\rho_j(r)w(r)dr + \int_{I_j} \frac{r^{1/2}\left(\varphi(\mu_j)-\mu_j\right)^{1/2}}{r^{1/2}(r-\mu_j)^{1/2}}\rho_j(r)w(r)dr = L_1 + L_2,$$

$$L_1 \leq C2^{j/2}n^{2/3}\int_{\varphi(\mu_j)} \frac{\rho_j(r)}{r^{1/2}(r-\mu_j)^{1/2}}w(r)dr \leq C2^{j/2}n^{2/3-2},$$

however $\left(\varphi(\mu_j)-\mu_j\right)^{1/2}\sim n^{1/6}$ this means

$$L_2 \leq Cn^{2/3-2}$$

in the last from Lemma7 the $\sup_{I_j} e^{-2n(\alpha(r)-\tanh\alpha(r))} \leq exp\left(-2^{2j/3}\right)$ then

$$\mathcal{J}_2 \leq \frac{C}{n^2}\sum_{I_j}\sup_{I_j} e^{-2n(\alpha(r)-\tanh\alpha(r))} \leq \frac{C}{n^2}$$

**Example8.** Each function $\rho_j \in L^1([0,\infty), w(r)dr)$ like $\rho_j(r) \equiv$ is constant if $r > r_0$ is the a Carleson symbol as $\mu > r_0$,

$$\int_\mu^\infty \widetilde{H}(r,\mu)\,\rho_j(r)w(r)dr \leq C\int_\mu^\infty \widetilde{H}(r,\mu)\frac{dr}{r^2} \leq C\left(\mu^{-1/3} + \frac{1}{\mu}\int_{\mu+\mu^{1/3}}^\infty \frac{dr}{(r-\mu)^{1/2}}\right) \leq C,$$

when $\widetilde{H}$ is defined in the proof of Theorem7. Speciallyat,

$$\rho_j(r) = \frac{1}{(r-a)^\alpha}\chi_{(a,\infty)}(r)\,, a > 0\,, 0 \leq \alpha < 1$$

is a Carleson symbol which boundless.

In effect,

$$\rho_j(r) \leq \frac{1}{(r-a)^\alpha}\chi_{(\alpha,a+1)}(r) + \chi_{(a+1,\infty)}(r).$$

**Remark9**.

1- Put $\rho_j \geq 0$ so that the sequence $\lambda_n$ is defined as

$$\lambda_n = \inf\left\{\rho_j(r): r \in \left[\frac{n}{\gamma}, \gamma n\right]\right\},$$

such as $\gamma > 1$ is boundless. Then $\rho_j$ is not define it a Carleson symbol. When as

$$n^2\int_0^\infty J_n^2(r)\rho_j(r)w(r)dr \geq n^2\int_{n/\gamma}^{\gamma n} J_n^2(r)\rho_j(r)w(r)dr \sim\lambda_n\int_{n/\gamma}^{\gamma n} J_n^2(r)dr \sim\lambda_n$$

hence the Lemma5 $\int_{n/\gamma}^{\gamma n} J_n^2(r)dr$ such that a constant

And in the particular, a Carleson measure $\rho_j$ is not satisfy a condition such as

$$\lim_{r\to\infty}\frac{\rho_j(r)}{\log r} \geq 0.$$

2- we noticed that if $\rho_j$ like

$$\left\{\int_n^{2n}\frac{\rho_j(r)}{r}dr\right\}_{n\in\mathbb{N}}$$

which boundless sequence, such $\rho_j$ that is not a Carleson symbol. Indeed that by Leama6, if $r > 2(n+1)$ then

$$J_n^2(r) + J_{n+1}^2(r) \sim \frac{2}{\pi r}\left(\cos^2\left(r - \frac{\pi n}{2} - \frac{\pi}{4}\right) + \cos^2\left(r - \frac{\pi(n+1)}{2} - \frac{\pi}{4}\right) + 0\left(\frac{1}{r}\right)\right) = \frac{2}{\pi r}\left(1 + 0\left(\frac{1}{r}\right)\right)$$

Hance

$$n^2\int_0^\infty\left(J_{\left[\frac{n}{2}\right]}^2(r) + J_{\left[\frac{n}{2}\right]+1}^2(r)\right)\rho_j(r)w(r)dr \geq n^2\int_n^{2n}\left(J_{\left[\frac{n}{2}\right]}^2(r) + J_{\left[\frac{n}{2}\right]+1}^2(r)\right)\rho_j(r)w(r)dr \sim \int_n^{2n}\frac{\rho_j(r)}{r}dr$$

3- The definition of the Toeplitz operators can be extended by using radial symbols in the location function $\rho_j$ of the positive Basel standard $v$ in $[0,\infty)$ that allows as

$$T_v u(x) = \int_{\mathbb{R}^2}\left(K(x,y)u(y) + \partial_\varphi K(x,y)\partial_\varphi u(y)\right)w(s)dv(s)d\varphi$$

Then immediately, if $v = \delta_a$ we have this $T_v$ is the diagonal operator $\mathrm{diag}(\gamma_n)$ which is basical related $\{e_n\}$ with $W^2$ and $\gamma_n = \frac{2\pi(1+n^2)}{\beta_n^2}J_n^2(a)w(a)$. Then $\left\|T_{\delta_a}\right\|$ uniformly limited $a \geq a_0 > 0$ in equation (30).

**Example10**. By studying these examples and believing that the Carlson symbols are not too big for the values in $r$. The following example shows the opposite. Thus, it is difficult to know that $\rho_j$ defines the Carlson symbols by

$$\rho_j(r) = \sum_{j=1}^\infty \frac{2^{j/2}}{(r-2^j)^{\frac{1}{2}}}\chi M_j(r),$$

when $M_j = \left(2^j + 2^{\frac{j}{3}}, 2^j + 2^{1+\frac{j}{3}}\right)$.

We review the following suggestions on the importance of the condition (8).

**Proposition11**. Let $\rho_j$ be a Carlesson symbol, where (15) holds

$$\mu^2\int_{2\mu}^\infty \frac{\rho_j(r)}{r^{1/2}(r-\mu)^{1/2}}w(r)dr \leq C_{\mu_0}, \tag{17}$$

where $\mu > \mu_0$.

**Proof:** Put $\rho_j$ to be a Carleson symbol. hence mentioned, for $r > 2(n+1)$ we have

$$J_n^2(r) + J_{n+1}^2(r) \sim \frac{2}{\pi r}\left(1 + 0\left(\frac{1}{r^2}\right)\right)$$

Then

$$\int_{2(n+1)}^{\infty} \frac{\rho_j(r)}{r^3} dr \le C \int_{2(n+1)}^{\infty} \frac{J_n^2(r)+J_{n+1}^2(r)}{r^2} \rho_j(r)\, dr \le C(\alpha_n + \alpha_{n+1}) \le \frac{C}{n^2}.$$

This is proves that

$$\int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^3} dr \le \frac{C}{\mu^2}.$$

So

$$\mu^2 \int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-\mu)^{1/2}} w(r) dr \le \mu^2 \int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^3} dr \le C$$

and (17) holds.

If $n \in \mathbb{N}$, then by Lemma 6 can find

$$\int_{n-n^{1/3}}^{n+n^{1/3}} \rho_j(r) w(r) dr \sim n^{2/3} \int_{n-n^{1/3}}^{n+n^{1/3}} J_n^2(r) \rho_j(r) w(r) dr \le \frac{C}{n^{4/3}}$$

if $r$ big, proving equation (15) at that case. Let $\mu = n + \alpha$ where $n \in \mathbb{N}$ and $0 < \alpha < 1$. Put $\varphi(\mu)$ and $\psi(\mu)$ in equation (16). Then equation (15) which comes from the previous state from that time

$$[\psi(\mu), \varphi(\mu)] \subset [\psi(n), \varphi(n)] \cup [\psi(n+1), \varphi(n+1)]. \tag{18}$$

**Remark12**. To note from the Proposition11 and assuming that $\rho_j$ satisfies that

$$\sup_{\mu \ge \mu_0} \frac{1}{\mu^{1/2}} \int_{\mu+\mu^{1/3}}^{2\mu} \frac{\rho_j(r)}{(r-\mu)^{1/2}} dr \le C, \tag{19}$$

so $\rho_j$ is a Carleson symbol if and only if (8) its holds.

So, quite simply, we find the radial functions $\rho_j$ such that

$$\sup_{\Omega} \frac{1}{b} \int_a^{a+b} \rho_j(r) dr < C \tag{20}$$

then

$$\Omega = \{\mu, a, b : \mu \ge \mu_0\, , \mu + \mu^{1/3} \le a \le 2\mu\, , \mu^{1/3} \le b \le \mu\}$$

with $\mu_0 > 0$ and $C$ is absolute constant makes in (19).

And especially if it is a fixed $b > 0$

$$\sup_{a>1} \frac{1}{b} \int_a^{a+b} \rho_j(r) dr < \infty, \tag{21}$$

then $\rho_j$ satisfies in (20).

Here we can study the question of how far condition (19) then to be (8) it to being nessary for $\rho_j$ its a Carleson symbol. We can prove the following result that

$$\frac{1}{\mu^{1/2}} \int_{\mu+\mu^{1/3}}^{2\mu} \frac{\rho_j(r)}{(r-\mu)^{1/2}} dr$$

that its an order $\log \mu$ if $\rho_j$ is a Carleson symsboll.

**Lemma13.** Let $\mu$, and $M \in \mathbb{N}$ and this $2^{M-1} \le 2\mu \le 2^M$,

$$(\mu + 2\mu^{1/3}, 2\mu) \subset \cup_{j=1}^{M-1} [\mu + 2^j \mu^{1/3}, \mu + 2^{j+1} \mu^{1/3}].$$

Such that $r \in (\mu + 2^j \mu^{1/3}, \mu + 2^{j+1}\mu^{1/3})$ and $j \in \{1,2, \dots, M-1\}$ it can have

$$\frac{2}{\sqrt{6}} \le \left|\theta(\mu, r) - \theta\left(\mu + \frac{\mu^{1/3}}{2^{j/2}}, r\right)\right| \le 2\sqrt{2} \tag{22}$$

Therefore

$$\cos 2\sqrt{2} \;\; \le \cos\left|\theta(\mu, r) - \theta\left(\mu + \frac{\mu^{1/3}}{2^{j/2}}, r\right)\right| \le \cos \frac{2}{\sqrt{6}}. \tag{23}$$

**Proof :** We put $f(t) = \theta(t, r) = (r^2 - t)^{\frac{1}{2}} - t\cos^{-1}\frac{t}{r} - \frac{\pi}{4}\, , r > t$

such that $f'(t) = -\cos^{-1}\frac{t}{r}$.

If that $r \in (\mu + 2^j \mu^{1/3}, \mu + 2^{j+1} \mu^{1/3})$ and some of $t \in \left(\mu, \mu + \frac{\mu^{1/3}}{2^{j/2}}\right)$ then

$$\frac{\mu^{1/3}}{2^{j/2}} \cos^{-1} \frac{\mu + \frac{\mu^{1/3}}{2^{j/2}}}{\mu + 2^j \mu} \le \left|\theta(\mu, r) - \theta\left(\mu + \frac{\mu^{1/3}}{2^{j/2}}, r\right)\right| = \frac{\mu^{1/3}}{2^{j/2}} \cos^{-1}\frac{t}{r} \le \frac{\mu^{1/3}}{2^{j/2}} \cos^{-1} \frac{\mu}{\mu + 2^{j+1}\mu}$$

Then $\cos^{-1} x = \int_x^1 \frac{1}{\sqrt{1-t^2}} dt$

then we have

$$\frac{\mu^{1/3}}{2^{j/2}} \cos^{-1} \frac{\mu}{\mu + 2^{j+1}\mu} \le \frac{\mu^{1/3}}{2^{j/2}} \int_{\frac{\mu}{\mu+2^{j+1}\mu}}^{1} (1-t^2)^{-1/2}\, dt \le 2\frac{\mu^{1/3}}{2^{j/2}} \left(1 - \frac{\mu}{\mu + 2^{j+1}\mu}\right)^{1/2} \le 2\sqrt{2}$$

In a similar way $\frac{\mu^{1/3}}{2^{j/2}} \cos^{-1} \frac{\mu + \frac{\mu}{2^{j/2}}}{\mu + 2^{j/2}\mu} \ge \frac{2}{\sqrt{6}}$

**Proposition14.** Assume that $\rho_j$ its a Carleson symbol, so there is $\mu_0$ then

$$\sup_{\mu \ge \mu_0} \frac{1}{\mu^{1/2} \log\mu} \int_{\mu+\mu^{1/3}}^{2\mu} \frac{\rho_j(r)}{(r-\mu)^{1/2}} dr \le C, \tag{24}$$

there is $C$ is the a universal constant.

When we need to prove the propostion14, requires a sharp study of the function $\theta(\mu, r)$ in lemma13, when a control to the zeros of Bessel functions in the transition interval $[\mu + \mu^{1/3}, 2\mu]$. More clearly, we can use the fact of that

$$\left|J_\mu(r)\right|^2+\left|J_{\mu+\frac{\mu^{1/3}}{2^{j/2}}}(r)\right|^2\geq\frac{C}{(r^2-\mu^2)^{\frac{1}{2}}},$$

When $r\in\left(\mu+2^j\mu^{1/3},\mu+2^{j+1}\mu^{1/3}\right), j\in\{1,2,\dots,M-1\}$, and $2^M\sim\mu^{2/3}$.

**Proof:** Let $\mu_0>\frac{1}{2}$ and $j_A\in\mathbb{N}$ satisfy that

a) $max\left[\frac{16A^2\log\mu_0}{\mu_0},\frac{8A^2}{\mu_0}\right]\leq\frac{1}{8}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}$

b) $\frac{8A^2}{2^{3j_A}}<\frac{1}{8}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}$

c) $j_A<\frac{\log\mu_0}{100}$

when A is a constant in (14).

That it is enough to prove that

$$\sup_{\mu\geq\mu_0}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr\leq C\log\mu, \tag{25}$$

In the correct number $M$ from lemma14.

Whether $\rho_j$ is a Carleson symbol it is a complete

$$\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\left|J_\mu(r)\right|^2\rho_j(r)dr+\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left|J_{\mu+\frac{\mu^{1/3}}{2^{j/2}}}(r)\right|^2\rho_j(r)dr\leq C.M\sim C\log\mu \tag{26}$$

In other words it can by using

$|a+b|^2\geq\frac{1}{2}|a|^2-|b|^2$

We can fine

$$\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\left|J_\mu(r)\right|^2\rho_j(r)dr+\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left|J_{\mu+\frac{\mu^{1/3}}{2^{j/2}}}(r)\right|^2\rho_j(r)dr$$
$$\geq\frac{1}{4\pi}\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\left(\cos^2\theta(\mu,r)+\cos^2\theta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right)\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$
$$-\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}|h(\mu,r)|^2\rho_j(r)dr-\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left|h\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right|^2\rho_j(r)dr.$$

From the Lemma6 we can assume that in paragraph (b) above, then

$$\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}|h(\mu,r)|^2\rho_j(r)dr\leq 2A^2\mu^4\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{1}{(r^2-\mu^2)^3}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{2A^2}{\mu^2}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\rho_j(r)dr$$

$$\leq 2A^2\mu^4\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\frac{1}{(r+\mu)^3(r-\mu)^3}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{2A^2}{\mu^2}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\rho_j(r)dr$$

$$\leq 2A^2\sum_{j=j_A}^{M-1}\frac{1}{2^{3j}}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{8A^2}{\mu}\int_{\mu+2^{j_A}\mu^{\frac{1}{3}}}^{\mu+2^M\mu^{\frac{1}{3}}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$

$$\leq\frac{8A^2}{2^{3j_A}}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{8A^2}{\mu}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr\leq\frac{1}{4}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr. \tag{27}$$

In the same way it can be prove

$$\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{\frac{1}{3}}}^{\mu+2^{j+1}\mu^{1/3}}\left|h\left(\mu+\frac{\mu^{\frac{1}{3}}}{2^{\frac{j}{2}}},r\right)\right|^2\rho_j(r)dr\leq\frac{1}{4}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}\int_{\mu+2^{j_A}\mu^{\frac{1}{3}}}^{\mu+2^M\mu^{\frac{1}{3}}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr \tag{28}$$

From the Lemma14

$$\cos^2\theta(\mu,r)+\cos^2\theta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)=1+\cos\left(\left|\theta(\mu,r)-\theta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right|\right).\cos\left(\left|\theta(\mu,r)+\theta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right|\right)\geq 1-\cos\frac{2}{\sqrt{6}}.$$

Then

$$\frac{1}{4\pi}\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left(\cos^2\theta(\mu,r)+\cos^2\theta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right)\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr\geq\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}\int_{\mu+2^{j_A}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr.$$

From that and equations (26), (27) follows it (25).

then by Proposition 11 and Proposition 13 we can obtain

**Remark15**. Let $\rho_j$ be a Carlesson symbol, hence

$$\sup_{\mu\geq\mu_0}\left\{\int_{(\mu-\infty)-(\mu+\mu^{1/3},2\mu)}\frac{\mu^2}{\mu^{2/3}+r^{1/2}(r-\mu)^{1/3}}\rho_j(r)w(r)dr+\frac{1}{\log\mu}\int_{(\mu+\mu^{1/3},2\mu)}\frac{\mu^2}{\mu^{2/3}+r^{1/2}(r-\mu)^{1/2}}\rho_j(r)w(r)dr\right\}<\infty.$$

We remmber that any Toeplitz operator $T_{\rho_j}$ its a diagonal operator $\{\gamma_n\}_{n\in\mathbb{Z}}$ with $\gamma_n$ where in (9) when respect to the basis $\{e_n\}$ is defined that. Then $T_{\rho_j}$ is a compact if and only if $\lim_n\gamma_n=0$. For the estimate in [15].

$$\left|J_\mu(r)\right|\leq\frac{r^n}{n!2^n}e^{\frac{r^2}{4}},$$

We can see that any function $\rho_j\in L^1([0,\infty),w(r)dr)$ with the bounded support defines a compact Toeplitz operator. and with there are a compact Toeplitz operators where is symbols have not bounded support where is in Proposition 21.

**Proposition16.** When $T_{\rho_j}\in L^1([0,\infty),w(r)dr)$. Where $T_{\rho_j}$ is a compact in $W^2$ implying $W^2\to W_\rho^2$ is compact.

**Proof :** From the sufficiency, we can discuss by contradiction. Let as $W^2\to W_{T_{\rho_j}}^2$ is compact. When $T_{\rho_j}$ is not compact there are a subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{Z}}$ such as $\{\gamma_{n_k}\}_{k\in\mathbb{Z}}\geq\varepsilon$ for any $\varepsilon>0$. Wich a corresponding sequence $\{e_{n_k}\}_{k\in\mathbb{Z}}$ it will have approximate equivalence in $W^2_{T_{\rho_j}}$. This is impossible since $\{e_{n_k}\}_{k\in\mathbb{Z}}$is orthogonal in $W^2_{\rho_j}$ and $\|e_{n_k}\|_{W^2_{\rho_j}} = \gamma_n \geq \varepsilon$. Hence $\{\gamma_n\}_{n\in\mathbb{Z}}$ is converges to 0.

We assume that $\gamma_n \to 0$. Put $\{u_n\}_{n\in\mathbb{Z}}$ is can a bouunded sequence in $W^2$. We can see that easily

$$u_n = \sum_{k\in\mathbb{Z}} a_{n,k} e_k$$

from the convergence in $W^2$. Then $\sup_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |a_{n,k}|^2 < \infty$, from this there is an incremental chain $\{l_n\}_{n\in\mathbb{Z}}$ in $\mathbb{N}$ and a sequence of the complex numbers $\{a_k\}_{k\in\mathbb{Z}}$ such that $\lim_{n\to\infty} a_{l_{n,k}} = a_k$ for any $k\in\mathbb{Z}$. From Lemma Fatou's we can $\sum_{k\in\mathbb{Z}}|a_k|^2 < \infty$ and if we put $u = \sum_k a_k e_k \in W^2$, we can have that

$$\|u_{I_n} - u\|_{W^2_{\rho_j}} = \sum_{k\in\mathbb{Z}} \gamma_k |a_{n,k} - a_k|^2$$

is converges to 0, gives proof of the inclusion $W^2 \to W^2_{\rho_j}$ is a compact.

To characterize the Hilbert- Smidt and Trace Class Toeplitz operators we define a Berezin type transform for the Toeplitz operators on $W^2$.

Put

$$R_x = \frac{1}{(1+|x|^2)^{1/2}}\left(\sum_{n\in\mathbb{Z}} (n^2+1)^{1/2} F_n(x) e_n(x)\right).$$

we can see there positive constant such that

$$c \leq \|R_x\|_{W^2} \leq C, \tag{29}$$

for any $x\in\mathbb{R}^2$. Actually, from (6) we have

$$\sum_{n\in\mathbb{Z}} (n^2+1)|F_n(x)|^2 = \sum_{n\in\mathbb{Z}} (n^2+1) J_n^2(r) = \sum_{n\in\mathbb{Z}} J_n^2(r) = \sum_{n\neq 0} n^2 J_n^2(r) = 1 + \sum_{n\neq 0} n^2 J_n^2(r)$$

By the summation formula in [15]

$$J_0(|x-y|) = \sum_{n\in\mathbb{Z}} J_n(r) J_n(s) e^{in(\theta-\varphi)},$$

$x = re^{i\theta}$, $y = se^{i\varphi}$, we have got $x \neq y$

$$\sum_{n\in\mathbb{Z}} n^2 J_n(r) J_n(s) e^{in(\theta-\varphi)} = \frac{\partial^2}{\partial\theta\partial\varphi} J_0(|x-y|) = \frac{J_1(|x-y|)}{|x-y|} x.y + \frac{J_2(|x-y|)}{|x-y|^2}(x.y^{\perp})^2.$$

It can follows $\sum_{n\neq 0} n^2 J_n^2(r) = r^2 J_1'(0) = r^2/2$, and

$$\sum_{n\in\mathbb{Z}} (n^2+1) J_n^2(r) \sim (1+r^2), \tag{30}$$

From (29) above.

**Definition17.** In order to get the a symbol $\rho_j \in L^1([0,\infty), w(r)dr)$ we can define the transform of Berezin

$$B\rho_j(r) = \frac{1}{1+r^2}\left(\sum_{n\in\mathbb{Z}} (n^2+1)\gamma_n J_n^2(r)\right)$$

We can notice that for any $x\in\mathbb{R}^2$ hence

$$B\rho_j(r) = \left(T_{\rho_j} R_x, R_x\right).$$

**Rremark18**.

a) Assuming that $\rho_j$ is the Carlson symbol, so the Berezin transform $B\rho_j$ is limited.

b) Assuming that $T_{\rho_j}$ is the compact operator, so $\lim_{r\to\infty} B\rho_j(r) = 0$.

**Proof:** In actuality Assuming that $T_{\rho_j}$ is a bounded operator, then from (29) then can have $\left(T_{\rho_j} R_x, R_x\right)$ is bounded in $\mathbb{R}^2$ that meens (a). Assuming that $T_{\rho_j}$ is the compact, then can have $(\gamma_n)$ is converges to 0. If given $\varepsilon > 0$, then from (30)

$$\frac{1}{1+r^2}\left(\sum_{|n|>M} (n^2+1)\gamma_n J_n^2(r)\right) \leq \frac{\varepsilon}{1+r^2}\left(\sum_{|n|>M} (n^2+1) J_n^2(r)\right) \leq C\varepsilon$$

if the $M$ large enough. Then

$$\lim_{r\to\infty} sup B\rho_j(r) = \lim_{r\to\infty} sup \left( \left(\sum_{|n|\leq M} + \sum_{|n|>M}\right)(n^2+1)\gamma_n J_n^2(r)\right) \leq C\varepsilon$$

Thus $\lim_{r\to\infty} B\rho_j(r) = 0$.

Now we can study the Toeplitz operators where is belonging to the Schatten classes $s_{1+\varepsilon}$. Remember that the bounded operator $T$ in a Hilbert space is belongs to the Schatten class $s_{1+\varepsilon}$ if $Tr|T|^P < \infty$, where $TrA$ stands for the trace of the operator $A$ and $|T| = \sqrt{T^*T}$ in [17]. Since $T_{1+\varepsilon}$ is a diagonal operator with respect to the basis $\{e_n\}_{n\in\mathbb{Z}}$, we can have $T_{1+\varepsilon} \in s_{1+\varepsilon}$ if and only if $\{\gamma_n\} \in \ell^P(\mathbb{Z})$. In this case $\|T_{1+\varepsilon}\|_{s_{1+\varepsilon}} = \|\gamma_n\|_{l_{1+\varepsilon}}$.

Then from the trace class $s_1$ and the Hilbert–Schmidt operators $s_2$ we can have the following theorems.

**Theorem19**. InThe following statements are is equivalent:

a) $T_{1+\varepsilon} \in s_1$

b) $\rho_j \in L^1\left((1+r^2)w(r)dr\right)$

c) $B\rho_j \in L^1\left((1+r^2)w(r)dr\right)$

and the quantities $\|T_{1+\varepsilon}\|_{s_1}, \|\rho_j\|_{L^1\left((1+r^2)w(r)dr\right)}, \|B\rho_j\|_{L^1\left((1+r^2)w(r)dr\right)}$ are the comparable.

**Proof :** From (30) we have been

$$\sum_{n\in\mathbb{Z}} \gamma_n \sim \sum_{n\in\mathbb{Z}} (n^2+1) \int_0^\infty \rho_j(r) J_n^2(r) w(r)dr = \int_0^\infty \rho_j(r) J_n^2(r) w(r)dr \sum_{n\in\mathbb{Z}} (n^2+1) J_n^2(r)dr \sim \int_0^\infty \rho_j(r)(1+r^2)w(r)dr$$

the proof of equation (a) and (b). From another side.

$$\sum_{n\in\mathbb{Z}}\gamma_n \sim \sum_{n\in\mathbb{Z}}\gamma_n \int_0^\infty (n^2+1)J_n^2(r)w(r)dr = \int_0^\infty B\rho_j(r)J_n^2(1+r^2)w(r)dr$$

where the equivalence of (a) and (c) follows.

**Theorem20**. Let $T_{1+\varepsilon} \in s_2$ if and only if $\rho_j B\rho_j \in L^1\big((1+r^2)w(r)dr\big)$, moreover $\|T_{1+\varepsilon}\|_{s_2}$ and $\|\rho_j B\rho_j\|_{L^1((1+r^2)w(r)dr)}^{1/2}$ are the comparable quantities.

**Proof:** Then we have

$$\sum_{n\in\mathbb{Z}}\gamma_n^2 \sim \sum_{n\in\mathbb{Z}}\gamma_n(n^2+1)\int_0^\infty \rho_j(r)J_n^2(r)w(r)dr = \int_0^\infty \sum_{n\in\mathbb{Z}}\gamma_n(n^2+1)J_n^2(r)\rho_j(r)w(r)dr \sim \int_0^\infty \rho_j(r)B\rho_j(r)(1+r^2)w(r)dr$$

This is completes the proof.

In general, we have the next part.

**Proposition21**. Let $\varepsilon \geq 1$. Then

a) If $\rho_j \in L^{1+\varepsilon}\big((1+r^2)w(r)dr\big)$, then $T_{1+\varepsilon} \in s_2$ then

$$\|T_{1+\varepsilon}\|_{s_{1+\varepsilon}} \leq C_{1+\varepsilon}\|\rho_j\|_{L^{1+\varepsilon}((1+r^2)w(r)dr)}. \tag{31}$$

b) If $T_{1+\varepsilon} \in s_{1+\varepsilon}$, then $B\rho_j \in L^{1+\varepsilon}\big((1+r^2)w(r)dr\big)$ then

$$\|B\rho_j\|_{L^{1+\varepsilon}((1+r^2)w(r)dr)} \leq C\|T_{1+\varepsilon}\|_{s_{1+\varepsilon}}. \tag{32}$$

**Proof:** When

$$\int_0^\infty J_n^2(r)w(r)dr \sim (n^2+1)^{-1}$$

we have by jensen's inequality and estimate in (30) that

$$\sum_{n\in\mathbb{Z}}\gamma_n^{1+\varepsilon} \leq C\sum_{n\in\mathbb{Z}}\left((n^2+1)\int_0^\infty \rho_j(r)J_n^2(r)w(r)dr\right)^{1+\varepsilon} \leq C\sum_{n\in\mathbb{Z}}\left(\frac{\int_0^\infty \rho_j(r)J_n^2(r)w(r)dr}{\int_0^\infty J_n^2(r)w(r)dr}\right)^{1+\varepsilon} \leq C\sum_{n\in\mathbb{Z}}\frac{\int_0^\infty {\rho_j}^{1+\varepsilon}(r)J_n^2(r)w(r)dr}{\int_0^\infty J_n^2(r)w(r)dr}$$
$$\leq C\int_0^\infty \sum_{n\in\mathbb{Z}}(n^2+1)J_n^2(r){\rho_j}^{1+\varepsilon}(r)w(r)dr \leq C\int_0^\infty {\rho_j}^{1+\varepsilon}(r)(1+r^2)w(r)dr,$$

Its prove (31).

By (30) we have by jensen's inequality again that

$$B^{1+\varepsilon}\rho_j(r) \leq \frac{1}{1+r^2}\left(\sum_{n\in\mathbb{Z}}\gamma_n^{1+\varepsilon}(n^2+1)J_n^2(r)\right).$$

Then to integrating this inequality with respect to $(1+r^2)w(r)dr$ we get (32).

We notice that the Berezin transform $B\rho_j$ is an integral operator on $[0,\infty)$ is proofed with the measure $(1+r^2)w(r)dr$ with given by

$$B\rho_j(r) = \int_0^\infty M(r,s)\,\rho_j(s)(1+s^2)w(s)ds$$

where $M(r,s)$ its to be the symmetric kernel

$$M(r,s) = \frac{2\pi}{(1+r^2)(1+s^2)}\sum_{n\in\mathbb{Z}}\frac{(n^2+1)^2}{\beta_n^2}J_n^2(r)J_n^2(s).$$

**Corollary22.** The Berezin transform $B\rho_j$ is to be a bounded integral operator on $L^{1+\varepsilon}\big((1+r^2)w(r)dr\big)$ for $1 \leq \varepsilon \leq \infty$.

**Proof :** Since $M$ is symmetric, that is enough to prove the statement for $\varepsilon \in [0,1]$. For $\varepsilon = 0$, this is part of Theorem 19. Now we prove it for $\varepsilon = 1$. In [18].

When $\rho_j \in L^2\big((1+r^2)w(r)dr\big)$, then $T_{\rho_j} \in s_2$ by Proposition 21. If $(\gamma_n)$ is the sequence corresponding to $T_{\rho_j}$, then by Proposition21 and Theorem20, we can have

$$\|B\rho_j\|_{L^2((1+r^2)w(r)dr)}^2 \leq \sum_{n\in\mathbb{Z}}\gamma_n^2 \leq C_{1+\varepsilon}\int_0^\infty \rho_j(r)B\rho_j(r)(1+r^2)w(r)dr \leq \|\rho_j\|_{L^2((1+r^2)w(r)dr)}\|B\rho_j\|_{L^2((1+r^2)w(r)dr)}.$$

It can follows that

$$\|B\rho_j\|_{L^2((1+r^2)w(r)dr)} \leq C_{1+\varepsilon}\|\rho_j\|_{L^2((1+r^2)w(r)dr)}.$$

In the end, we reach the result $\varepsilon \in [0,1]$ by the inter interpolation.

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