THE LOGISTIC-EXPONENTIAL POWER DISTRIBUTION WITH STATISTICAL PROPERTIES AND APPLICATIONS

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Abstract**:** In this article, we have presented a three-parameter univariate continuous distribution called Logistic-Exponential power distribution. We have discussed some mathematical and statistical properties of the distribution such as the probability density function, cumulative distribution function and hazard rate function, survival function, quantile function, the skewness, and kurtosis measures. The model parameters of the proposed distribution are estimated using three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods. All the computations are performed using R software. The goodness of fit of the proposed distribution is also evaluated by fitting it in comparison with some other existing distributions using two real data sets.

Keywords: **Logistic distribution, Exponential power distribution, Hazard function, LSE, CVME.**

I. INTRODUCTION

In probability theory and applied statistics, the exponential model plays a crucial role in analyzing the life testing data. It is the probability distribution of the time between trials in a Poisson point process, i.e., a progression in which events occur continuously and independently at a constant average rate. It is a special case of the gamma distribution. It is the continuous analog of the geometric distribution, and memoryless is the important properties of this distribution. In addition to being applied for the analysis of Poisson point processes, it is found in various other contexts. A generated survival model that take account of the different shapes like Increasing, decreasing, bathtub-shaped, and inverted Bathtub-Shaped failure rate in a single model would be beneficial in survival study. Such a model would provide considerable flexibility and goodness of fit for fitting a broad variety of lifetime data sets. Such a survival model might also be taken to establish the distribution class from which the data is selected, by constructing confidence interval over its parameters. The logistic exponential power distribution introduced here satisfies these criteria.

The logistic distribution is a single variate continuous probability distribution and both of its PDF and CDF functions have been used in many different fields such as logistic regression, logit models and neural networks. It has been used in the physical sciences, biological sciences, sports modeling, and recently in finance as well as insurance. The logistic distribution has thicker tails than a normal distribution so it is more consistent with the underlying data and provides good insight into the likelihood of extreme events.

Suppose Y be a positive random variable follows the logistic distribution with shape parameter $\beta > 0$, and its cumulative distribution function is given by

$$F\left(y;\beta\right)=\frac{1}{1+e^{-\beta y}};\quad \beta>0, y\in\Re \tag{1.3}$$

and its corresponding PDF is

$$f\left(y;\beta\right)=\frac{\beta e^{-\beta y}}{\left(1+e^{-\beta y}\right)^{2}};\quad \beta>0, y\in\Re \tag{1.4}$$

Tahir et al. (2016) has defined a new generating family of continuous distributions generated from a logistic random variable called the *logistic-X family*. The probability density function of this distribution can be symmetrical, positively skewed, negatively skewed and reversed-J shaped, and can have increasing, decreasing, bathtub and upside-down bathtub hazard rates shaped. Mandouh (2018) has introduced Logistic-modified Weibull distribution which is flexible for survival analysis as compared to modified Weibull distribution. Joshi & Kumar (2020) have introduced the Lindley exponential power distribution having a more flexible hazard rate function. Mansoor et al. (2019) have introduced a three-parameter extension of the exponential distribution which contains as sub-models the exponential, logistic-exponential and Marshall-Olkin exponential distributions. The distribution is very flexible and its associated density function can be decreasing or unimodal. Lan and Leemis (2008) has presented an approach to define the logistic compounded model and introduced the logistic–exponential survival distribution. This has numerous useful probabilistic properties for lifetime modeling. Unlike most distributions in the bathtub and upside down bathtub classes, the logistic–exponential distribution exhibit closed-form density, hazard, cumulative hazard, and survival functions. The survival function of the logistic–exponential distribution is

$$S(x;\lambda)=\frac{1}{1+\left(e^{\lambda x}-1\right)^{\alpha}};\quad \alpha>0,\lambda>0,x\geq 0 \tag{1.5}$$

Applying the similar approach used by (Lan & Leemis, 2008) we have introduced the new distribution called Logistic- exponential power (LEP) distribution. The key objective of this paper is to establish a more flexible distribution by adding just one extra parameter to the exponential power distribution to attain a better fit to the lifetime data sets. We have presented some mathematical and statistical properties and its applicability. The different sections of the proposed study are arranged as follows. We have presented the Logistic- Exponential power (LEP) distribution and its various mathematical and statistical properties in section 2. We have make use of three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. Two real data sets have been considered to explore the applicability and capability of the proposed distribution in section 4. In this section, we present the estimated value of the parameters and log-likelihood, AIC, BIC and CAIC criterion for ML, LSE, and CVME also the goodness of fit of the proposed distribution is evaluated by fitting it in comparison with some other existing distributions using two real data sets. Finally, in Section 5 we present some concluding remarks.

II. THE LOGISTIC- EXPONENTIAL POWER (LEP) DISTRIBUTION

Adopting the similar approach used by (Lan & Leemis, 2008) we have created a new distribution called Logistic- exponential power (LEP) distribution. Let X be a non negative random variable with a positive shape parameters α and β and a positive scale parameter λ then CDF of logistic- exponential power distribution can be defined as

$$F(x)=1-\frac{1}{1+\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha}};\quad (\alpha,\beta,\lambda>0),\quad x>0 \tag{2.1}$$

The PDF of the logistic-exponential power distribution is

$$f(x)=\alpha\beta\lambda\frac{x^{\beta-1}e^{\lambda x^{\beta}}\exp\left(e^{\lambda x^{\beta}}-1\right)\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha-1}}{\left[1+\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha}\right]^{2}};\quad (\alpha,\beta,\lambda>0),\quad x>0 \tag{2.2}$$

This CDF function be similar to the log logistic CDF function with the second term of the denominator being changed in its base to an exponential power function, hence we named it logistic- exponential power distribution.

Reliability function

The reliability function of Logistic- exponential power distribution is

$$R(x)=1-F(x)$$

$$=\frac{1}{1+\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha}};\quad (\alpha,\beta,\lambda>0),\quad x>0 \tag{2.3}$$

Hazard function

The failure rate function of Logistic- exponential power distribution can be defined as,

$$h(x)=\frac{f(x)}{R(x)}=\alpha\beta\lambda\frac{x^{\beta-1}e^{\lambda x^{\beta}}\exp\left(e^{\lambda x^{\beta}}-1\right)\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha-1}}{\left[1+\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha}\right]};\quad (\alpha,\beta,\lambda>0),\quad x>0 \tag{2.4}$$

In Figure 1, we have displayed the plots of the PDF and hazard rate function of LEP distribution for different values of α, β and λ.

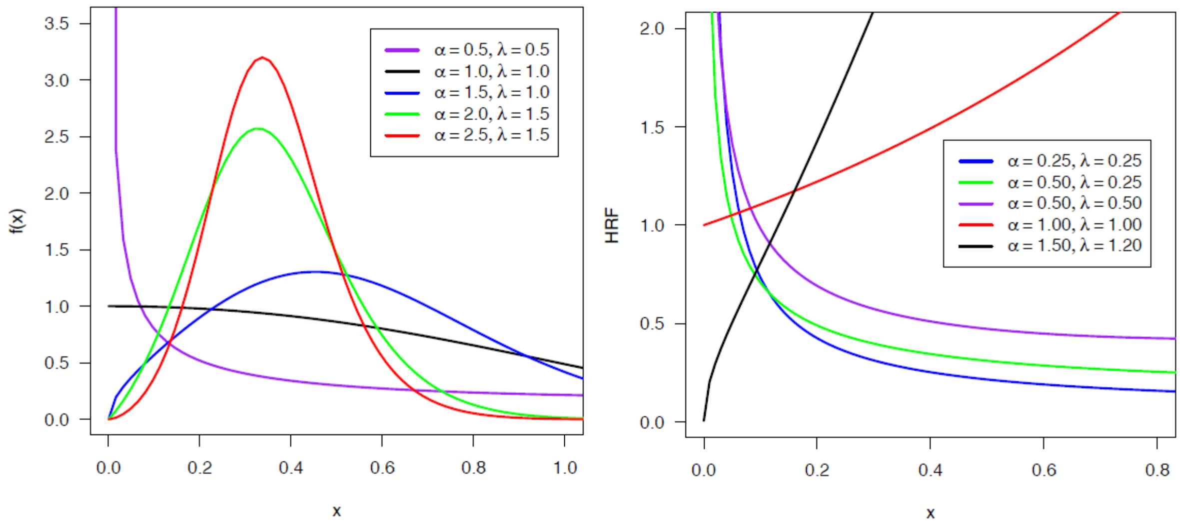

Figure 1. Plots of PDF (left panel) and hazard function (right panel) for different values of α, β and λ.

Reverse hazard rate function

The reverse hazard rate function of LEP distribution can be defined as

$$R_{HRF}(x)=\alpha\beta\lambda\frac{x^{\beta-1}e^{\lambda x^{\beta}}\exp\left(e^{\lambda x^{\beta}}-1\right)\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha-1}}{\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha}\left[1+\left\{\exp\left(e^{\lambda x^{\beta}}-1\right)-1\right\}^{\alpha}\right]};\ (\alpha,\beta,\lambda>0),\ \ x>0 \qquad (2.5)$$

Quantile function

The Quantile function of Logistic exponential power distribution can be expressed as

$$Q(p)=\left[\frac{1}{\lambda}\ln\left\{\ln\left\{\left(\frac{p}{1-p}\right)^{1/\alpha}+1\right\}+1\right\}\right]^{1/\beta}\ ;0<p<1 \qquad (2.6)$$

Median:

The median of Logistic exponential power distribution can be expressed as

$$Median=\left[\frac{1}{\lambda}\ln\left\{\ln 2+1\right\}\right]^{1/\beta}$$

Here we notice that median is depends up on scale parameter λ and shape parameter β only.

Skewness and Kurtosis:

The measures Skewness and Kurtosis based on quantiles can be calculated as, Bowley's coefficient of skewness can be computed by using

$$Skewness(B)=\frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(0.75)-Q(0.25)} \text{ and} \qquad (2.7)$$

Coefficient of kurtosis based on octiles which was defined by (Moors, 1988) is

$$Kurtosis(M)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)} \qquad (2.8)$$

III. ESTIMATION OF THE MODEL PARAMETERS

Here, the parameters of the proposed distribution are estimated by utilizing some well-known estimation methods which are as follows

3.1. Maximum Likelihood Estimates

For the estimation of the parameters of LEP distribution, the maximum likelihood method is the most commonly used method introduced by (Casella & Berger, 1990). Let, $x_1, x_2, \dots, x_n$ be a random sample from $LEP(\alpha, \beta, \lambda)$ and the likelihood function, $L(\alpha, \beta, \lambda)$ is given by,

$$L(\varpi; x_1, x_2 \dots, x_n) = f(x_1, x_2, \dots, x_n / \varpi) = \prod_{i=1}^{n} f(x_i / \varpi)$$

$$L(\alpha,\beta,\lambda) = \alpha\beta\lambda \prod_{i=1}^{n} \frac{x_i^{\beta-1} e^{\lambda^{x_i^{\beta}}} \exp\left(e^{\lambda^{x_i^{\beta}}} - 1\right)\left\{\exp\left(e^{\lambda^{x_i^{\beta}}} - 1\right) - 1\right\}^{\alpha-1}}{\left[1 + \left\{\exp\left(e^{\lambda^{x_i^{\beta}}} - 1\right) - 1\right\}^{\alpha}\right]^2}; \ (\alpha,\beta,\lambda) > 0, \ x > 0$$

Now log-likelihood density is

$$\ell(\alpha,\beta,\lambda \mid \underline{x}) = n\ln(\alpha\beta\lambda) - n + (\beta-1)\sum_{i=1}^{n}\ln x_i + \lambda\sum_{i=1}^{n} e^{\lambda x_i^{\beta}} + \lambda\sum_{i=1}^{n} x_i^{\beta} + (\alpha-1)\sum_{i=1}^{n}\ln\left[\exp\left\{\left(e^{\lambda x_i^{\beta}} - 1\right)\right\} - 1\right] - 2\sum_{i=1}^{n}\ln\left\{1 + \left[\exp\left\{\left(e^{\lambda x_i^{\beta}} - 1\right)\right\} - 1\right]^{\alpha}\right\} \qquad (3.1.1)$$

Differentiating (3.1.1) with respect to α, β and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(D(x_i)\right) - 2\sum_{i=1}^{n}\frac{\left(D(x_i)\right)^{\alpha}\ln\left(D(x_i)\right)}{1 + D(x_i)^{a}}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln x_i + \lambda\sum_{i=1}^{n} x_i^{\beta}\ln x_i + \lambda\sum_{i=1}^{n} x_i^{\beta} \mathrm{e}^{\lambda x_i^{\beta}}\ln x_i + (\alpha-1)\lambda\sum_{i=1}^{n}\frac{x_i^{\beta}\mathrm{e}^{\lambda x_i^{\beta}} D(x_i)\ln x_i}{D(x_i)} - 2\alpha\lambda\sum_{i=1}^{n}\frac{x_i^{\beta}\ln(x_i) D(x_i)^{\alpha-1}\mathrm{e}^{\mathrm{e}^{\lambda x_i^{\beta}} + \lambda x_i^{\beta} - 1}}{1 + D(x_i)^{a}}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} x_i^{\beta}\mathrm{e}^{\lambda x_i^{\beta}} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\beta}\mathrm{e}^{\lambda x_i^{\beta}}\mathrm{e}^{\mathrm{e}^{\lambda x_i^{\beta}} + \lambda x_i^{\beta} - 1}}{D(x_i)} - 2\alpha\sum_{i=1}^{n}\frac{x_i^{\beta} D(x_i)^{\alpha-1}\mathrm{e}^{\mathrm{e}^{\lambda x_i^{\beta}} + \lambda x_i^{\beta} - 1}}{1 + D(x_i)^{a}}$$

Where $D(x_i) = e^{e^{\lambda x_i^{\beta}} - 1} - 1$

Equating above three non linear equations to zero and solving simultaneously for α, β and λ, we get the maximum likelihood estimate $\hat{\alpha}, \hat{\beta}$ *and* $\hat{\lambda}$ of the parameters α, β and λ. By using computer software like R, Matlab, Mathematica etc for maximization of (3.1.1) we can obtain the estimated value of α, β and λ. For the confidence interval estimation of α, β and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α, β and λ can be obtained as,

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Where

$$A_{11}=\frac{\partial^2 l}{\partial\alpha^2},\ A_{12}=\frac{\partial^2 l}{\partial\alpha\partial\beta},\ A_{13}=\frac{\partial^2 l}{\partial\alpha\partial\lambda}$$
$$A_{21}=\frac{\partial^2 l}{\partial\beta\partial\alpha},\ A_{22}=\frac{\partial^2 l}{\partial\beta^2},\ A_{23}=\frac{\partial^2 l}{\partial\beta\partial\lambda}$$
$$A_{31}=\frac{\partial^2 l}{\partial\lambda\partial\alpha},\ A_{32}=\frac{\partial^2 l}{\partial\beta\partial\lambda},\ A_{33}=\frac{\partial^2 l}{\partial\lambda^2}$$

Let $\Theta=(\alpha,\beta,\lambda)$ represent the space for the model parameters and the corresponding MLE of Θ as $\hat{\Theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$, then $\left(\hat{\Theta}-\Theta\right)\to N_3\left[0,\left(A(\Theta)\right)^{-1}\right]$ where $A(\Theta)$ is the Fisher's information matrix. Using the Newton-Raphson algorithm to maximize the likelihood creates the observed information matrix and hence the variance-covariance matrix can be calculated as,

$$\left[A(\Theta)\right]^{-1}=\begin{pmatrix}\mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda})\\ \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta},\hat{\lambda})\\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\beta},\hat{\lambda}) & \mathrm{var}(\hat{\lambda})\end{pmatrix} \quad (3.1.2)$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for the model parameters α, β and λ can be constructed as,

$$\hat{\alpha}\pm Z_{\alpha/2}SE(\hat{\alpha})\ ,\ \hat{\beta}\pm Z_{\alpha/2}SE(\hat{\beta})\ \text{and,}\ \hat{\lambda}\pm Z_{\alpha/2}SE(\hat{\lambda})$$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate

3.2. Method of Least-Square Estimation (LSE)

Swain et al. (1988) have introduced the ordinary least square estimators and weighted least square estimators to estimate the parameters of Beta distributions. Here the same technique is adopted for the estimation of the parameters of the LEP distribution. The least-square estimators of the unknown parameters α, β and λ of LEP distribution can be calculated by minimizing

$$M\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2 \quad (3.2.1)$$

with respect to unknown parameters α, β and λ.

Consider $F(X_i)$ denotes the distribution function of the ordered random variables $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ where $\{X_1,X_2,\ \ldots,X_n\}$ is a random sample of size n from a distribution function F(.). The least-square estimators of α, β and λ say $\hat{\alpha},\hat{\beta}, and\, \hat{\lambda}$ respectively, can be obtained by minimizing

$$M\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{i}{n+1}\right]^2;x\geq 0,\alpha>0,\beta>0\ \lambda>0. \quad (3.2.2)$$

with respect to α, β and λ.

Differentiating (3.2.2) with respect to α, β and λ we get,

$$\frac{\partial M}{\partial\alpha}=-2\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\ln\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]}{\left\{1+\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial M}{\partial \beta}=-2\alpha\lambda\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{x_i^{\beta}\ln(x)e^{\lambda x_i^{\beta}}\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha-1}\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)\right]}{\left\{1+\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\right\}^{2}}$$

$$\frac{\partial M}{\partial \lambda}=-2\alpha\sum_{i=1}^{n}x_i^{\beta}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha-1}e^{\lambda x_i^{\beta}}\exp\left(e^{\lambda x_i^{\beta}}-1\right)}{\left\{1+\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\right\}^{2}}$$

Similarly the weighted least square estimators can be obtained by minimizing

$$M\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right]^{2}$$

with respect to α, β and λ. The weights w_i are $w_i=\frac{1}{Var(X_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$

Hence, the weighted least square estimators of α, β and λ respectively can be obtained by minimizing,

$$M\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{i}{n+1}\right]^{2} \quad (3.2.3)$$

with respect to α, β and λ.

3.4. Method of Cramer-Von-Mises estimation (CVME)

The CVME estimators of α, β and λ are obtained by minimizing the function

$$C\left(X;\alpha,\beta,\lambda\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^{2}$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]^{2} \quad (3.4.1)$$

Differentiating (3.4.1) with respect to α, β and λ we get,

$$\frac{\partial C}{\partial \alpha}=-2\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]\frac{\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\ln\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]}{\left\{1+\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\right\}^{2}}$$

$$\frac{\partial C}{\partial \beta}=-2\alpha\lambda\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]\frac{x_i^{\beta}\ln(x)e^{\lambda x_i^{\beta}}\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha-1}\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)\right]}{\left\{1+\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\right\}^{2}}$$

$$\frac{\partial C}{\partial \lambda}=-2\alpha\sum_{i=1}^{n}x_i^{\beta}\left[1-\frac{1}{1+\left(\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]\frac{\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha-1}e^{\lambda x_i^{\beta}}\exp\left(e^{\lambda x_i^{\beta}}-1\right)}{\left\{1+\left[\exp\left(e^{\lambda x_i^{\beta}}-1\right)-1\right]^{\alpha}\right\}^{2}}$$

Solving $\frac{\partial C}{\partial \alpha}=0, \frac{\partial C}{\partial \beta}=0 \text{ and } \frac{\partial C}{\partial \lambda}=0$ simultaneously we will get the CVM estimators.

IV. APPLICATION TO REAL DATASET

In the following section, we illustrate the applicability of the LEP distribution by using two different real datasets used by previous researchers. To illustrate the goodness of fit of the Lindley inverse exponential distribution, we have taken some well known distribution for comparison purpose which are listed blew,

A. Generalized Exponential Extension (GEE) distribution:

The probability density function of GEE introduced by (Lemonte, 2013) having upside down bathtub-shaped hazard function distribution with parameters α, β and λ is

$$f_{GEE}(x;\alpha,\beta,\lambda)=\alpha\beta\lambda\,(1+\lambda x)^{\alpha-1}\,exp\left\{1-(1+\lambda x)^{\alpha}\right\}$$
$$\left[1-exp\left\{1-(1+\lambda x)^{\alpha}\right\}\right]^{\beta-1}\quad ;\; x\geq 0.$$

B. Lindley-Exponential (LE) distribution:

The probability density function of LE (Bhati, 2015) can be expressed as

$$f_{LE}(x)=\lambda\left(\frac{\theta^2}{1+\theta}\right)e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}\left\{1-\ln\left(1-e^{-\lambda x}\right)\right\};\; \lambda,\theta>0,\; x>0$$

C. Generalized Exponential (GE) distribution

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999)

$$f_{GE}(x;\alpha,\lambda)=\alpha\,\lambda\,e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1};(\alpha,\lambda)>0, x>0\cdot$$

D. Chen distribution

The probability density function of Chen distribution (Chen, 2000) is

$$f_{CN}(x;\lambda,\beta)=\lambda\,\beta\,\;x^{\beta-1}\,e^{x^{\beta}}\,exp\left\{\lambda\left(1-e^{x^{\beta}}\right)\right\}\quad ;(\lambda,\beta)>0, x>0\cdot$$

E. Exponential power (EP) distribution

The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) is

$$f_{EP}(x)=\alpha\,\lambda^{\alpha}\;x^{\alpha-1}\,e^{(\lambda x)^{\alpha}}\;\exp\left\{1-e^{(\lambda x)^{\alpha}}\right\}\quad ;(\alpha,\lambda)>0,\quad x\geq 0\cdot$$

where α and λ are the shape and scale parameters, respectively.

Dataset-I (NP data)

We illustrate the applicability of the LC model using a real dataset used by former researchers. We have taken 100 observations on breaking the stress of carbon fibers (in Gba) (Nichols & Padgett, 2006).

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

To estimate the MLEs we are utilizing the optim() function in R software (R Core Team, 2020) and (Ming, 2019) by maximizing the likelihood function (3.1). By maximizing the likelihood function in (3.1) we have obtained $\hat{\alpha}$ = 1.8940, $\hat{\beta}$ = 1.2276, $\hat{\lambda}$ = 0.3268 and corresponding Log-Likelihood value is l = -141.2223. We have presented the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and λ in Table 1.

Table 1: MLE and SE and 95% confidence interval for α, β and λ

Parameter	MLE	SE	95% ACI
alpha	1.8940	0.5296	(0.8560, 2.9321)
beta	1.2276	0.3077	(0.6245, 1.8307)
lambda	0.1650	0.0516	(0.0640, 0.2661)

.

We have displayed the graph of the profile log-likelihood function of α, β, and λ in Fig. 2 (Kumar & Ligges, 2011) and observed that the MLEs are unique.

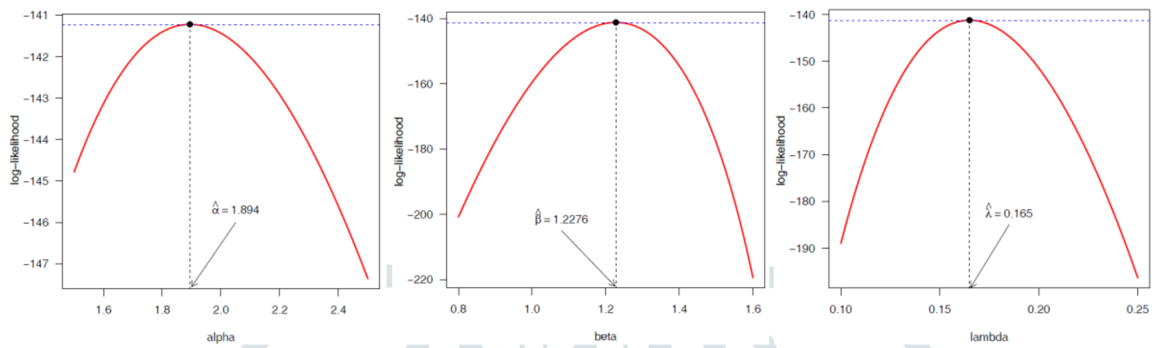

Figure 2. Graph of profile log-likelihood function of α, β, and λ (dataset-I).

Dataset-II (Lee and Wang)

The second real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang, 2003): sorted data

0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

By maximizing the likelihood function in (3.1) we have obtained $\hat{\alpha}$ = 3.7321, $\hat{\beta}$ = 0.2591, $\hat{\lambda}$ = 0.3268 and corresponding Log-Likelihood value is l = -409.4238. We have presented the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and λ in Table 2.

Table 2: MLE and SE and 95% confidence interval for α, β and λ

Parameter	MLE	SE	95% ACI
alpha	3.7321	1.6241	(0.5489, 6.9153)
beta	0.2591	0.1088	(0.0459, 0.4723)
lambda	0.3268	0.0675	(0.1945, 0.4591)

. We have displayed the graph of the profile log-likelihood function of α, β, and λ in Fig. 3 and observed that the MLEs are unique.

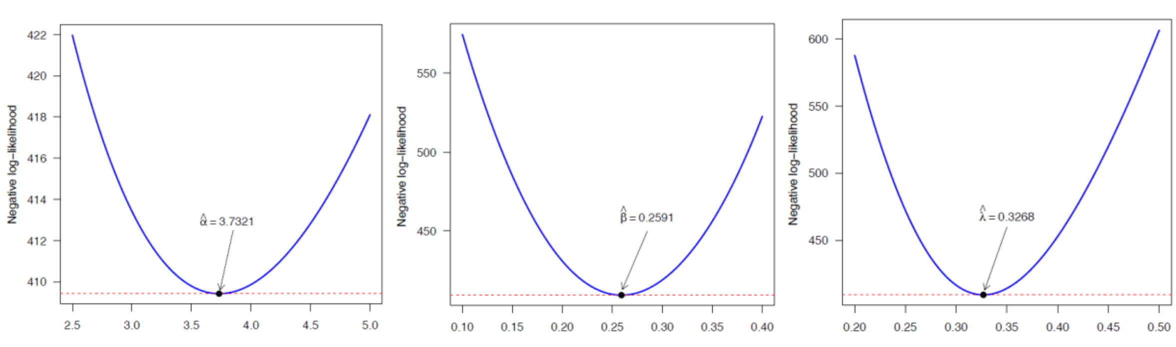

Figure 3. Graph of profile log-likelihood function of α, β, and λ (dataset-II).

In Fig. 4 we have presented the Q-Q plot (empirical quantile against theoretical quantile) for both data sets.

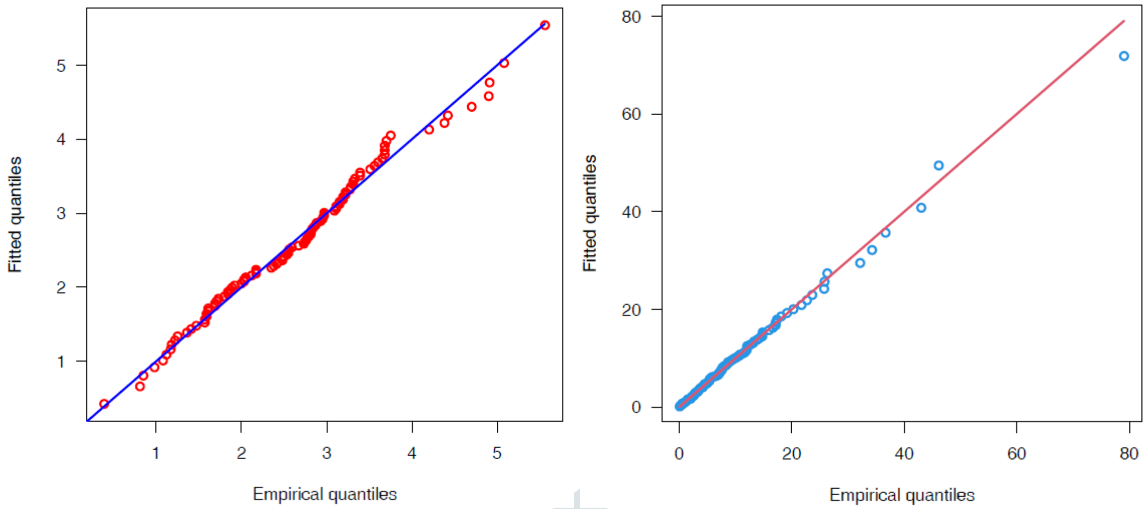

Figure 4. The Q-Q plots of the data set-I (left panel) and data set-II (right panel).

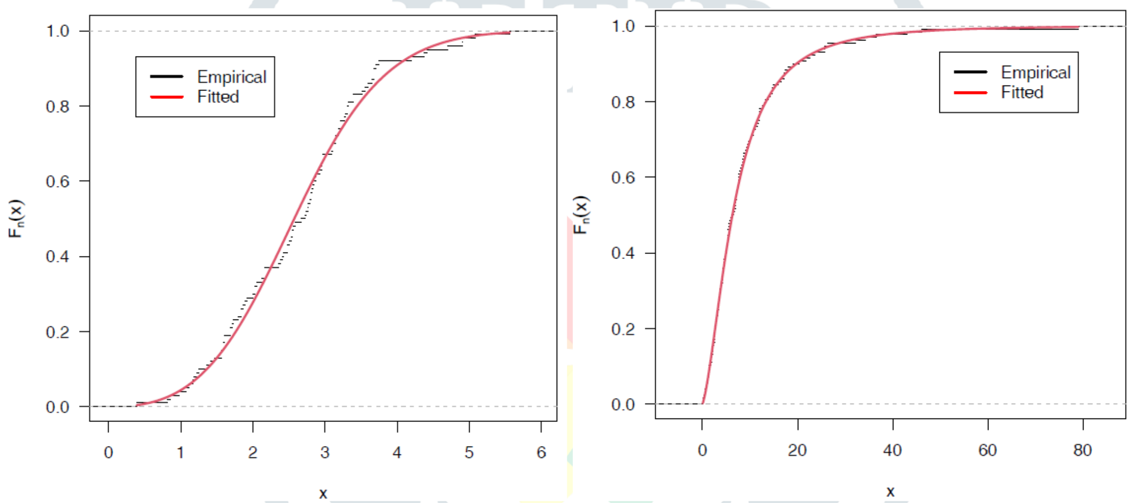

Figure 5. CDF plots (empirical distribution function against theoretical distribution function) of data set-I (left panel) and data set-II (right panel).

For the estimation of the parameters of the LEP distribution, we have made use of MLE, LSE and CVME methods. We have also used negative log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information criterion (CAIC) and Hannan-Quinn information criterion (HQIC), statistic for the comparison of goodness of fit propose. The expressions to compute AIC, BIC, CAIC and HQIC are listed below:

i. $AIC = -2l(\hat{\theta}) + 2k$

ii. $BIC = -2l(\hat{\theta}) + k\log(n)$

iii. $CAIC = AIC + \dfrac{2k(k+1)}{n-k-1}$

iv. $HQIC = -2l(\hat{\theta}) + 2k\log\left[\log(n)\right]$

where k is the number of parameters and n is the size of the sample in the model under consideration. Further, in order to assess the fits of the LEP distribution with some other distributions, the Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistic are used. These statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. These statistics are calculated as

$$KS = \max_{1\le i\le n}\left(d_i - \frac{i-1}{n}, \frac{i}{n} - d_i\right)$$

$$W = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i + \ln\left(1 - d_{n+1-i}\right)\right]$$

$$A^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - d_i\right]^2$$

where $d_i = CDF(x_i)$; the x_i's being the ordered observations.

In Table 3 and Table 4 we have presented the estimated value of the parameters of Logistic exponential power distribution using MLE, LSE and CVME method and their corresponding log-likelihood, AIC and KS statistic with p-value.

Table 3: Estimated parameters, log-likelihood, AIC and KS statistic with p-value (Dataset-I)

Estimation Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-ll	AIC	KS(p-value)
MLE	1.8940	1.2276	0.1650	141.2223	288.4445	0.0623(0.8327)
LSE	0.9915	2.2896	0.0581	158.9975	323.9951	0.0588(0.8802)
CVME	0.9803	2.3506	0.0546	163.0648	332.1297	0.0610(0.8511)

Table 4: Estimated parameters, log-likelihood, AIC and KS statistic with p-value(Dataset-II)

Estimation Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-ll	AIC	KS(p-value)
MLE	3.7321	0.2591	0.3268	409.4238	824.8476	0.0316(0.9995)
LSE	5.8651	0.1654	0.3891	409.7016	825.4033	0.0305(0.9998)
CVME	5.8717	0.1672	0.3878	409.6965	825.3931	0.0310(0.9997)

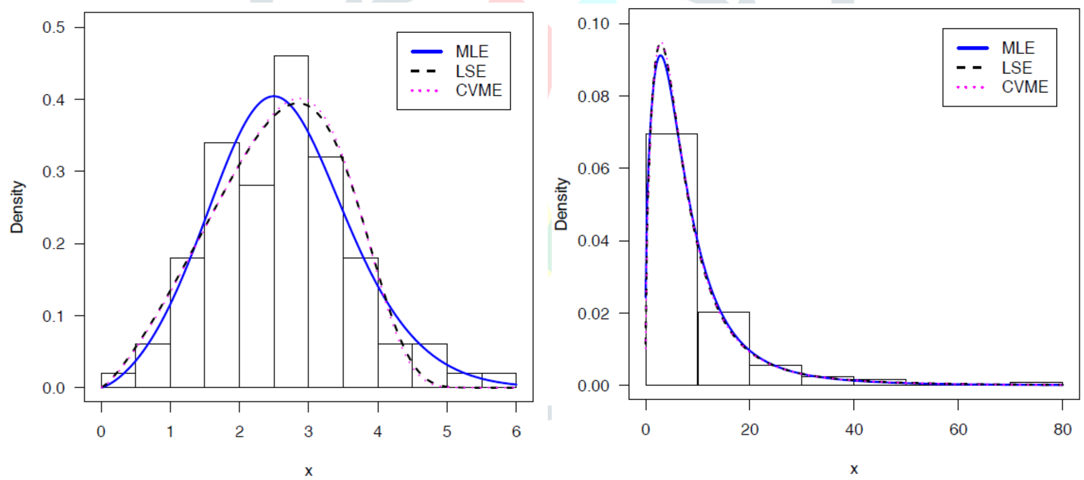

Figure 4. The Histogram and the density function of fitted distributions of dataset-I (left panel) and dataset-II (right panel) of MLE, LSE and CVME methods.

We have calculated the value of Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) for the assessment of goodness of fit of the proposed model, which are displayed in Table 4 and Table 5 for dataset I and II respectively.

Table 4: Log-likelihood (LL), AIC, BIC, CAIC and HQIC (dataset-I)

Model	-LL	AIC	BIC	CAIC	HQIC
LEP	141.2223	288.4445	296.2600	288.6945	291.6076
GEE	141.3708	288.7416	296.5571	288.9916	291.9047
LE	143.2473	290.4946	295.7049	290.6183	292.6033
EP	145.9589	295.9179	301.1282	296.0391	298.0266
GE	146.1823	296.3646	301.5749	296.4883	298.4733
Chen	148.9044	301.8089	307.0192	301.9326	303.9176

Table 5: Log-likelihood (LL), AIC, BIC, CAIC and HQIC (dataset-II)

Model	-LL	AIC	BIC	CAIC	HQIC
LCD	409.4238	824.8476	833.4037	825.0411	828.3240
GEE	410.6013	827.2026	835.7586	827.3961	830.6789
LE	412.6254	829.2507	834.9548	829.3467	831.5683
GE	413.0776	830.1552	835.8592	830.2512	832.4728
EP	426.6474	857.2948	862.9989	857.3893	859.6124
Chen	431.1625	866.3251	872.0291	866.4211	868.6427

In fig. 5, we have displayed the histogram and the density function of fitted distributions and empirical distribution function with estimated distribution function of LEP distribution and some selected distributions taken for comparisons.

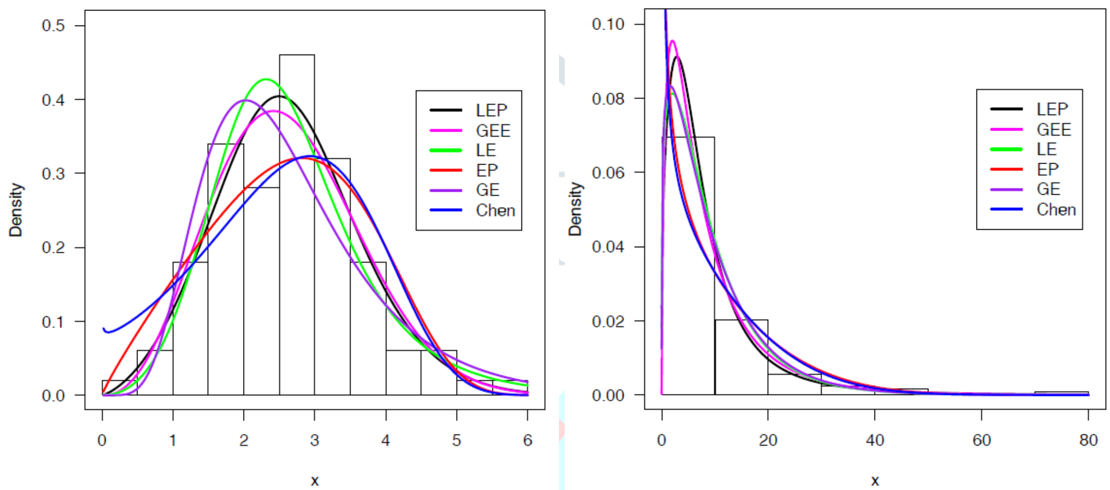

Figure 5. The Histogram and the density function of fitted distributions for the dataset-I (left panel) and dataset-II (right panel). Empirical distribution function with estimated distribution function for the dataset-I (left panel) and dataset-II (right panel).

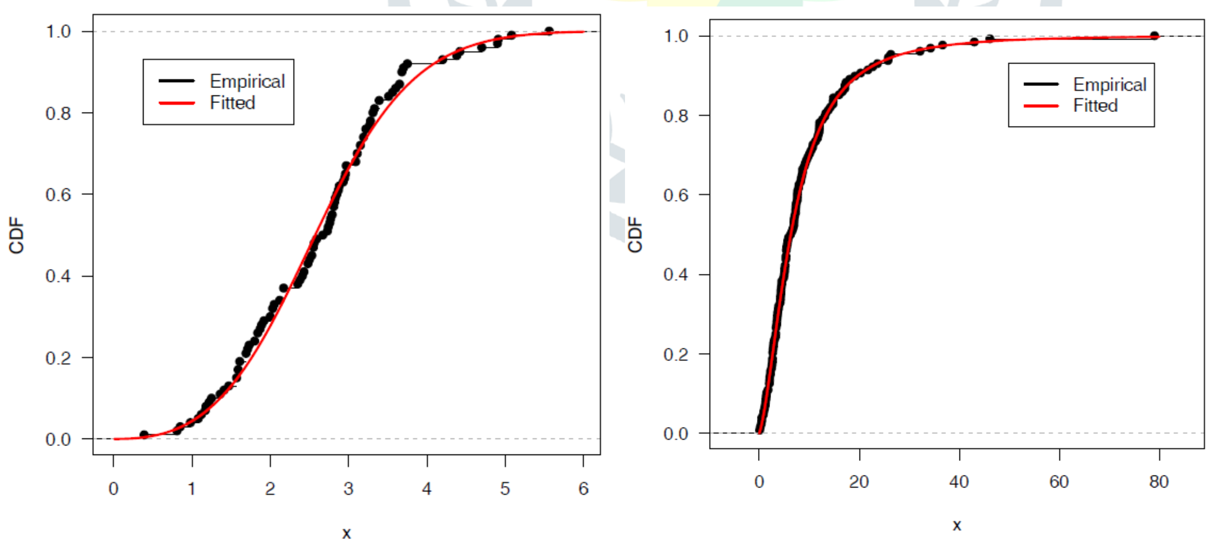

Figure 6. Empirical distribution function with estimated distribution function for the dataset-I (left panel) and dataset-II (right panel).

For the comparison of goodness-of-fit of the LEP distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 6 and Table 7 for the both data sets. We observe that the LEP distribution has the minimum value of the test statistic and higher *p*-value hence we conclude that the LEP distribution gets quite better fit and more consistent and reliable results then all the models taken for assessment.

Table 6: The goodness-of-fit statistics and their corresponding p-value (dataset-I)

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LCD	0.0623(0.8327)	0.0642(0.7885)	0.3829(0.8652)
GEE	0.0654(0.7862)	0.0723(0.7385)	0.4202(0.8281)
LE	0.0838(0.4836)	0.1225(0.4860)	0.7042(0.5549)
GE	0.0993(0.2771)	0.1861(0.2963)	1.3081(0.2297)
EP	0.1078(0.1959)	0.2293(0.2174)	1.2250(0.2581)
Chen	0.0945(0.3336)	0.2180(0.2353)	1.6938(0.1364)

Table 7

The goodness-of-fit statistics and their corresponding p-value (dataset-II)

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LCD	0.0316(0.9995)	0.0146(0.9997)	0.0961(0.9998)
GEE	0.0442(0.9636)	0.0394(0.9367)	0.2630(0.9631)
LE	0.0691(0.5740)	0.1131(0.5252)	0.6276(0.6219)
GE	0.0725(0.5115)	0.1279(0.4652)	0.7137(0.5472)
EP	0.1199(0.0503)	0.5993(0.0223)	3.6745(0.0126)
Chen	0.1426(0.0108)	0.6879(0.0135)	4.3878(0.0057)

V. CONCLUSIONS

In this study, we have introduced a three-parameter univariate continuous distribution named Logistic-exponential power (LEP) distribution. Some mathematical and statistical properties of the LEP distribution are presented such as the shapes of the probability density, cumulative density and hazard rate functions, survival function, reverse hazard rate function quantile function, the skewness, and kurtosis measures are derived and established and found that the proposed model is flexible and inverted bathtub shaped hazard function. We have employed three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods to estimate the model parameters and we concluded that the MLEs are quite better than LSE, and CVM. Two real data sets are considered to explore the applicability and suitability of the proposed distribution and found that the Logistic-exponential power model is quite better than other lifetime model taken into consideration. We hope this model may be an alternative in the field of survival analysis, reliability theory and applied statistics.

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