Review on computing variance on PoRB Function by using hypergeometric function

Madhav Prasad Poudel,
School of Engineering, Pokhara University, Kaski, Nepal.

Abstract

Among many tools to solve the functions of many variables, Radial Basis Function (RBF) is one of them. RBF were first introduced by Powell for multivariate interpolation. The scientific computing with RBF focuses on construction and solution of partial differential equations.

Wendland functions are the special case of RBF. Generally RBF follow the Poisson equation and the covariance can be calculated. This research aims to review the variance of Poisson based RBF with gamma prior by using hypergeometric function.

Keyword: Radial Basis Function, Wendland Function, hypergeometric function, variance

Introduction

There are many ways to approximate a function with many variables. Almost all of them have their own limitations. If the dimension of the problem is large, it is difficult to choose the function to apply. In this case tensor product method may be the option. The tensor product methods in high dimensions always require large data. In this situation, the only option is radial basis function. Due to its excellent approximation properties, at any rate it is universally applicable independent of dimension. [1]

Radial Basis Functions (RBF) are means to approximate multivariable functions by linear combinations of terms based on a single univariate function is called radial basis function. They are usually applied to approximate functions or data which are only known at a finite number of points. RBF methods are modern techniques to approximate multivariate functions, especially in the absence of grid data. [1] RBFs were introduced in [2] and formed a primary tool for multivariate interpolation. Hardy [3] showed that multi quadrics (MQs) are related to a consistent solution of the biharmonic potential problem and thus they have a physical foundation. Buhmann and Micchelli [4] have shown that RBFs are related to pre wavelets. Kansa [5] illustrated firstly the idea of using RBFs collocation method for solving partial differential equations (PDEs) and forming a class of truly mesh-free method.

Scientific computing with Radial Basis Functions focuses on construction of unknown functions from the known data. The functions are in general multivariate, and they may be solutions of partial differential equations satisfying certain initial conditions[5]. Compactly supported Radial Basis Functions, with polynomials were considered by Wu and Schaback[6] and are now called Wendland functions[7]. These functions are piecewise polynomials, and Wendland's construction provided functions with minimal degrees under the given conditions.

The Wendland functions are a class of compactly supported radial basis functions with a user-specified smoothness parameter that converge uniformly to a Gaussian function as the smoothness of the parameter approaches infinity[8].

Coker, B.[9] introduced the hypergeometric function to solve the poison process radial basis function netwok for homogenous gamma prior and obtained the satisfactory result In that case the hypergeometric function used was ${}_2F_1(a, b, c, z)$. Similarly, Chernih, A. [10] used the hypergeometric function $F(a, b, c, z)$ $c = 0$ to solve the Fourier transform of the Wendland function to disseminate the Gaussian equation. Bang Farnberg used the hypergeometric series ${}_0F_1$ to solve the partial differential equation on oscillatory radial basis function [11].The result obtained was satisfactory and was in parallel with the other methods used to solve the differential equation.
The aim of this paper is to review the hypergeometric series ${}_2F_1(a, b, c, z)$; $c \neq 0$ to solve the Fourier Transform of Wendland function by a gamma prior function in RBF.

Literature review

RBF is a real-valued function ψ whose value depends only on distance between the input and some fixed point, either the origin, so that $\psi(x) = \psi(\|x\|)$, or some other fixed point c, called a center. So that $\psi(x) = \psi(\|x - c\|)$. Any function ψ that satisfies the property $\psi(x) = \psi(\|x\|)$ is a radial function. The distance is usually Euclidean distance. Radial basis functions are means to approximate multivariable functions by linear combinations of terms based on a single univariate function (the radial basis function). This is interpreted so that in can be used in more than one dimension. They are usually applied to approximate functions or data in mathematics

In mathematical studies, functions of many variables we often need to approximate by the help of other functions that are better understood or more readily evaluated. May be for generating in computer graphics or other practical use. Radial basis functions are one efficient, frequently used way to do this. Further applications include the important fields of neural networks and learning theory. Since they are radially symmetric functions which are shifted by points in multidimensional Euclidean space and then linearly combined, they form data-dependent approximation spaces. This data-dependence makes the spaces so formed suitable for providing approximations to large classes of given functions. It also opens the door to existence and uniqueness results for interpolating scattered data by radial basis functions in very general settings

The radial basis function network is an artificial neural network that uses radial basis functions as activation functions. The output of the network is a linear combination of radial basis functions of the inputs and neuron parameters. Radial basis function networks have many uses, including function approximation, time series prediction, classification, and system control. (9)

Poisson Process Radial Basis Function Networks (PoRB-Nets), belong to the interpretable family of RBFNs that employ a Poisson process prior over the center parameters in an RBFN. Intuitively, PoRB-Nets work by trading off between the concentration and scale of the radial basis functions. Consider that a higher concentration of basis functions allows for a smaller length scale but also a larger variance, since the basis functions add up. By making the scale of the basis functions depend inversely on their concentration, PoRB-Nets undo the impact on the variance.(9)

A Poisson process (PP) on R^D is a stochastic process characterized by a positive real-valued intensity function λ(c). For any set $C \in R^D$, the number of points in C follows a Poisson distribution with parameter $\int \lambda(c)dc$ over c. The process is inhomogeneous if λ(c) is non-constant. We use a PP as a prior on the center parameters of an RBFN.(9)

The term "hypergeometric function" sometimes refers to the generalized hypergeometric function. The first person to use the term "hypergeometric series" was by John Wallis in his 1655 book Arithmetica Infinitorum. Then the further studies on Hypergeometric series was done by Leonhard Euler and was systematically developed by Carl Friedrich Gauss (1813). In mathematics, the Gaussian or ordinary hypergeometric function $_2F_1(a,b;c;z)$ is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE). Every second-order linear ODE with three regular singular points can be transformed into this equation.

The hypergeometric function is defined for $|z| < 1$ by the power series

$$_2F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}$$

It is undefined (or infinite) if c equals a non-positive integer. Here $(q)_n$ is the (rising) Pochhammer symbol which is defined by: $(q)_n = \begin{Bmatrix} 1 & n = 0 \\ q(q+1)(q+2)(q+(n-1)) & n > 0 \end{Bmatrix}$

The series terminates if either a or b is a non positive integer. In this case the function reduces to a polynomial

$$_2F_1(-m, b, c, z) = \sum_{n=0}^{\infty} (-1)^n \binom{m}{n} \frac{(b)_n z^n}{(c)_n}$$

The hypergeometric function can be expressed as $_pF_q = \sum_{n=0}^{\infty} \frac{(a_1)(a_2)(a_3)(a_4)---------(a_p)}{n!(b_1)(b_2)(b_3)-----------(b_q)} Z^n$, where n is the number of parameter a_i and q is the number of parameter b_j. It is the solution of the differential equation $z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0$ [13, 14] which has three regular singular points: 0,1 and ∞. The generalization of this equation to three arbitrary regular singular points is given by Riemann's differential equation. Any second order differential equation with three regular singular points are converted to the hypergeometric differential equation by a change of variables. It is used to solve the differential equations with the wide number of variables.

Solutions to the hypergeometric differential equation are built out of the hypergeometric series ${}_2F_1(a,b;c;z)$. The equation has two linearly independent solutions. At each of the three singular points, 0, 1, ∞, there are usually two special solutions of the form x^s times a holomorphic function of x. And s is one of the two roots of the indicial equation and x is a local variable vanishing at the regular singular point.

The property of the function of retaining its form where two variables are linearly transformed is called the covariance. The word variance refers to the statistical measurement that is diversified then the original set. More specifically, it measures the deviations from the mean and the values in the given set.

In tensor product problems, many functions are solved by means of RBF. Rayleigh-Ritz applications was applied to solve the partial differential equation by using radial basis function. It is because RBF replace other meshless tools like multi quadrics. They generate sparse and well-conditioned matrices, which is relevant for boundary element techniques.

Mirinejad, H.[15] tried to solved the optimal control problem by means of RBF. His work presents two direct methods based on the radial basis function interpolation and arbitrary discretization for solving continuous-time optimal control problems: RBF Collocation Method and RBF-Galerkin Method. Both methods take advantage of choosing any global RBF as the interpolant function and any arbitrary points (meshless or on a mesh) as the discretization points. Huaiquing, Z. et al [16] applied the basic linear function to solve the non linear equations and they observed that the superior interpolation performance of multi quadratic function, the method can acquire higher accuracy with fewer nodes, so it takes obvious advantage over the Gaussian RBF method which can be revealed from the numerical results.

The entire theory of the RBF of compact support are piecewise polynomial definite. This theory encompasses recursions for the coefficients when they are expanded in linear combinations of powers and truncated powers of lower order with convergence results, and minimalism of polynomial degree for given dimension and smoothness

Cho, Y.[17] extended the positivity for integrals of Bessel function and Buhmann's radial basis function by using the hypergeometric function. He used Saalschutzian series , Wipples transformation to evaluate the integral of Bessel'sfunction and Buhmann's Radial Basis Function. The Saalschutzian series, Wipples transformation are the important tools in hypergeometric series.

Coker, B. et. al[9] have extended radial basis function networks (RBFNs) that allows for independent specification of functional amplitude variance and length scale (i.e., smoothness), where the inverse length scale corresponds to the concentration of radial basis functions. When the length scale is uniform over the input space, the result was consistent and approximate variance was achieved. The model's behavior have been compared to standard BNNs and Gaussian processes using synthetic and real examples. To calculate the variance under Poisson process radial basis function network for homogeneous process with gamma prior it, found that:

$$V(x_1, x_2) = \iint \phi(s(x_1-c)) \iint \phi(s(x_2-c))p(c/\lambda)p(\lambda)d\lambda dc$$

where λ is the center parameter which is uniformly distributed over the domain c.

$$= \iint \exp\left\{-\frac{1}{2}({s_o}^2\lambda^2(x_1-c))^2\right\}\exp\left\{-\frac{1}{2}({s_o}^2\lambda^2(x_2-c))^2\right\}\frac{1}{\mu(c)}\frac{\beta^\alpha}{\gamma(\alpha)}\lambda^{2(\alpha-1)}e^{-\beta\lambda^2}d\lambda dc$$

$$=\frac{1}{\mu(c)}\frac{\beta^\alpha}{\gamma(\alpha)}\iint \lambda^{2(\alpha-1)}\exp\left\{-\lambda^2[\frac{1}{2}\ {s_o}^2(x_1-c)^2+\frac{1}{2}\ {s_o}^2(x-c)+\beta]\right\}d\lambda dc$$

$$=\frac{1}{\mu(c)}\frac{\beta^\alpha}{\gamma(\alpha)}\iint \lambda^{2(\alpha-1)}\exp\{-\lambda^2\breve{\beta}(C)\}d\lambda dc$$

$$\text{Where } \breve{\beta}(C)=\frac{1}{2}{S_0}^2(x_1-c)^2+\frac{1}{2}{S_0}^2(x-c)+\beta \text{ ----} \qquad (1)$$

$$=\frac{\beta^\alpha}{\mu(c)}[\int\frac{1}{\gamma(\alpha)}(\int\lambda^{2(\alpha-1)}\ \exp\{-\lambda^2\breve{\beta}(C)\}d\lambda]dc$$

$$= \frac{\beta^{\alpha}}{\mu(C)} \int \breve{\beta}^{-\alpha}(C)dC \quad ---------------------- \quad (2)$$

Since the equation (2) is the gamma probability density function, it was solved with the inner integral.

$$\text{Now } \breve{\beta}(C) = \frac{1}{2}{S_0}^2(x_1 - c)^2 + \frac{1}{2}{S_0}^2(x - c) + \beta$$

$$= {S_0}^2\left[\frac{1}{2}(x_1 - c)^2 + \frac{1}{2}(x - c)\right] + \beta$$

$$= {S_0}^2\left[\frac{1}{2}({x_1}^2 - 2x_1c + c^2 + \frac{(x-c)}{2}\right] + \beta \quad ----- \quad (3)$$

If $x= ({x_2}^2+c)^2+c$ then equation (3) reduces to

$$\breve{\beta}(C) = {S_0}^2\left[\frac{1}{2}({x_1}^2 - 2x_1c + c^2 + \frac{1}{2}(({x_2}^2 + c)^2 + c) - c\right] + \beta$$

$$= \frac{1}{2}{S_0}^2[2c^2 - 2x_1c + 2x_2c + {x_1}^2 + {x_2}^2] + \beta$$

$$= {S_0}^2\left[c^2 - 2c\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2}({x_1}^2 + {x_2}^2)\right] + \beta \quad ------------ \quad (4)$$

Let $x_m = \frac{x_1+x_2}{2}$ Then equation (4) reduces to

$$\breve{\beta}(C) = {S_0}^2(c - x_m)^2 + {S_0}^2(\frac{x_1-x_2}{2})^2 + \beta$$

$$= u^2 + r^2$$

where $u^2 = {S_0}^2(c - x_m)^2$ and $r^2 = {S_0}^2(\frac{x_1-x_2}{2})^2$

$$\text{Then } \int \breve{\beta}^{-\alpha}(C)dC = \int_{U_0}^{U_1}(u^2 + r^2)^2 dU$$

$$= {U_r}^{-2\alpha}\, {}_2F_1\left[\frac{1}{2,} \quad \alpha, \quad \frac{3}{2}, \quad -\frac{u^2}{r^2}\right]_{u_o}^{u_1}$$

where the hypergeometric function ${}_2F_1(a, b, c, z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!}$

$$\text{And } (q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)(q+2)(q+(n-1)) & n > 0 \end{cases}$$

By using the commutative property of hypergeometric, function

$${}_2F_1\left[\frac{1}{2,} \quad \alpha, \quad \frac{3}{2}, \quad -\frac{u^2}{r^2}\right] = {}_2F_1\left[\alpha \quad \frac{1}{2,}, \quad \frac{3}{2}, \quad -\frac{u^2}{r^2}\right] \text{ it can be written as}$$

$$\int \breve{\beta}^{-\alpha}(C)dC = {U_r}^{-2\alpha}\, {}_2F_1\left[\alpha \quad \frac{1}{2,}, \quad \frac{3}{2}, \quad -\frac{u^2}{r^2}\right]_{u_o}^{u_1} \quad ----------------------- \quad (5)$$

Here it is noted that $U_o={S_o}^2(c_0-x_m)$ and $U_1={S_o}^2(c_1-x_m)$

Replacing the values of (5) in (2), the variance is obtained as

$$V(x_1, x_2) = \frac{1}{\mu(c)}\left(\frac{\beta}{r^2}\right)^{-\alpha}\left[\frac{1}{(x-c_0)}\{{}_2F_1(\alpha, \quad \frac{1}{2,}, \quad \frac{3}{2}, \quad -\frac{s^2(c_o-x)^2}{r^2}\} + \frac{1}{(c_1-x)}\{{}_2F_1(\alpha, \quad \frac{1}{2,}, \quad \frac{3}{2}, \quad -\frac{s^2(c_1-x)^2}{r^2}\}\right]$$

This is the solution of the variance of homogeneous Poisson process radial basis function for the gamma prior.

Results

Through the series of derivations, the variance of the homogeneous Poisson process radial basis (PoRB) function by using the transitive property on gamma prior was found out. It was initially found out by Coker, B. et. al[9], which provided the sufficient results for covariance without gamma prior as well. There is a slight change in the result and this was due to the property applied in the hypergeometric function. Thus the result obtained is

$$V(x_1, x_2) = \frac{1}{\mu(c)}\left(\frac{\beta}{r^2}\right)^{-\alpha}\left[\frac{1}{(x-c_0)}\{{}_2F_1(\alpha, \quad \frac{1}{2,}, \quad \frac{3}{2}, \quad -\frac{s^2(c_0-x)^2}{r^2}\} + \frac{1}{(c_1-x)}\{{}_2F_1(\alpha, \quad \frac{1}{2,}, \quad \frac{3}{2}, \quad -\frac{s^2(c_1-x)^2}{r^2}\}\right]$$

Conclusion

The computation of variance in RBF is mainly based in the work of Coker on PoRB Nets. In general, PoRB nets allows for

a) Easy specification of length scale and amplitude variance information.
b) Learning of an input independent length scales.

So the computation of variance has significant importance in the network functions.

Author Contribution

Madhav Prasad Poudel is the sole author of this work. The conceptualization, methodology, analysis and preparation of the manuscript have been done by myself.

Funding

There is not any specific funding available for this research work.

Conflicts of Interest

The author declares no conflict of interest.

Bibliography

(1) Buhmann, Martin D. "Radial basis functions." Acta numerica 9 (2000): 1-38

(2) R. L. Hardy, “Multiquadric equations of topography and other irregular surfaces,” Journal of Geophysical Research, vol. 176, pp. 1905–1915, 1971.

(3) Hardy, Rolland L. "Theory and applications of the multiquadric-biharmonic method 20 years of discovery 1968–1988." Computers & Mathematics with Applications 19.8-9 (1990): 163-208.

(4) M. D. Buhmann and C. A. Micchelli, “Multiquadric interpolation improved,” Computers & Mathematics with Applications, vol. 24, no. 12, pp. 21–25, 1992.

(5) E. J. Kansa, “Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics— II. Solutions to parabolic, hyperbolic and elliptic partial differential equations,” Computers & Mathematics with Applications, vol. 19, no. 8-9, pp. 147–161, 1990.

(6) Schaback, Robert. "A practical guide to radial basis functions." Electronic Resource 11 (2007): 1-12.

(7) Argáez, Carlos, Sigurdur Hafstein, and Peter Giesl. "Wendland functions a C++ code to compute them." Proceedings of the 7th International Conference on Simulation and Modeling Methodologies, Technologies and Applications, 2017, Madrid, Spain. SCITEPRESS, 2017

(8) Schaback, Robert, and Holger Wendland. "Using compactly supported radial basis functions to solve partial differential equations." WIT Transactions on Modelling and Simulation 23 (1970).

(9) Coker, Beau, Melanie Fernandez Pradier, and Finale Doshi-Velez. "PoRB-Nets: Poisson Process Radial Basis Function Networks." Conference on Uncertainty in Artificial Intelligence. PMLR, 2020.

(10) Chernih, Andrew, Ian H. Sloan, and Robert S. Womersley. "Wendland functions with increasing smoothness converge to a Gaussian." Advances in Computational Mathematics 40.1 (2014): 185-200

(11). H. Wendland, Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree, Adv. Comput. Math., 4 (1995), pp. 389–396

(12) Fornberg, Bengt, Elisabeth Larsson, and Grady Wright. "A new class of oscillatory radial basis functions." Computers & Mathematics with Applications 51.8 (2006): 1209-1222.

(13) Rainville, Earl D. "Special functions." (1971).

(14) Seaborn, James B. Hypergeometric functions and their applications. Vol. 8. Springer Science & Business Media, 2013.

(15) Mirinejad, Hossein. "A radial basis function method for solving optimal control problems." (2016).

(16) Zhang, Huaiqing, et al. "Application of radial basis function method for solving nonlinear integral equations." Journal of Applied Mathematics 2014 (2014).

(17) Cho, Yong-Kum, Seok-Young Chung, and Hera Yun. "An extension of positivity for integrals of Bessel functions and Buhmann's radial basis functions." arXiv preprint arXiv:1805.11863 (2018).