SOME SPHERICALLY SYMMETRIC NON-STATIC MODELS IN GENERAL RELATIVITY

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Abstract:

The present paper provides some solutions of Einstein's field equations of non-static spherically symmetric metric. Pressure and density for the model have been found. These solutions are of special interest as they afford suitable models of a universe which is assumed to consist of isotropic and homogeneous matter.

Key Words: Pressure, density, isotropic, homogeneous, non-static models, energy-momentum tensor.

1. INTRODUCTION:

As a matter of fact Solutions of Einstein's field equations in general relativity is much discussed & interesting problem. Solutions giving an isotropic and homogeneous distribution of matter in space have since long been known in differential geometry. Such solutions have special interest in general relativity as they afford suitable models of a universe which is assumed to consist of isotropic and homogeneous matter. Such a model was considered by Friedmann and Lemaitre in their solutions for the expanding universe. The field of a static fluid sphere of constant density ε_0 was obtained by Schwarzachild [10] in the form

(1.1) $$ds^2 = -\frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left\{A - B/(1-\frac{r^2}{R^2})\right\}^2 dt^2$$

Where A and B are constants and $R^2 = \frac{3}{8\pi\varepsilon_0}$. Narlikar [4] gave a generalization of it in the form

(1.2) $$ds^2 = -R^2\left\{dX^2 + \sin^2 X\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} + s^2 dt^2$$

Where R = R (t) and S = S (X). The interesting conclusion is that relativity permits non-static spherical distribution of matter with a static gravitational potential, the only restriction being that either the time rate of change of the radius must be small or that it must be constant. A method for treating Einstein's field equation applied to static sphere of fluid to provide solutions in terms of known analytic functions was developed by Tolman [11]. Leibovitz ([2], [3]) has extensively discussed the static and non-static solutions of Einstein's field equations for the spherically symmetric distributions. The significance of the Weyl conformed curvature tensor in relation to distribution of spherical symmetry, has been investigated by Narlikar and Singh [4, 5].

Tolman [11] developed a method for tracing Einstein's field equations applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic function. A number of new solutions were thus obtained and the properties of three of them were examined in detail. These solutions were used by Oppenheimer and Volkof [6] in the study of massive neutron cores. Krori [1] obtained exact solutions for some dense massive spheres pointed out their astrophysical implication.

J.P. Deleon [8] has presented two new exact analytical solutions to Einstein's field equations representing static fluid spheres with anisotropic pressure while Yadav and Saini[13] have obtained an exact, static spherically symmetric solution of Einstein's field equation for the perfect fluid with p = ?. Some other workers in this field are Yadav et. al. [14] Pant and Sah [7], Sah and Chamalra [10 (a)] and Singh and Kumar [9].

Here in this paper we have considered some solution of Einstein's field equation for non-static spherically symmetric metric. Pressure and density have been calculated. As these solutions afford suitable models of a universe which is assumed to consist of isotropic and homogeneous matter, so these are of special interest in general relativity.

2. SOLUTIONS OF THE FIELD EQUATIONS:

Here we consider the non-static metric in the form given by

(2.1) $$dS^2 = e^{\beta}dt^2 - e^{\alpha}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)$$

Where α, β are functions of r and t (i.e. non-static case). The two similar solutions are given by

(2.2) $$e^{\alpha} = \frac{16e^{\psi(t)}}{\left(4+\frac{r^2}{R^2}\right)^2},$$

(2.3) $$e^{\alpha} = \frac{16R^2}{A^2(4+Kr^2)^2},\ R = R(t)$$

Clearly e^{β} is functions of t only. By using a simple transformation $e^{\beta}\,dt^2$ can be transformed as dt^2 and thus e^{α} has to be expressed in the form

(2.4) $$e^{\alpha} = \mu_1(t)\mu_2(r).$$

The usual condition of isotropy is obtained by using (2.4) in

(2.5) $$e^{\alpha/2}\left(\alpha'' - \frac{1}{2}\alpha'^2 - \frac{\alpha'}{r}\right) = \zeta(r)$$

We get the solution (2.1) and (2.4) by suitable adjustment of constant if we choose $\zeta(r) = 0$.

When $\zeta(r) = 0$, we get

(2.6) $$\alpha'' - \frac{1}{2}\alpha'^2 - \frac{\alpha'}{r} = 0$$

With the condition (2.6), the most general possible solutions are given above. The first solution gives the Friedman Lemaitre model of the expanding universe and the second one gives the solutions due to Tolman [12]. We can also obtain some other solution of (2.5). Let us consider one such solution given by

(2.7) $e^{-\alpha/2} = \upsilon(c^2t^2 - r^2)$

Use of (2.7) in (2.5) yields

(2.8) $\frac{d^2\upsilon}{dZ^2} + \frac{\zeta(r)}{8r^2}.\upsilon^2 = 0$

Where $Z = t^2 - \frac{r^2}{A^2}$.

Case (a)

If $\zeta(r) = 0$ then υ is given by

(2.9) $\gamma(A^2t^2 - r^2) = cz + d$

Where e and d are integrating constants and

$A^2z = A^2t^2 - r^2$

The metric in this case represents Milne's model and is given by

(2.10) $ds^2 = dt^2 - (cz + d)^{-2}(dr^2 + r^2d\theta^2 + r^2 + \sin^2\theta d\phi^2)$

Case (b)

By choosing $\zeta(r) = 12\ r^2$ and putting in (2.8) we get

(2.11) $\frac{d^2\upsilon}{dz^2} = \frac{-3}{2}\upsilon^2$

Whose solution after some readjustment is given in terms of elliptic functions as

(2.12) $\upsilon = 1 + \sqrt{3}\frac{an(cz+d)-1}{an(cz+d)+1}$

This in terms of a series can be written as

(2.13) $$\upsilon = 1 + \sqrt{3}\pi / K\Sigma \frac{\frac{\cos(2n-1)(cz+d)\pi/2k}{\cos h(2n-1)\pi K'/2k} - K}{\pi K\Sigma\frac{\cos(2n-1)(cz+d)\pi/2k}{\cosh(2n-1)\pi K'/2k} + k}$$

Where k is the modulus, K is the quarter period and K' is complementary to K. The surviving components of the energy – momentum tensor t^i_j usually expressed as the isotropic pressure P and density Ɛ are given by

(2.14) $8\pi P = -4r^2\left\{2\upsilon.d^2\upsilon/dx^2 - 3\left[d\upsilon/dX\right]^2\right\} - \frac{3}{4} + \eta$

And

(2.15) $8\pi\varepsilon = 4r^2\left\{2.d^2/dX^2 - 3\left\{d/dX\right\}^2\right\} - 12d/dx + \frac{3}{4} - \eta$

These can be written for case (a) and case (b) as follows :

Case (a)

(2.16) $$8\pi P = 12c^2r^2 - \frac{3}{4} + \eta$$

(2.17) $$8\pi\varepsilon = -12c(cd^2t^2 + d) + \frac{3}{4} - \eta$$

These expressions satisfy the relation

(2.18) $$\frac{\delta \in}{\delta r} + \frac{1}{2}\left(P + \in\right)\frac{\delta\beta}{\delta r} = 0$$

Which is the relativistic analogue of dependence of pressure on gravitational potential in Newtonian theory [12].

Case (b)

Here the pressure and density are expressed in terms of elliptic functions, namely

(2.19) $$8\pi P = 16\sqrt{3}\,c^2r^2 \frac{\{2gn^2(cz+d-an(cz+d)\}}{[an(cz+d)+1]^2}$$

$$+3\frac{sn^2(cz+d)[gn^2(cz+d)+an(cz+d)]}{[an(cz+d)+1]^4} + \frac{3}{4} - \eta$$

(2.20) $$8\pi\varepsilon = -16\sqrt{3}\,c^2r^2 \frac{\{2gn^2(cz+d)-an(cz+d)}{[an(cz+d)+1]^2}$$

$$+\frac{\sqrt{3}\,sn^2(cz+d)(gn^2(cz+d)+an(cz+d)]}{[an(cz+d)+1]^2}$$

$$+24\sqrt{3}\;c\left\{1+\sqrt{3}\,\frac{an(cz+d)-1}{an(cz+d)+1}\right\}\frac{sn(cz+d)an(cz+d)}{[an(cz+d)+1]^2} - \frac{3}{4} + \eta$$

The relation (2.18) though not satisfied in this case renders the left hand side of its into a perfect differential of the form

$$\frac{d}{dx}\left\{\log\left(\frac{1}{\upsilon^2}\,d^2\upsilon/dz^2\right)\right\} - \frac{d}{dr}\left(\frac{1}{2r}\right)$$

Besides (a) and (b) the differential equation (2.8) has apparently no other solution possible, that is, $\zeta(r) = 0$ and $\zeta(r) = 12r^2$ afford the only solutions given above. For let us take another metric of a model having spherical symmetry, namely

(2.21) $$ds^2 = -e^n dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 + e^\beta dt^2$$

Where η and β are functions of r and t only.

The relation (2.18) is satisfied and it gives the following differential equation

(2.22) $$\frac{\delta}{\delta r}\left[\frac{1}{r}e^{\beta/2-n/2}\,\delta\beta/\delta r\right] = \frac{2}{r^3}\left(1-e^\eta\right)\exp\left(\frac{\beta}{2}-\frac{\eta}{2}\right)$$

Further in addition to this, if we suppose that the condition of isometric is also satisfied then we get the following differential equation

(2.25) $$\frac{\delta}{\delta r}\left[\frac{1}{r^2}\frac{\delta\beta}{\delta r}+\frac{1}{r^3}\right]\exp\left(\frac{\beta}{2}-\frac{\eta}{2}\right)=\frac{-2(1+e^{\eta})\exp(\beta/2-\eta/2)}{r^4}$$

Solving (2.22) and (2.23) for η and β we get

(2.24) $$e^{\eta}=\frac{1}{1-m^2r^4}$$

and

(2.25) $$e^{\beta}=A^2\cos^2\left\{\frac{1}{2}\sin^{-1}(mr^2)+\alpha\right\}$$

Here the isotropic pressure and density are found to be

(2.26) $$8\pi P=\left[\frac{1}{r^2}-m^2r^4/r^2m^2r\left(1-m^2r^4\right)^{1/2}\tan\left\{\frac{1}{2}\sin^{-1}(mr)^2+\alpha\right\}\right]-\frac{1}{r^2}+\eta$$

and

(2.27) $$8\pi\varepsilon=-\frac{(1-6m^2r^4+m^4r^8)}{r^2(1-m^2r^4}+\frac{1}{r^2}-\eta$$

REFERENCES

1. Krori, K.D. (1970): Indian J. Pure and Appl. Phys. 8, 588.
2. Leibovitz, c. (1969): Phys. Rev., 185, 1664.
3. Leibovitz, C. (1971): Phys. Rev., D4, 2949.
4. Narlikar, V.V. (1936): Phil. Mag., 22, 767.
5. Narlikar, V.V. and Singh, K.P. (1950) : Philos, Mag. 41, 152.
6. Oppenheimer, J. R. and Volkoff, G.M. (1939): Phys. Rev. 55, 374.
7. Pant, D.N. and Sah, A. (1982), Phys. Rev. D, 96, 1254.
8. J.P. Deleon (1987): J. Math. Phys., 28 (5), 1114.
9. Singh, S and Kumar, M. (2019), Res. Rev. J, 4, p. 1684-88.
10. Schwarzschild, K. (1916): Sitz. Preuss akad Wess, 189.

10. (a) Sah, A and Chandra, PC (2019), Int. J. Astronomy Astrophys., 6, 494.

11. Tolamn, R.C. (1939): Phys. Rev., 55, 364.
12. Tolman : Relativity Thermodynamics and Cosmology, 364 and 243.
13. Yadav, R.B. S. and Saini, S.L. (1991): Astrophysics and space science 186, 331-336.
14. Yadav, R.B.S. and Kumar, J. (2019), Res. Rev. J., 5, pp. 2300-2306.