An Interactive Approach For Integer Bilevel Programming Problem with Quadratic Constraints

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Abstract

The aim of this paper is to present an interactive fuzzy programming method for solving integer bilevel programming problems (IBLPP) involving nonlinear conflicting goals with quadratic constraints. The problem is solved in two phases. Feasible region of the integer decision variables is obtained first. In phase I, fuzzy programming model of the problem is formulated using the concept of membership function. Finally the concept of ratio of satisfactory degree defined by Lai[7] is applied to generate a satisfactory solution. Illustrative numerical example is provided to demonstrate the feasibility of the approach.

Keywords: Bilevel Programming, Fuzzy Programming, Membership Function, Integer Programming

1. Introduction

Bilevel programming problems (BLPP) concerns with a large hierarchical system with two decision makers. There are two hierarchical levels in BLPP. One decision maker (DM) is located at each level. The upper level decision maker called the leader and the lower level decision maker called the follower independently controls a set of decision variables. Both the leader and follower wish to maximize their objective functions on the given constraint set. The decision of the leader affects the decision made by the follower. Also the decision of the follower influences the degree of achievement of the leader's decision.

The Stackelberg strategy has been employed to obtain a solution of the BLPP. In Stackelberg strategy the leader first declares his/her decision and then the follower tries to maximize his/her objective function. The DM's hardly communicate while employing Stackelberg solution. Even if they communicate there is no agreement which binds them. The problem is a non-convex problem even if we consider linear objective functions with linear constraints. Various approaches such as vertex enumeration approach [Candler and Townsley 1982, Wen and Bialas 1980], Kuhn-Tucker approach [Candler and Townsley 1982, Cruz 1978] and the multiple objective linear programming techniques [Bard 1984] have been used extensively for obtaining the Stackelberg solution. BLPP are NP-hard problems and are non-cooperative in nature.

The general BLPP [Bard 1991, Bialas and Karwan 1984, Candler and Townsley 1982] is formulated as

(P1) $\underset{X_1}{\text{Max}}\, F_1(X_1, X_2)$

where for a given X_1, X_2 solves

$\underset{X_2}{\text{Max}}\, F_2(X_1, X_2)$

subject to

$(X_1, X_2) \in S$

where the feasible region $S(\neq \varphi)$ is assumed to be bounded. The vector of decision variables $X_1 \in R^{n_1}$ and $X_2 \in R^{n_2}$ are under the control of leader and follower respectively, $n_1, n_2 \geq 1$. F_1 and F_2 are the objective functions of the leader and the follower respectively.

According to the leader – follower Stackelberg game and mathematical programming, the definitions of the solution for the model BLPP are:

Definition 1 : For any X_1 ($X_1 \in S_o = \{X_1 \mid (X_1, X_2) \in S\}$) given by the leader, if the decision making variable X_2 ($X_2 \in S_1 = \{X_2 \mid (X_1, X_2) \in S\}$) at the lower level is an optimal solution of the follower then (X_1, X_2) is a feasible solution of the model BLPP.

Definition 2 : If (X_1^*, X_2^*) is a feasible solution of BLPP and there does not exist any other feasible solution $(X_1, X_2) \in S$ such that $F_1(X_1^*, X_2^*) \leq F_1(X_1, X_2)$ then (X_1^*, X_2^*) is the optimal solution of the model BLPP.

To avoid the problem arising from Stackelberg strategy, the two DM's must show their willingness to cooperate with each other so that the minimum level of satisfaction is also taken into consideration. This will lead to a solution which is satisfactory for both the DM's. Keeping in view the overall satisfaction of the BLPP, Lai[7] has introduced the concept of fuzzy set theory to solve BLPP's and multi-level programming problems in 1996. Shih.et.al[10] have extended this concept. Their solution methodology is different from Stackelberg method, as all the DM's cooperate with each other and decisions of all the DM's are taken into consideration. First the higher level DM define a membership function for their fuzzy goals or preferences of their objective functions and decision variables controlled by them. Then the lower level DM optimizes the objective function with a constraint on membership function of the higher level DM. This may lead to a solution which is not desired by all the DM's because the fuzzy goals of the objective functions and the decision variables may be inconsistent.

To eliminate the drawback of Lai.et.al method, Sakawa.et.al [9] proposed interactive fuzzy programming approach to obtain a satisfactory solution of bilevel programming problem. Numerous method have been developed for solving linear bilevel programming problems [1,8,13]. However, methods for obtaining a satisfactory solution for nonlinear bilevel programming problem are very few.

In this paper, we present an interactive fuzzy programming approach for solving bilevel integer programming problems with non-linear objective functions and quadratic constraints. In order to overcome the difficulty in the methods of Lai.et.al., after finding the feasible region of the decision variables and eliminating the fuzzy goals for the decision variables, the interactive approach method for bilevel integer programming problems with non-linear objective functions is applied. In our interactive approach, after determining the fuzzy goals of the DM at both the levels, a satisfactory solution is derived efficiently by updating the satisfactory degrees of DMs with consideration of overall satisfactory balance between both the levels. Illustrative numerical example is provided to demonstrate the feasibility of the proposed approach.

2. Problem Formulation

We consider an integer non-linear programming problem with quadratic constraints. The problem can be formulated as

(P2) $\underset{X_1}{\text{Max}}\, F_1(X_1, X_2)$

where for a given X_1, X_2 solves

$\underset{X_2}{\text{Max}}\, F_2(X_1, X_2)$

subject to

$(X_1, X_2) \in S = \{(X_1,X_2) \in R^N / h_i(X_1,X_2) \le b_i,\ i = 1,2,\ldots,M,\ X_1, X_2 \ge 0,$

X_1 and X_2 are integers, $b \in R^M\}$

where the vector of decision variables $X_1 \in R^{n_1}$ and $X_2 \in R^{n_2}$ are under the control of leader and follower respectively, $n_1, n_2 \ge 1$, $N = n_1 + n_2$. $X = (x_1, x_2, \ldots, x_n)$. F_1 and F_2 are non-linear objective functions of the leader and the follower respectively. h_i, i = 1, 2, …, M are quadratic functions. b_i is the right hand side of the i^{th} resource constraint. The feasible region S($\neq\varphi$) is assumed to be bounded.

3. Solution Procedure

The problem (P2) is solved in two phases:

Phase 1: An auxiliary problem is designed for generating feasible region of the decision variables.

Phase 2: An interactive fuzzy programming approach is applied to obtain a satisfactory solution.

3.1 Solution Methodology of Phase 1

Since the achievement of the objective functions of the leader and follower depends on the constraint set

$$h_i\,(X_1,X_2) \leq b_i, \quad i = 1,2,\ldots,M \qquad \textbf{(1)}$$

$$X_1, X_2 \geq 0 \text{ , } X_1 \text{ and } X_2 \text{ are integers}$$

So, we first we apply the transformations to convert the variables in the form of 0,1.

The following transformation is employed and the integer variables x_k are replaced by the sum of a number of zero-one variable

$$x_k = \sum_{n=1}^{N_k} 2^{n-1} y_n, \text{ where } y_n \text{ are 0 or 1} \qquad \textbf{(2)}$$

The upper limit on the variable x_k is used to determine the value of N_k Every zero-one variable x_n with positive exponent is replaced by that variable to the power one. That is

$$x_n^{y} = x_n \qquad \textbf{(3)}$$

Also, every product of 0-1 variables is changed to a linear 0-1 function as follows:

Let u = x_i x_j. Introduce the following constraints

$$x_i + x_j - u \leq 1 \qquad \textbf{(4)}$$

$$- \; x_i - x_j + 2u \leq 0 \quad \text{where } u = 0, 1$$

Using these transformations, the constraint set (1) is linearized to form an auxiliary problem. The feasible region of the auxiliary problem is called the extended feasible region.

All the extreme points can be obtained by using the phase I method.

The optimal solution of the problem (P2) may correspond to any one of the extreme point or nearer to any of them in the extended feasible region S. We obtain the feasible region of the decision variables as follows:

Find all the extreme point solution of the problem. Let X^{ν} (= $x_1^{\nu}, x_2^{\nu}, \ldots, x_n^{\nu}$) be the extreme point solution corresponding to the ν^{th} iteration. Find the convex combination of these extreme point solutions as

$$\sum_{\nu=1}^{\nu} \lambda_{\nu} X^{\nu} \in S \qquad \textbf{(5)}$$

where $\sum_{\nu=1}^{V} \lambda_{\nu} = 1,\ 0 \le \lambda_{\nu} \le 1$ and $\lambda_{\nu} (v = 1, \ldots . V)$ are scalars.

Let X^w (w=1, …, W), W ≥ V, be the different solutions obtained for different sets of values of λ_{ν} (ν = 1, 2, …, V) in the augmented feasible region. Then, the feasible regions of the decision variables appear as

$$S = \{ X^w ; X^w = (x_1^w, x_2^w, \ldots, x_n^w),\ w = 1, 2, \ldots, W\} \qquad \textbf{(6)}$$

The set S is called the effective solution set

3.2 Solution Methodology of Phase 2

In the BLPP under consideration, since both the DMs would like maintain a balance of decision powers, they would have to relax their individual optimal decision. In such a case, the objective functions F_1, F_2 and the decision vector X_1 are transformed into fuzzy goals which are quantified by eliciting the corresponding membership functions.

Characterization of Membership functions

Using the concept of fuzzy sets, the membership functions can be defined based on the following steps given by Zimmermann [Zimmermann, 1978].

Step1: Find the individual best solutions (F_k^{max}) for each of the objective, where $F_k^{max} = \max_{X \varepsilon S} F_k(X)$, k = 1,2. **(7)**

Step2: Find the individual worst solution (F_k^{min}) for each of the objective where $F_k^{min} = \min_{X \varepsilon S} F_k(X)$, k = 1,2 **(8)**

Step3: Decision maker k, k=1,2 determines the membership function μ_{F_k} using the variation ratio of the degree of satisfaction in the interval between the individual maximum (7) and the individual minimum (8). The membership function for the objective $F_k(X)$ is defined as

$$\mu_{F_k} = \begin{cases} 1 & \text{if } F_k(X) > F_k^{max} \\ \dfrac{F_k(X) - F_k^{min}}{F_k^{max} - F_k^{min}} & \text{if } F_k^{min} \le F_k(X) \le F_k^{max} \\ 0 & \text{if } F_k(X) < F_k^{min} \end{cases} \quad \textbf{(9)}$$

Decision maker k specifies the value F_k^{min} of the objective function for which the degree of satisfaction is 0 and the value F_k^{max} of the objective function for which the degree of satisfaction is 1. If the value is smaller than F_k^{min} then $\mu_{F_k} = 0$ and if the value is larger than F_k^{max} then $\mu_{F_k} = 1$.

4. Formulation of Fuzzy Programming Model

After eliciting the membership functions, DM1 specifies a minimal satisfactory level $\delta \in [0, 1]$ for the membership function μ_{F_1} and the DM2 maximizes the membership function μ_{F_2} subject to the condition that DM1's membership function μ_{F_1} is larger than or equal to δ together with the extended feasible constraint set, that is DM2 solves the following problem:

(P3) Max μ_{F_2}

subject to

$\mu_{F_1} \ge \delta$

$X \in S$

Constraints on the fuzzy goals for decision variables are eliminated in our formulation (P3).

If the optimal solution to problem (P3) exists, then the DM1 obtains a satisfactory solution having a satisfactory degree larger than or equal to δ specified by DM1's own self. However, if the DM1 assigns a larger minimal satisfactory degree, the DM2 achieves a smaller satisfactory degree. Thus, a relative difference between the satisfactory degrees of DM1 and DM2 becomes larger and the overall satisfactory balance between both levels may not be achieved.

To take account of the overall satisfactory balance between both the levels, DM1 needs to compromise with DM2 on DM1's minimal satisfactory level.

The optimal solution (X*, λ*) to problem (P3) obtained by solving the problem satisfies the condition $\mu_{F_1} \geq \delta$, it follows that the solution is satisfactory for DM1. It may happen that the solution obtained does not always maintain overall satisfactory balance between both levels. Then the ratio of satisfactory degree between both the levels $\Delta = \dfrac{\mu_{F_2}(X^*)}{\mu_{F_1}(X^*)}$ which is defined by Lai [Lai, 1996] is useful. Let Δ_L and Δ_U denote the lower and upper bound of δ specified by DM1, respectively. If $\Delta > \Delta_U$ i.e. $\mu_{F_2}(X^*) > \Delta_U\ \mu_{F_1}(X^*)$ then the DM1 increase the value of δ. DM2 solves the problem (P3) with the updated value $\hat{\delta}$ and the DM1 obtains a larger satisfactory degree whereas the DM2 accepts a smaller satisfactory degree. Conversely, if $\Delta < \Delta_L$ i.e. $\mu_{F_2}(X^*) < \Delta_L\ \mu_{F_1}(X^*)$, then the DM1 decreases the minimal satisfactory level δ and the DM1 accepts a smaller satisfactory degree and the DM2 obtains a larger satisfactory degree.

At the kth iteration , let $\mu_{F_i}^k$,i = 1, 2 denote satisfactory degrees of DMi, i = 1,2 and let $\Delta^k = \dfrac{\mu_{F_2}^k(X^*)}{\mu_{F_1}^k(X^*)}$ denote the satisfactory degrees of the upper and the lower levels. Let a corresponding solution be X^{*k}. When DM2 proposes the solution to DM1 and the following condition is satisfied, DM1 concludes the solution as a satisfactory solution and the iterative interactive process terminates.

Termination condition of the interactive process for bilevel programming problems

The interactive process terminates if the ratio of satisfactory degree at k_{th} iteration $\Delta^k \in [\Delta_L, \Delta_U]$. This condition maintains the overall satisfactory balance between both the levels.

Procedure for updating the minimal satisfactory level δ

If the ratio Δ^k exceeds its upper bound, the leader increases its minimal satisfactory level. Conversely, if the ratio Δ^k is below its lower bound, then the leader decreases its minimal satisfactory level.

5. Algorithm

The outline of the above procedure is summarized in the following algorithm:

The Algorithm of the interactive fuzzy programming for BLPP

Step 0 (Pre-process)

DM1 specifies the lower and the upper bounds of the ratio of satisfactory degrees Δ.

Step 1 Obtaining effective feasible solution set

Obtain the effective feasible solution set S as given in phase 1.

Step 2 Problem Formulation

2a) DM i, i = 1, 2 obtains the membership functions for the objective functions of the fuzzy goal for their objective functions.

2b) DM1 specifies the minimal satisfactory level δ.

Set l= 1.

Step 3 Formulation of FPP

DM2 formulates the fuzzy programming problem (P3).

Step 4

DM2 solves the problem (P3) and then proposes a solution X^l and Δ^l to DM1

Step 5 Termination Condition

If the solution proposed by the DM2 to DM1 satisfies the termination conditions, DM1 concludes the solution as a satisfactory solution and the algorithm stops. Otherwise set l = l+1.

Step 6 Updating minimal satisfactory level

DM1 updates the minimal satisfactory level δ in accordance with the procedure of updating minimal satisfactory level. Go to Step 3.

6. Numerical Example

In this section, we provide illustrative numerical example for bilevel programming problem to demonstrate the feasibility of the proposed method.

Example: Consider the following integer bilevel programming problem with nonlinear objective function and quadratic constraints.

$$\text{(P4)} \quad \underset{x_1}{\text{Max}}\, F_1 = -x_1 + 2x_2^2 + 3x_3$$

$$\underset{x_2, x_3}{\text{Max}}\, F_2 = (x_1+2)^2 + x_2 + x_3^2$$

subject to

$$x_1^2 + 4x_2 \leq 4$$

$$x_1 + x_2^2 + 2x_3 \leq 4$$

$x_1, x_2, x_3 \geq 0$ and x_1, x_2, x_3 are integers

Step 0 **S**uppose the DM1specifies the lower and the upper bounds of Δ as [0.6, 1].

Step 1 Obtaining the effective solution set

Upper limit of x_1 is 2, x_2 is 1 and that of x_3 is 2.

Let $x_1 = y_1 + 2y_2$ $x_2 = y_3$ $x_3 = y_4 + 2y_5$

where $y_i = 0, 1$; $i = 1,2,3,4,5$

Constraint set in (P4) becomes equivalent to

$$y_1^2 + 4y_2^2 + 4y_1y_2 + 4y_3 \leq 4$$

$$y_1 + 2y_2 + y_3^2 + 2y_4 + 4y_5 \leq 4 \qquad \textbf{(10)}$$

$$y_i = 0, 1; \; i = 1,2,3,4,5$$

By linearization technique of section 3.1, the above constraint set (10) becomes

$$y_1 + 4y_2 + 4u_1 + 4y_3 \leq 4$$

$$y_1 + 2y_2 + y_3 + 2y_4 + 4y_5 \leq 4$$

$$y_1 + y_2 - u_1 \leq 1$$

$$-y_1 - y_2 + 2u_1 \leq 0 \qquad \textbf{(11)}$$

$u_1 = 0, 1,\ u_1 = y_1 y_2$

$y_i = 0, 1;\ \ i = 1,2,3,4,5$

The extreme points are obtained by solving the problem in (11) without considering the integrability condition using phase I method. Now by taking convex combination of these extreme points the feasible solution set with integral values for (10) appear as

$(y_1, y_2, y_3, y_4, y_5)$ = {(1,0,0,0,0), (1,0,0,1,0), (0,1,0,0,0,), (0,1,0,1,0), (0,0,1,0,0), (0,0,1,1,0), (0,0,0,1,0), (0,0,0,2,0), (0,0,0,0,0,), (0,0,0,0,1)}

S_1 = {(x_1, x_2, x_3) = (1,0,0), (1,0,1), (2,0,0,), (2,0,1), (0,1,0), (0,1,1), (0,0,1),(0,0,2),(0,0,0)}

Step 2 Problem Formulation

The individual best and the worst solutions for the DMs are found as

$F_1^{max} = 6 \qquad F_1^{min} = -2$

$F_2^{max} = 17 \qquad F_2^{min} = 4$

Now the membership functions can be built by (9) as

$$\mu_{F_1} = \begin{cases} 1 & \text{if } -x_1 + 2x_2^2 + 3x_3 \geq 6 \\ \dfrac{-x_1 + 2x_2^2 + 3x_3 + 2}{6+2} & \text{if } -2 \leq -x_1 + 2x_2^2 + 3x_3 \leq 6 \\ 0 & \text{if } -x_1 + 2x_2^2 + 3x_3 \leq -2 \end{cases}$$

$$\mu_{F_2} = \begin{cases} 1 & \text{if } (x_1+2)^2 + x_2 + x_3^2 \geq 17 \\ \dfrac{(x_1+2)^2 + x_2 + x_3^2 - 4}{17-4} & \text{if } 4 \leq (x_1+2)^2 + x_2 + x_3^2 \leq 17 \\ 0 & \text{if } (x_1+2)^2 + x_2 + x_3^2 \leq 4 \end{cases}$$

Suppose the DM1 determines the initial minimal satisfactory level as $\delta = 1$

Step 3 Formulation of Fuzzy Programming Problem

The fuzzy problem for this numerical example can be formulated as

Max μ_{F_2}

subject to

$\mu_{F_1} \geq \delta$

$X \in S$

Step 4

DM2 solves this problem. The solution obtained is (0,0,2) with $\mu_{F_1} = 1$ and $\mu_{F_2} = 4/13$. $\Delta = 0.3076 \notin [0.6, 1]$

Step 5

Since $\Delta < \Delta^L$, so DM1 decreases δ. Consequently, DM1 changes the minimal satisfactory level.

Step 6

Suppose the DM1 changes the minimal satisfactory level from 1 to 0.8.

Step 3

The problem is reformulated as

Max μ_{F_2}

subject to

$\mu_{F_1} \geq 0.8$

$X \in S$

Step 4

DM2 solves the above problem. The optimal solution is (0,0,2) with $\mu_{F_1} = 1$ and $\mu_{F_2} = 4/13$. $\Delta = 0.3076 \notin [0.6, 1]$

Step 5 and 6

Since $\Delta < \Delta^L$, so DM1 decreases δ, so the DM1 changes the minimal satisfactory level to 0.6.

Step 3 & 4

DM2 formulates and solves the corresponding problem.

The optimal solution to this problem is (0,0,2) with $\mu_{F_1} = 1$ and $\mu_{F_2} = 4/13$. $\Delta = 0.3076 \notin [0.6, 1]$

Step 5 & 6

Since $\Delta < \Delta^L$, so DM1 decreases δ, so the DM1 changes the minimal satisfactory level to 0.5.

Step 3, 4 & 5

DM2 formulates and solves the corresponding problem. The optimal solution to this problem is (1,0,1) with $\mu_{F_1} = 0.5$ and $\mu_{F_2} = 0.4613$. $\Delta = 0.923 \in [0.6, 1]$

Hence the satisfactory solution for the given bilevel programming problem is (1,0,1) with $F_1 = 2$, $F_2 = 10$, $\mu_{F_1} = 0.5$ and $\mu_{F_2} = 0.4613$.

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