STATISTICAL AND UNIFORM STATISTICAL CONVERGENCE OF SEQUENCES OF FUNCTIONS AND APPLICATIONS

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Abstract: The purpose of this paper is to introduce a statistical convergence and uniform Statistical convergence of sequence of real valued functions. Also Examine the validity of some theorems on Riemann- Integrable functions using statistical convergence and uniform statistical convergence of sequence of real valued functions.

Key words: Statistical convergence, Uniform statistical convergence, Statistical Cauchy Sequence, Riemann Integrable function.

1 Introduction

The concept of statistical convergence was introduce by Fast[2] and Steinhaus[3]and later reintroduce by Schoenberg[4] independently. Some applications of statistical convergence in number theory and mathematical analysis can be found in [[1], [5],[6]]. Furthermore Gungor et al.[8] introduce the concept of a uniform statistical Cauchy sequence for functional sequence and show that it is equivalent to uniform statistical convergence of sequence of real-valued functions. Omer et al.[9] obtain a statistical version of Lebesgue bounded convergence theorem and examine the validity of the classical theorem of Measure Theory for statistical convergence. The concept pointwise and uniform statistical convergence of order α for sequence of real valued functions is introduce by Cinar et al.[10]. Fridy[11] focus on statistically limit superior and limit inferior. Balcerzak [12] discussed on a statistical convergence and ideal convergence for Sequence of functions, Salat [13] guided about statistical convergence sequence of real numbers, Goldberg [7] helps for obtaining some results. In this paper view of sequence of functions are Riemann-integral. The Riemann-integral is discussed in terms of Statistical convergence and Uniform statistical convergence.

2 Preliminaries

This section is allocated to recall the definitions that will be needed in this manuscript.

Definition 2.1 A subset A of the ordered set N of natural numbers is said to have density $d(A)$.

If $\lim_{n\to\infty} \frac{|A_n|}{n} = A$ where, $A(n) = \{k \le n : k \epsilon A\}$ and $|A|$ denotes the cardinality of the set $A \subset N$.

Clearly finite set has zero density and $d(A') = 1 - d(A)$ where $d(A') = N - A$.

If a property $P(k)$ holds for all $k \epsilon A$ with $d(A) = 1$. We say that P holds for almost all k, i.e. a.a.k.

Definition 2.2 A sequence of function $\{f_k\}$ is statistically convergent to f on a set M, if for every $\epsilon > 0$;

$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |f_k(x) - f(x)| \ge \epsilon, \ for\ every\ x\ \epsilon M\}| = 0$, i.e for every $x\ \epsilon$ M,

$|f_k(x) - f(x)| < \epsilon$, a.a.k ; In this case we write: $st\text{-}\lim f_k(x) = f(x)$ or $f_k(x) \overset{st}{\to} f(x)$

Definition 2.3 A sequence of function $\{f_k\}$ is statistically Cauchy sequence provided that for every $\epsilon > 0$ there is number $n > N$ such that $\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |f_k(x) - f_n(x)| \ge \epsilon, \ for\ every\ x\ \epsilon M\}| = 0$

Theorem 2.1 Let $\{f_k\}$ be a sequence of functions defined on $\mathbb{R}$ is statistically convergent if and only if it is statistically Cauchy Sequence.

Proof. Let $\{f_k\}$ be a sequence of functions defined on $\mathbb{R}$ is statistically converges to f.

i.e. $st\text{-}\lim f_k(x) = f(x)$

$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : |f_k(x) - f_n(x)| \ge \frac{\epsilon}{2}, for\ every\ x \in M, n \ge N\right\}\right| = 0$

i.e. $|f_k(x) - f_n(x)| < \frac{\epsilon}{2}, n \ge N,$ a. a. k.

$\therefore$ for $n \ge N$, $|f_k(x) - f_n(x)| \le |f_k(x) - f(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, a.a.k.

Hence $\lim_{n\to\infty}\frac{1}{n}|\{k \le n : |f_k(x) - f_n(x)| \ge \epsilon, for\ every\ x \in M\}| = 0$

i.e. sequence $\{f_k\}$ is statistically Cauchy Sequence.

Conversly: let $\{f_k\}$ is statistically Cauchy Sequence. To show $\{f_k\}$ is statistically convergent.

It is sufficient to show that, $\liminf\{f_k\} = \limsup\{f_k\}$ (2.1)

Since , $\liminf\{f_k\} \le \limsup\{f_k\}$, and $\{f_k\}$ is statistically Cauchy Sequence.

$\therefore \lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : |f_k(x) - f_n(x)| \ge \frac{\epsilon}{2}, for\ every\ x \in M, n \ge N\right\}\right| = 0$

$\therefore \lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : |f_N(x) - f_n(x)| \ge \frac{\epsilon}{2}, for\ every\ x \in M\right\}\right| = 0$

$\Rightarrow f_N + \frac{\epsilon}{2}$ is upper bound and lower bound is $f_N - \frac{\epsilon}{2}$ for the set $\{f_N, f_{N+1}, \dots\}$

$\therefore f_N - \frac{\epsilon}{2} \le$ g.l.b. $\{f_n, f_{n+1}, \dots\} \le$ l.u.b. $\{f_n, f_{n+1}, \dots\}$ $f_N + \frac{\epsilon}{2}$.

$\Rightarrow$ l.u.b. $\{f_n, f_{n+1}, \dots\}$ $-$ g.l.b. $\{f_n, f_{n+1}, \dots\} \le \epsilon$

$\Rightarrow$ l.u.b. $\{f_n, f_{n+1}, \dots\}$ $\le$ g.l.b. $\{f_n, f_{n+1}, \dots\} + \epsilon$;

We obtain $\limsup\{f_k\} \le \liminf\{f_k\} + \epsilon$. Since ϵ was arbitrary. This establish (2.1). □

3. Definitions:

Definition 3.1 Let f be a bounded function on closed and bounded interval [a, b]. Let $\sigma = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$ be the subdivision of [a, b].

We define $U[f, \sigma]$ called upper sum for f corresponding σ as $U[f, \sigma] = \sum_{k=1}^{n} M[f, I_k]\ |I_k|$ note that M [f, [a ,b]] = l. u. $b_{x \in [a,b]} f(x)$.

and $L[f, \sigma]$ called a lower sum for f corresponding σ as $L[f, \sigma] = \sum_{k=1}^{n} m[f, I_k]\ |I_k|$, note that m [f, [a ,b]] = g. l. $b_{x \in [a,b]} f(x)$.

The $I_1, I_2, I_3, \dots, I_k$ are the component intervals of σ and $|I_k|$ is length of I_k.

Definition 3.2 A function f be bounded on closed bounded interval [a, b] is said to be Riemann integrable on [a, b], if $\int_{\underline{a}}^{b} f(x)\,dx =$ l. u. b. $L[f, \sigma]$ and $\int_{a}^{\overline{b}} f(x)\,dx =$ g. l. b. $U[f, \sigma]$ are equal.

That means $\int_{\underline{a}}^{b} f(x)\,dx =$ l. u. b. $L[f, \sigma] =$ g. l. b. $U[f, \sigma] = \int_{a}^{\overline{b}} f(x)\,dx = \int_a^b f(x)dx$.

We write as $f \in \mathbb{R}[a, b]$.

Definition 3.3 A sequence of function $\{f_k\}$ is said to be uniformly statistically Convergent to f on a set M, if for every $\epsilon > 0$; $\lim_{n\to\infty}\frac{1}{n}|\{k \le n : |f_k(x) - f(x)| \ge \epsilon, for\ every\ x \in M\}| = 0$,

i.e. for every $x \in$ M, $|f_k(x) - f(x)| < \epsilon$, a. a. k ;

In this case we write: st- $\lim f_k(x) = f(x)$ uniformly on M or $f_k(x) \overset{st}{\to} f(x)$ uniformly on M.

Furthermore it is clear that uniformly statistical converges implies statistical convergence with the same limit point on the set M. But converse is not true. □

Theorem 3.1 Let $\{f_k\}$ be a sequence of real valued functions defined on a metric space M which is uniformly statistical convergent to function f on M. If each f_k ($k \in I$) is continuous at a ϵ M . Then f is also continuous at a.

Proof: Let $\{f_k\}$ be uniformly statistical convergent to function f on M. By definition (3.3) for every $\epsilon > 0$

$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : |f_k(x) - f(x)| \ge \frac{\epsilon}{3}, for\ every\ x \in M\right\}\right| = 0$

i.e. $|f_k(x) - f(x)| < \frac{\epsilon}{3}$ a.a.k.

Since each f_N ($N \in I$) is continuous at a then there exists $\delta > 0$ such that

$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : |f_N(x) - f_N(a)| \ge \frac{\epsilon}{3}, \rho\,|x, a| < \delta, for\ every\ x, a \in M\right\}\right| = 0,$

where ρ is metric for M.

We have $|f(x)-f(a)| \leq |f(x)-f_N(x)|+|f_N(x)-f_N(a)|+|f_N(a)-f(a)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$

Thus $\lim_{n\to\infty}\frac{1}{n}|\{|f(x)-f(a)| \geq \epsilon,\ \rho\,|x,a| < \delta, for\ every\ x, a\ \epsilon\ M\}| = 0$, hence f is continuous at a. □

Corollary 3.1: A function f is continuous at almost every point in [a; b] then f is ℝ −. integrable.

Corollary 3.2: If $\{f_k\}$ is sequence of continuous real-valued functions on metric space M that uniformly statistically converges to f on M then f is also continuous on M.

Theorem 3.2 If $\{f_k\}$ is Sequence of functions in ℝ [a; b] and if uniformly statistically converges to f on [a; b] then f is also in ℝ [a; b].

Proof: For $\epsilon = 1$, there exists $N \in I$ such that $\lim_{n\to\infty}\frac{1}{n}|\{k \leq n : |f_k(x)-f(x)| \geq 1, k \geq N;\ x\ \epsilon\ [a,b]\}| = 0$

$\Rightarrow \lim_{n\to\infty}\frac{1}{n}|\{k \leq n : |f_N(x)-f(x)| \geq 1,\ x\ \epsilon\ [a,b]\}| = 0$

Now $|f(x)| \leq |f_N(x)|+|f(x)-f_N(x)| < |f_N(x)|+1,$ for all $x\ \epsilon\ [a,b]$ |

Each $f_N(x)$ is bounded and $f_N\ \epsilon$ ℝ [a; b]

Clearly f is also bounded and continuous on [a; b]

∴ f is also in ℝ [a; b]. □

If $\{f_k\}$ is sequence of functions which is ℝ [a; b] and statistically convergent to a function f on [a, b], if $f\ \epsilon$ ℝ [a; b]. Then it is true that $\left\{\int_a^b f_k\right\}$ statistically converges to $\left\{\int_a^b f\right\}$.

Other words, $\lim_{k\to\infty} f_k(x) = f(x)$

That means $\lim_{k\to\infty}\frac{1}{n}|\{k \leq n : |f_k(x)-f(x)| \geq \epsilon,\ x\ \epsilon\ [a,b]\}| = 0$

Is $\lim_{k\to\infty}\int_a^b f_k(x)dx = \int_a^b f(x)dx$?

This is equivalent to asking if $\lim_{k\to\infty}\int_a^b f_k(x)dx = \int_a^b \lim_{k\to\infty} f_k(x)dx$ (3.1)

Means is it permissible to interchange limit and integration.

For example, let,

$f_k(x) = 2k\ ; \frac{1}{k} \leq \frac{2}{k}$

$= 0$; for all other $x\ \epsilon\ [0,1]$

Now $\int_0^1 f_k(x)\,dx = \int_{\frac{1}{k}}^{\frac{2}{k}} 2\text{k dx} = 2\text{k}\left(\frac{2}{\text{k}}-\frac{1}{\text{k}}\right) = 2 = \lim_{k\to\infty}\int_0^1 f_k(x)\,dx$

But $\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} 2k = 0$, for all other $x\ \epsilon\ [0,1]$ and $f_k(0) = 0;\ k\ \epsilon\ I$

Also for $x > 0$, $f_N(x) = f_{N+1}(x) = \cdots = 0$ if $\frac{2}{N} < x$

∴ $\int_0^1 \lim_{k\to\infty} f_k(x)\,dx = 0$. Hence equation (3.1) does not holds for given sequence $\{f_k\}$. □

Theorem 3.3 Let $\{f_k\}$ is Sequence of functions in ℝ [a; b] which is uniformly statistically converges to f on [a; b] then $f\ \epsilon$ ℝ [a; b] and $\lim_{k\to\infty}\int_a^b f_k(x)dx = \int_a^b f(x)dx$

Proof: By theorem (3.2) $f\ \epsilon$ ℝ [a; b]. By definition (3:3) for given $\epsilon \geq 0$ there exists $N\ \epsilon\ I$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n : |f_k(x)-f(x)| \geq \frac{\epsilon}{b-a}, k \geq N;\ x\ \epsilon\ [a,b]\right\}\right| = 0 \tag{3.3.1}$$

Now $\left|\int_a^b f_k(x)\ dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b [f_k(x) - \int_a^b f(x)]\,dx\right| \leq \int_a^b |f_k(x)-f(x)|\ dx$

Hence by equation (3.3.1) we obtain

$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n : \left|\int_a^b f_k(x)\,dx - \int_a^b f(x)\,dx\right| \geq \int_a^b \frac{\epsilon}{\text{b}-\text{a}}\,dx = \epsilon\ , k \geq N;\ x\ \epsilon\ [\text{a},\text{b}]\right\}\right| = 0$

That means $\left\{\int_a^b f_k(x)\,dx\right\}$ statistically converges to $\int_a^b f(x)\,dx$.

That means $\lim_{n\to\infty}\int_a^b f_k(x)\,dx = \int_a^b f(x)\,dx$. □

If a sequence $\{f_k(x)\}$ uniformly statistically convergent to f and f'_k , f' exists for all $x\ \epsilon$ [a, b] , it may happened that $\{f'_k\}$ does not statistical converges to f' at some x.

For example, If $f_k(x) = \frac{x^k}{k}$, $0 \le x \le 1$ then $\{f'_k\}$ uniformly statistically converges to $f = 0$.

But $\{f'_k(1)\}$ does not statistically converges to $f'(1)$.

Thus $\lim_{k\to\infty} f'_k(x) = (\lim_{k\to\infty} f_k)'(x)$ does not holds for $x = 1$.

Theorem 3.4 If $f'_k(x)$ exists for each $x \in [a,b]$, for each $k \in I$. If f'_k is continuous on [a, b]. If $\{f_k\}_{k=1}^{\infty}$ statistically converges on [a; b] to f, and if $\{f'_k\}$ uniformly statistically converges on [a; b] to g then $g(x) = f'(x)$; $x \in [a,b]$. i.e. $\lim_{k\to\infty} f'_k(x) = f'(x)$, $x \in [a,b]$.

Proof: Since $\{f'_k\}$ uniformly statistically converges to g on [a; b]. By Corollary-2 sequence $\{f'_k(x)\}$ uniformly statistically converges to g on [a; y], where $y \in [a,b]$

by theorem (3.3) $\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \left|\int_a^y f'_k(x)\,dx - \int_a^y g(x)\,dx\right| \ge \epsilon,\ y \in [a,b]\right\}\right| = 0$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \lim_{k\to\infty}\left|[f_k(y) - f_k(a)] - \int_a^y g(x)\,dx\right| \ge \epsilon,\ y \in [a,b]\right\}\right| = 0 \quad (3.2)$$

But by hypothesis $\lim_{k\to\infty} f_k(y) = f(y)$ and $\lim_{k\to\infty} f_k(a) = f(a)$

∴ equation (3.2) becomes as

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \lim_{k\to\infty}\left|[f(y) - f(a)] - \int_a^y g(x)\,dx\right| \ge \epsilon,\ y \in [a,b]\right\}\right| = 0$$

By fundamental theorem of calculus $\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |[f'(y) - g(y)]| \ge \epsilon,\ y \in [a,b]\}| = 0$

$\Rightarrow f'(y) = g(y),\ y \in [a,b]$

$\Rightarrow f'(x) = g(x),\ x \in [a,b]$

$\Rightarrow \lim_{k\to\infty} f'_k(x) = f'(x),\ x \in [a,b]$. □

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