THE EXISTENCE OF THE UNIQUE SOLUTION OF A CLASS OF NONLINEAR EQUATIONS IN SUPERMATRIC SPACE

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Abstract : In this paper we intend to introduce some simple criteria about the existence of solution of a class of nonlinear functional equations by the method of approximate iteration procedure in supermatric spaces following appropriate examples. In this paper we have also reduced the constraints occured in Sen and Mukherjee's derivation have been reduced and then improved it to some extent.

Keywords : Supermetric space, Hammerstein equation, Monotonically decomposable operators, Schwartz inequality.

Introduction

Iterative processes are common for the solution of equations in diverse fields. But when dealing with abstract spaces, the method becomes a little reflection, such as supermetric space, metric space, Hilbert space, etc. The approximate iteration solution of a class of functional equations by Joshi [6], Sen [3], Sen and Mukherjee [5, 6] and others and its uniqueness in the supermetric space is presented.

We intend to introduce some simple criteria about the existence of solution of a class of nonlinear functional equations by the method of approximate iteration procedure in supermatric spaces following appropriate examples. In this paper we have also reduced the constraints occured in Sen and Mukherjee's derivation have been reduced and then improved it to some extent.

Some general concepts

A space X is a set of elements f, g, … In the applications, these elements may be real or complex numbers, vectors, matrices, functions of one or more variables etc. Ordinarily the spaces that occur in the applications are linear and hence we shall consider linear spaces only.

In applications these elements can be real or complex numbers, vectors, matrices, one or more variables, etc. Ordinarily spaces occurring in applications are linear and hence we shall consider only linear spaces. **A** linear space is characterized by the following properties:

(i) An operation is defined, which we call addition and this follows the rules of ordinary addition; if f and g are elements of X, then $f + g \in X$ and $\exists$ a null element $\theta \in X$ s.t. $\theta + f = f, \forall f \in X$.

(ii) A multiplication is defined between the elements f and scalars c of a field F. This multiplication obeys the rule of ordinary vector algebra, i.e. if $f \in X$ and $c \in F$, then $cf \in X$. The field F is usually the field of rational, real or complex numbers.

A metric linear space X is supermetric [2] when the distance ρ satisfies the relation

$$\rho(f_1, f_2 + f_3) = \rho(f_1 + f_2, f_3),$$

for any three elements $f_1, f_2, f_3 \in X$.

Example. The space X of all complex numbers z with the distance

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|}{1 + |z_1 - z_2|}$$

is supermetric. For, if $z_1, z_2, z \in X$, then

$$\begin{aligned}\rho(z_1, z_2 + z_3) &= \frac{|z_1 - (z_2 + z_3)|}{1 + |z_1 - (z_2 + z_3)|} \\ &= \frac{|(z_1 - z_2) - z_3|}{1 + |(z_1 - z_2) - z_3|} \\ &= \rho(z_1 - z_2, z_3).\end{aligned}$$

Solvability of nonlinear functional equations in supermatric space

In solving the physical problems, the presence of more than one solution sometimes creates difficulty to find a true solution of the problem and thus a lack of agreement of the solution with the experimental result often occurs. Prof. Sen and Mukherjee [1, 2] and Prof. Sen [4] devised some methods to overcome these difficulties.

We have considered a nonlinear complete super-metric space X [2] and f is a number of X. A is a nonlinear operator mapping X into X. Our problem is to solve the nonlinear equation of the type

$$u = Au + f, f \in X. \qquad (1)$$

Let us consider the iterate of the form

$$u_{n+1} = Au_n + f, (n = 0, 1, 2, \dots) \qquad (2)$$

where u_0 pre-choosen

In following paper contains the convergence theorem and again we have exposed an example satisfying the method.

Theorem 1 (Convergence Theorem). Let the following conditions be fulfilled:

L is the bounded linear operator mapping X into X such that

1. (i) $\rho(Au, Av) \le \rho(Lu, Lv) \forall u, v \in X$

(ii) $\rho(L^m u, L^m v) \le q\rho(u,v) \forall u, v \in X$ and $0 < q < 1$ for fixed m.

2. f belongs to the range of $(I - A)$.

Proof. In the condition derived by Sen and Mukherjee there exists a $u^* \in X$ such as

$$u^* = Au^* + f \tag{3}$$

The space being supermetric, using the above condition, we get

$$\begin{aligned}\rho(u_{m+1}, u^*) &= (Au_m + f, u^*) \\ &= \rho(Au_m + f, Au^* + f) \\ &= \rho(Au_m + f, Au^* + f) \qquad (4) \\ &\le \rho(Lu_m + Lu^*) \\ &\le q\rho(u_m + Lu^*) \\ &\le q^m(u_0 + u^*) \qquad \text{[using (i) and (ii)]} \qquad (5) \\ &\to 0 \text{ as } m \to \infty, 0 < q < 1.\end{aligned}$$

To prove the uniqueness of the solution, let us suppose that v^* is another solution of (1). Then

$$\begin{aligned}\rho(u^*, v^*) &= \rho(Au^* + f, Av^* + f) \\ &= \rho(Au^*, Av^*) \\ &\le q\rho(u^*, v^*) \\ &= q\rho(Au^* + f, Av^* + f) \\ &\le q^n \rho(u^*, v^*), 0 < q < 1 \\ &\to 0 \text{ as } m \to \infty.\end{aligned}$$

Hence $u^* = v^*$.

Thus when f belongs to the range of $(I - A)$, $\{u_n\}$ converges uniquely to the solution u^* of (1).

Example: As in Sen and Mukherjee derivation we have consider the Hammerstein equation

$$u(x) = 1 + \int_0^1 |x - t| \left[u(t) - \frac{1}{2} u^2(t) \right] dt \tag{6}$$

in $C(0, 1)$.

Using the theory of monotonically decomposable operator, Collatz [2] proved the existence of a unique solution $u(x)$ of (6) with

$$2(x - x^2) \le u(x) \le 2(1 - x - x^2) \tag{7}$$

Let $X = \left[\frac{u(x)}{2(x - x^2)} \le u(x) \le 2(1 - x - x^2) \right]$

Here $Au = \int_0^1 |x - t| \left[u(t) - \frac{1}{2} u^2(t) \right] dt$ (8)

Let us choose $Lu = \int_0^1 |x - t| u(t) dt$

and $\rho(u, v) = \|u - v\| = \max_{0 \le x \le 1} |u(x) - v(x)|, \forall u(x), v(x) \in C(0,1)$.

Let us consider that the metric in X is induced by the metric in $C(0, 1)$ and is complete in X w. r. t. the induced metric so that

$$\begin{aligned}Au - Av &= \int_0^1 |x - t| \left[u(t) - \frac{1}{2} u^2(t) \right] dt - \int_0^1 |x - t| \left[v(t) - \frac{1}{2} v^2(t) \right] dt \\ &= \int_0^1 |x - t| [u(t) - v(t)] dt - \frac{1}{2}[u(\xi) + v(\xi)] \int_0^1 |x - t| u(t) dt - \frac{1}{2}[u(\eta) + v(\eta)] \int_0^1 |x - t| v(t) dt\end{aligned} \tag{9}$$

where $0 < \xi < 1$, $0 < \eta < 1$

Now $\max_{0 \le x \le 1} \left| 1 - \frac{u(x) + v(x)}{2} \right| \le 1, \forall u(x), v(x) \in X$ (10)

$$\rho(Lu, Lv) = \max_{0 \le x \le 1} \left| u(\bar{\xi}) - v(\bar{\eta}) \right| \int_0^1 |x - t| dt, \tag{11}$$

where $0 < \bar{\xi} < 1, 0 < \bar{\eta} < 1$

$$\int_0^1 |x - t| v(t) dt \le \frac{v(\bar{\eta})}{u(\bar{\xi}) - v(\bar{\eta})} \rho(Lu, Lv) \tag{12}$$

Neglecting the quantities of second and higher orders in ξ and η, we get

$$|Au - Av| \le \left[1 - \frac{1}{2}[u(\xi) + v(\xi)] + v(\eta) \right] \rho(Lu, Lv)$$

Now $1-\frac{1}{2}\left[u(\xi)+v(\xi)\right]+v(\overline{\eta})=1-2(\xi-\xi^2)+2(1-\overline{\eta}+\overline{\eta}^2)\cong 1$ (13)

Therefore, $\rho(Au,Av)\le\rho(Lu,Lv)\forall u,v\in X$ (14)

Using Schwartz inequality, we have

$$\rho(L^2u,L^2v)\le\frac{1}{3}\rho(u,v) \tag{15}$$

Hence by theorem 1, the sequence $\{u_n\}$ defined by
u_{n+1} = A$u_n + f$, $(n = 0, 1, 2, \ldots)$

$$u_{n+1}=1-\int_0^1|x-t|\left[u_n(t)-\frac{1}{2}u_n^2(t)\right]dt, \qquad n=0,1,2,\ldots$$

converges to the unique solution of the equation (6) in X.

REFERENCES

[1] Joshi, M. (Sept. 1983): An approximate solvability scheme for a class of nonlinear equations, Proc. Indian Acad. Sci. (Math. Sci.), 92(1):61-65.

[2] Collatz, L. (1966): Functional Analysis and Numerical Mathematics, Academic Press, New York.

[3] Collatz, L. (1971): Nonlinear Functional Analysis and Applications. (Ed. L. B. Rall), Academic Press, New York.

[4] Sen, R. *(*1971): Approximate Iterative Process in a Supermetric Space. Bull. Cal. Moth. Soc., 63:121-123.

[5] Sen, R. and Mukherjee, S. (1983): On Iterative Solution of Nonlinear Functional Equations in a Metric Space. Int. J. Math. and Math. Sci., 6:161-170.

[6] Sen, R. and Mukherjee, S. (1988): A Note on the Unique Solvability of a class of Nonlinear Equations. Int. J. Math. and Math. Sci. Vol. II No. 1:201-204.