A NEW INTEGRAL TRANSFORM AND ITS APPLICATIONS IN ELECTRIC CIRCUITS AND MECHANICS

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Abstract

In this paper, few fundamental properties of a new integral transform have been taken into consideration. The main purpose of this paper is to show the applicability of a new integral transform to electric circuits and Mechanics with verification by Laplace transform which require utilization of differential equations and solutions in time domain.

Keywords: New integral transform, Laplace transform, Electric circuit, Differential equation.

1. Introduction

Laplace transform found to be very applicable in many areas of Mathematics such as ordinary differential equations, partial differential equations, integral equations, electric circuits in Physics and electrical engineering. A new integral transform was introduced by Artion Kashuri and Akli Fundo [1] who showed that a new integral transform is applicable in solving ordinary and partial differential equations in the time domain. Artion Kashuri and Akli Fundo also concluded that there is a much deeper connection between Laplace and a new integral transform and other relations of a new integral transform can be found by this connection [1]. A.R. Vasishtha and R. K. Gupta [2] showed the applications of Laplace transform to electric circuits and Mechanics. Janki Vashi and M. G. Timol [3] showed the applications of Laplace and Sumudu transforms in Physics and electric circuits. Tarig M. Elzaki and Salih M. Elzaki [4] showed the connection between Laplace and Elzaki transforms. The main purpose of this paper is to show the applications of a new integral transform to electric circuits and Mechanics and to verify that there is a deeper connection between Laplace and a new integral transform.

2. Preliminaries

Definition.[1] Consider the class of functions F, where

$$F = \left\{ f(t) \mid \exists\, M, k_1, k_2 > 0 \text{ such that } |f(t)| \le M e^{\frac{|t|}{k_i^2}}, \text{if } t \in (-1)^i \times [0,\infty) \right\} \tag{1}$$

For a given function in the set F, the constant M must be finite number, k_1, k_2 may be finite or infinite. A new integral transform denoted by the operator $\mathcal{K}(.)$ is defined by

$$\mathcal{K}[f(t)] = H(v) = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} f(t)\,dt, t \ge 0, -k_1 < v < k_2 \tag{2}$$

New integral transform of some special functions

- **i)** $\mathcal{K}[1] = v$
- **ii)** $\mathcal{K}[t^n] = n!\, v^{2n+1}$
- **iii)** $\mathcal{K}[e^{at}] = \frac{v}{1-av^2}$
- **iv)** $\mathcal{K}[sinat] = \frac{av^3}{1+a^2v^4}$
- **v)** $\mathcal{K}[cosat] = \frac{v}{1+a^2v^4}$
- **vi)** $\mathcal{K}[\sinh(at)] = \frac{av^3}{1-a^2v^4}$
- **vii)** $\mathcal{K}[\cosh(at)] = \frac{v}{1-a^2v^4}$

Theorem 1.1.[1] Let $G(v)$ be a new integral transform of $f(t)$, then

- **i)** $\mathcal{K}[f'(t)] = \frac{G(v)}{v^2} - \frac{f(0)}{v}$

ii) $\mathcal{K}[f''(t)] = \frac{G(v)}{v^4} - \frac{f(0)}{v^3} - \frac{f'(0)}{v}$

Proof.

i) $\mathcal{K}[f'(t)] = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} f'(t)dt$. Integrating by parts to find that:

$$\mathcal{K}[f'(t)] = \frac{G(v)}{v^2} - \frac{f(0)}{v}$$

ii) Let $g(t) = f'(t)$,then

$$\mathcal{K}[g'(t)] = \frac{\mathcal{K}[g(t)]}{v^2} - \frac{g(0)}{v}$$

By using i) we find that

$$\mathcal{K}[f''(t)] = \frac{G(v)}{v^4} - \frac{f(0)}{v^3} - \frac{f'(0)}{v}$$

Theorem 1.2. [First translation theorem] [1] Let $f(t) \in F$ with a new integral transform $A(v)$. Then:

$$\mathcal{K}[e^{at} f(t)] = \left(\frac{1}{\sqrt{1-av^2}}\right) A\left[\frac{v}{\sqrt{1-av^2}}\right] \quad (3)$$

Proof. From definition of a new integral transform we have:

$$\mathcal{K}[f(t)] = v\int_0^\infty e^{-t} f(tv^2)dt \quad (4)$$

Which is an equivalent form of,

$$\mathcal{K}[f(t)] = A(v) = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} f(t)dt$$

$$\Rightarrow \mathcal{K}[e^{at} f(t)] = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} e^{at} f(t)dt = \frac{1}{v}\int_0^\infty e^{\frac{-t(1-av^2)}{v^2}} f(t)dt$$

Let

$$u = \frac{t(1-av^2)}{v^2} \Rightarrow du = \left[\frac{(1-av^2)}{v^2}\right] dt$$

Then,

$$\mathcal{K}[e^{at} f(t)] = \frac{1}{v}\int_0^\infty e^{-u} f\left(\frac{uv^2}{1-av^2}\right)\left(\frac{v^2}{1-av^2}\right) du$$

$$= \left(\frac{v}{1-av^2}\right)\int_0^\infty e^{-u} f\left[u\left(\frac{v}{\sqrt{1-av^2}}\right)^2\right] du$$

$$= \left(\frac{1}{\sqrt{1-av^2}}\right)\left(\frac{v}{\sqrt{1-av^2}}\right)\int_0^\infty e^{-t} f\left[t\left(\frac{v}{\sqrt{1-av^2}}\right)^2\right] dt$$

$$= \left(\frac{1}{\sqrt{1-av^2}}\right) A\left[\frac{v}{\sqrt{1-av^2}}\right]$$

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Here we find new integral transform of $e^{-at} sin\omega t$, $e^{-at} t$ and $e^{-at} \sinh kt$ for further reference:
By first translation theorem, we have

$$\mathcal{K}[e^{-at} sin\omega t] = \left(\frac{1}{\sqrt{1+av^2}}\right) \frac{\omega\left(\frac{v}{\sqrt{1+av^2}}\right)^3}{\left[1+\omega^2\left(\frac{v}{\sqrt{1+av^2}}\right)^4\right]}$$

$$= \frac{\omega v^3}{\left(\sqrt{1+av^2}\right)^4}\left\{\frac{1}{1+\frac{\omega^2 v^4}{\left(\sqrt{1+av^2}\right)^4}}\right\}$$

$$= \frac{\omega v^3}{(1+av^2)^2 + \omega^2 v^4}$$

$$= \frac{\omega v^3}{1 + 2av^2 + (a^2+\omega^2)v^4}$$

(5)

And

$$\mathcal{K}[e^{-at}t] = \left(\frac{1}{\sqrt{1+av^2}}\right)\left(\frac{v}{\sqrt{1+av^2}}\right)^3$$
$$= \frac{v^3}{1+2av^2+a^2v^4}$$

(6)

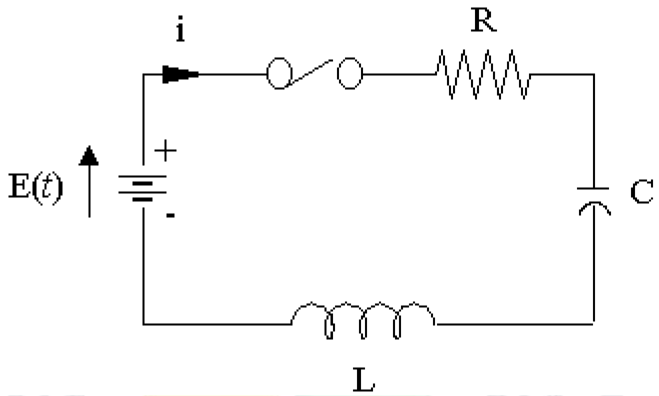
Also,

$$\mathcal{K}[e^{-at}\sinh kt] = \left(\frac{1}{\sqrt{1+av^2}}\right)\frac{k\left(\frac{v}{\sqrt{1+av^2}}\right)^3}{\left[1-k^2\left(\frac{v}{\sqrt{1+av^2}}\right)^4\right]}$$
$$= \frac{kv^3}{1+2av^2+(a^2-k^2)v^4}$$

3. Main Results

In this section we discuss the applications of a new integral transform to electric circuits and Mechanics and also provide the verification by Laplace transform which proved to be very applicable in electric circuits and Mechanics.

3.1. Applications to Electric circuit



Consider a series RLC circuit which consists of a resistor, an inductor and a capacitor with a **constant** driving electro-motive force (emf) E. When the circuit is completed, a charge Q will flow to the capacitor plates. The time rate of flow of charge is given by $\frac{dQ}{dt} = i$ and is called the current.

Applying the second Kirchhoff's law in the above shown circuit, we obtain a differential equation for determination of current as:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E$$

Which is equivalent to

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = E$$

Where $L\frac{di}{dt}$ = Voltage drop across an inductor having inductance L and
$Ri =$ Voltage drop across a resistance R.

EXAMPLES

Ex.1.[2] *At time $t = 0$, a constant voltage E is applied to a L-C-R series circuit. The current and the initial charge on the condenser are zero. Find the current at any time $t > 0$, distinguishing the three cases*

$$R^2 <,=,> \frac{4L}{C}$$

Solution by new integral transform:
The differential equation for determination of i is

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = E$$

(7)

Where

$$i = \frac{dQ}{dt}$$

(8)

Also, at $t = 0, i = 0 = Q$

Applying new integral transform to each term on both sides of (7) and (8), we obtain

$$\mathcal{K}\left[L\frac{di}{dt}\right] + \mathcal{K}[Ri] + \mathcal{K}\left[\frac{Q}{C}\right] = \mathcal{K}[E]$$

$$\Rightarrow L\mathcal{K}[i'(t)] + R\mathcal{K}[i(t)] + \frac{1}{C}\mathcal{K}[Q(t)] = vE$$

$$\Rightarrow L\left\{\frac{\mathcal{K}[i(t)]}{v^2} - \frac{i(0)}{v}\right\} + R\mathcal{K}[i(t)] + \frac{1}{C}\mathcal{K}[Q(t)] = vE$$

Using initial conditions, we obtain

$$\mathcal{K}[i(t)]\left[\frac{L}{v^2} + R\right] + \frac{1}{C}\mathcal{K}[Q(t)] = vE \qquad (9)$$

And

$$\mathcal{K}[i(t)] = \mathcal{K}[Q'(t)] = \frac{\mathcal{K}[Q(t)]}{v^2} - \frac{Q(0)}{v} = \frac{\mathcal{K}[Q(t)]}{v^2}$$

$$\Rightarrow \mathcal{K}[Q(t)] = v^2\mathcal{K}[i(t)]$$

Then from (7), we get

$$\left[\frac{L}{v^2} + R + \frac{v^2}{C}\right]\mathcal{K}[i(t)] = vE$$

$$\Rightarrow \mathcal{K}[i(t)] = \frac{v^3 E}{L\left[1 + \frac{Rv^2}{L} + \frac{v^4}{LC}\right]}$$

$$= \frac{E}{L}\left\{\frac{v^3}{1 + 2\frac{R}{2L}v^2 + \left[\left(\frac{R^2}{4L^2}\right) + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)\right]v^4}\right\}$$

$$= \frac{E}{L}\left[\frac{v^3}{1 + 2av^2 + (a^2 + n^2)v^4}\right]$$

Where

$$a = \frac{R}{2L}, n^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

Applying the inverse of a new integral transform on both sides of (10), we get

$$i(t) = \frac{E}{L}\mathcal{K}^{-1}\left[\frac{v^3}{1 + 2av^2 + (a^2 + n^2)v^4}\right] \qquad (10)$$

Case-I: When n^2 is positive *i.e.* $R^2 < \frac{4L}{C}$, then from (10), we have

$$i(t) = \frac{E}{L}e^{\frac{-R}{2L}t}\frac{1}{n}\sin nt$$

Case-II: When $n^2 = 0$, *i.e.* $R^2 = \frac{4L}{C}$, then from (10), we have

$$i(t) = \frac{E}{L}\mathcal{K}^{-1}\left[\frac{v^3}{1 + 2av^2 + a^2v^4}\right] = \frac{E}{L}te^{\frac{-R}{2L}t}$$

Case-III: When n^2 is negative say $n^2 = -k^2$ where k^2 is positive

i.e. $R^2 > \frac{4L}{C}$, then from (10), we have

$$i(t) = \frac{E}{L}\mathcal{K}^{-1}\left[\frac{v^3}{1+2av^2+(a^2-k^2)v^4}\right] = \frac{E}{kL}(\sinh kt)e^{\frac{-R}{2L}t}$$

Verification by Laplace transform

Applying Laplace transform to each term on both sides of (7) and (8) and using fundamental results, we obtain

$$\bar{\iota}(p) = \frac{E}{Lp^2 + pR + \frac{1}{C}}$$

Where $\bar{\iota}(p) = L[i(t)]$

$$\Rightarrow \bar{\iota}(p) = \frac{E}{L\left[\left(p+\frac{R}{2L}\right)^2 + n^2\right]}$$

Where $n^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

$$\therefore i(t) = L^{-1}\left\{\frac{E}{L\left[\left(p+\frac{R}{2L}\right)^2 + n^2\right]}\right\} \quad (11)$$

Considering the above three cases for (11), we obtain the same solution.

Ex.2.[2] *An alternating e. m. f.* $E\sin\omega t$ *is applied to an inductance* L *and a capacitance* C *in series. Show that the current in the circuit is*

$$\frac{E\omega}{(n^2-\omega^2)L}(\cos\omega t - \cos nt), \text{ where } n^2 = \frac{1}{LC}$$

Solution by new integral transform:

The differential equation for the determination of the current i in the circuit is given by

$$L\frac{di}{dt} + \frac{Q}{C} = E\sin\omega t \;\; [\because R = 0] \quad (12)$$

Where $i = \frac{dQ}{dt}$ (13)

Also, at $t = 0, i = 0 = Q$

Taking new integral transform on both sides of (12) and (13), we have

$$\mathcal{K}\left[L\frac{di}{dt}\right] + \mathcal{K}\left[\frac{Q}{C}\right] = \mathcal{K}[E\sin\omega t]$$

$$\Rightarrow L\left[\frac{A(v)}{v^2} - \frac{i(0)}{v}\right] + \frac{1}{C}\mathcal{K}[Q] = \frac{E\omega v^3}{1+\omega^2 v^4}$$

Where $A(v)$ is new integral transform of $i(t)$.

$$\Rightarrow L\frac{A(v)}{v^2} + \frac{1}{C}\mathcal{K}[Q] = \frac{E\omega v^3}{1+\omega^2 v^4} \quad (14)$$

And

$$\mathcal{K}[i] = \mathcal{K}\left[\frac{dQ}{dt}\right] = \frac{\mathcal{K}[Q]}{v^2} - \frac{Q(0)}{v}$$

$$\Rightarrow \mathcal{K}[Q] = v^2\mathcal{K}[i] = v^2 A(v)$$

Therefore from (14), we have

$$L\frac{A(v)}{v^2} + \frac{1}{C}v^2 A(v) = \frac{E\omega v^3}{1+\omega^2 v^4}$$

$$\Rightarrow \left(\frac{L}{v^2}+\frac{v^2}{C}\right)A(v)=\frac{E\omega v^3}{1+\omega^2 v^4}$$

Solving above equation for $A(v)$, we obtain

$$A(v)=\frac{E\omega}{L}\left[\frac{v^5}{\left(1+\frac{1}{LC}v^4\right)(1+\omega^2 v^4)}\right]$$

Or

$$A(v)=\frac{E\omega}{L}\left[\frac{v^5}{(1+n^2v^4)(1+\omega^2v^4)}\right]$$

Where $n^2=\frac{1}{LC}$.

$$\Rightarrow A(v)=\frac{E\omega}{L}\frac{1}{(n^2-\omega^2)}\left[\frac{v}{1+\omega^2v^4}-\frac{v}{1+n^2v^4}\right]$$

Applying inverse of a new integral transform, we obtain

$$i(t)=\frac{E\omega}{L}\frac{1}{(n^2-\omega^2)}[\cos\omega t-\cos nt].$$

Verification by Laplace transform

Applying Laplace transform on both sides of equations (12) and (13) and using initial conditions, we obtain

$$\bar{\imath}(p)=\frac{E\omega}{\left(Lp+\frac{1}{Cp}\right)(p^2+\omega^2)}$$

$$=\frac{E\omega}{L}\frac{p}{(p^2+n^2)(p^2+\omega^2)}$$

Where $n^2=\frac{1}{LC}$

$$=\frac{E\omega}{L}\frac{1}{(n^2-\omega^2)}\left[\frac{p}{p^2+\omega^2}-\frac{p}{p^2+n^2}\right]$$

Applying inverse Laplace transform, we get

$$i(t)=\frac{E\omega}{L}\frac{1}{(n^2-\omega^2)}[\cos\omega t-\cos nt].$$

3.2. Applications to Mechanics

Ex.3.[2]*A particle P of mass 2 grams moves on the X axis and is attracted towards origin O with a force numerically equal to 8X. If it is initially at rest at* $X=10$, *find its position at any subsequent time assuming*

i) no other force acts
ii) a damping force numerically equal to 8 times the instantaneous velocity acts.

Solution by new integral transform:

i) By Newton's law, the equation of motion of the particle is

$$2\frac{d^2X}{dt^2}=-8X$$

Or

$$\frac{d^2X}{dt^2}+4X=0 \qquad (15)$$

With initial conditions $X(0)=10$ and $X'(0)=0$.

Applying new integral transform on both sides of (15), we get

$$\mathcal{K}\left[\frac{d^2X}{dt^2}\right]+4\mathcal{K}[X]=0$$

$$\Rightarrow \frac{\mathcal{K}[X]}{v^4} - \frac{X(0)}{v^3} - \frac{X'(0)}{v} + 4\mathcal{K}[X] = 0$$

Using initial conditions, we obtain

$$\frac{\mathcal{K}[X]}{v^4} - \frac{10}{v^3} + 4\mathcal{K}[X] = 0$$

$$\Rightarrow \mathcal{K}[X] = \frac{10v}{1 + 4v^2}$$

Applying inverse of a new integral transform, we obtain

$$X = 10 \cos 2t$$

ii) In this case, the equation of motion of the particle is

$$2\frac{d^2X}{dt^2} = -8X - 8\frac{dX}{dt}$$

Or

$$\frac{d^2X}{dt^2} + 4\frac{dX}{dt} + 4X = 0 \quad (16)$$

Applying new integral transform on both sides of (16), we get

$$\mathcal{K}\left[\frac{d^2X}{dt^2}\right] + 4\mathcal{K}\left[\frac{dX}{dt}\right] + 4\mathcal{K}[X] = 0$$

$$\Rightarrow \frac{\mathcal{K}[X]}{v^4} - \frac{X(0)}{v^3} - \frac{X'(0)}{v} + 4\left\{\frac{\mathcal{K}[X]}{v^2} - \frac{X(0)}{v}\right\} + 4\mathcal{K}[X] = 0$$

Using initial conditions, we obtain

$$\mathcal{K}[X] = \frac{10v}{1 + 2v^2} + \frac{20v^3}{(1 + 2v^2)^2}$$

Applying inverse of a new integral transform, we obtain

$$X = 10e^{-2t} + 20te^{-2t}$$

Solution by Laplace transform of above example is given by [2].

4. Conclusion

The author found that the new integral transform is very applicable to electric circuits and Mechanics and also the author verified that there is a much deeper connection between Laplace transform and a new integral transform.

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