MOCK THETA FUNCTIONS OF THE THIRD ORDER

Praveen Kumar[1], Dr. Ramanand Raman[2] and Dr. Ranjana[3]

[1]Research Scholar

University Department of Mathematics

T.M. Bhagalpur University, Bhagalpur

[2]Guest Faculty

Department of Mathematics

Marwari College, Bhagalpur

T.M. Bhagalpur University, Bhagalpur

[3]Professor and Head

University Department of Mathematics

T.M. Bhagalpur University, Bhagalpur

ABSTRACT

In this paper Ramanujan introduced the notion of a mock theta function, and he offered some alleged examples. we remark that other transformations listed above may be used to derive some new transformations for three of the third order mock theta functions of Ramanujan and one of the third order mock theta functions. Recent work has elucidated the theory encompassing these examples. Here we prove that Ramanujan's examples do indeed satisfy his original definition.

***Keywords* :** *Mock Theta Function, Hypergeometric, Radially etc.*

Introduction

The third order mock theta functions stated by Ramanujan are the following basic hypergeometric series:

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q,-q;q)_n},\qquad \phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n},$$

$$\chi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n},\qquad \psi(q)=\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}.$$

All of the third order mock theta functions of Ramanujan, as well as those stated later by Watson [1] and Gordon and McIntosh [2], may be expressed in terms of the function $g(x, q)$, where

$$g(x,q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(x,q/x;q)_{n+1}} = x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(q/x;q)_n}\right). \qquad (1.1)$$

This was shown by Hickerson and Mortenson [3] this function was also defined by Gordon and McIntosh [4], where it was labelled "$g_3(x, q)$"). For completeness, we consider a generalization, namely the series

$$G_3(s,t,q) := 1+\sum_{n=1}^{\infty}\frac{s^n t^n q^{n^2}}{(sq,tq;q)_n}, \qquad (1.2)$$

which was defined and state a number of transformation formulae for this function. Note that the connection with the third order mock theta functions is that

$$G_3(x,q/x,q) = (1-x)(1-q/x)g(x,q). \qquad (1.3)$$

Proposition 1.1. *Let G_3 (s,t,q) be as defined at (1.2) above. Then*

$$G_3(s,t,q) = -\sum_{r=1}^{\infty}(s^{-1},t^{-1};q)_r q^r + \frac{(q/s,q/t;q)_\infty}{(sq,tq;q)_\infty}\sum_{r=-\infty}^{\infty}(s,t;q)_r q^r; \qquad (1.4)$$

$$= \quad -\sum_{r=1}^{\infty}(s^{-1},t^{-1};q)_r q^r + \frac{(q/t;q)_\infty}{(sq,q;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(t;q)_r(-s)^r q^{r(r+1)/2}}{(tq;q)r}; \qquad (1.5)$$

$$= \quad -\sum_{r=1}^{\infty}(s^{-1},t^{-1};q)_r q^r + \frac{(q/s,q/t;q)_\infty}{(stq,q/(st),q;q)_\infty}\sum_{r-\infty}^{\infty}\frac{(1-stq^{2r})(s,t;q)_r(st)^{2r}q^{2r^2}}{(1-st)(sq,tq;q)_r} \qquad (1.6)$$

Proof.

The transformations at (1.4) and (1.5) will follow as special cases of two more general identities. Replace z with zq/ac, let $a, c \to \infty$ and set $b = sq$ and $d = tq$ in, respectively, to get that

$$\sum_{n=-\infty}^{\infty}\frac{z^n q^n}{(sq,tq;q)_n} = \frac{(sq/z,tq/z;q)_\infty}{(sq,tq;q)_\infty}\sum_{r=-\infty}^{\infty}(z/s,z/t;q)_r\left(\frac{stq}{z}\right)^r, \qquad (1.7)$$

$$= \quad \frac{(qs/z;q)_\infty}{(sq,tq;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(z/s;q)_r(-s)^{r(r+1)/2}}{(tq;q)_r} \qquad (1.8)$$

Lastly, replace z with st, and use on the terms of negative index in the new series on the left sides.

We also prove a generalization of the transformation at (1.6) first, by letting e, $f \rightarrow \infty$, and then replacing a with z, c with z/s and d with z/t, to get

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(sq,tq;q)_n} = \frac{(sq/z,tq/z;q)_\infty}{(zq,q/z,stq/z;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-zq^{2r})(z/s,z/t;q)_r (zst)^r q^{2r^2}}{(1-z)(sq,tq;q)_r} \quad (1.9)$$

The identity at (1.14) follows after replacing z with st.

The identities (1.4) - (1.6) may be more concisely expressed using the function

$$G_3^*(s,t,q) := \sum_{n=-\infty}^{\infty} \frac{s^n t^n q^{n^2}}{(sq,tq;q)_n} \quad (1.10)$$

as follows :

$$G_3^*(s,t,q) = \frac{(q/s,q/t;q)_\infty}{(sq,tq;q)_\infty} G_3^*(s^{-1},t^{-1},q), \quad (1.11)$$

$$= \frac{(q/t;q)_\infty}{(sq,q;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(t:q)_r (-s)^r q^{r(r+1)/2}}{(tq;q)_r}; \quad (1.12)$$

$$= \frac{(q/s,q/t;q)_\infty}{(stq,q/(st),q;q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-stq^{2r})(s,t;q)_r (st)^{2r} q^{2r^2 r}}{(1-st)(sq,tq;q)_r}. \quad (1.13)$$

The identity at (1.4) was also proved by Choi [5], and stated previously by Ramanujan.

We will employ (1.5) to derive some results on explicit radial limits, as mentioned earlier. Before coming to that, we remark that other transformations listed above may be used to derive some new transformations for three of the third order mock theta functions of Ramanujan and one of the third order mock theta functions of Watson similar results may be derived for the other third order mock theta functions of Watson [6] and those of Gordon and McIntosh [2]. Before stating the next theorem, we recall Watson's [6] third order mock theta function $v(q)$,

where

$$v(q)=\sum_{r=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

Theorem 1.2 If $|q|<1, then$

$$f(q)=-\sum_{n=1}^{\infty}(-1,-1;q)_n q^n+4\frac{(-q;q)_\infty^2}{(q;q)_\infty^3}\sum_{r=-\infty}^{\infty}\frac{q^{2r^2+r}(4rq^r+1)}{(1+q^r)^2}. \qquad (1.14)$$

$$\phi(q)=-\sum_{n=1}^{\infty}(-1,;q^2)_n q^n+4\frac{(-q^2;q^2)_\infty}{(q;q)_\infty^3}\sum_{r=-\infty}^{\infty}\frac{q^{2r^2+2r}(2rq^{2r}+1)}{(1+q^{2r})^2}. \qquad (1.15)$$

$$v(q)=-\sum_{n=0}^{\infty}(-q;q^2)_n q^n+4\frac{(-q;q^2)_\infty}{(q;q)_\infty^3}\sum_{r=-\infty}^{\infty}\frac{q^{2r^2+2r}(r+1)}{(1+q^{2r+1})^2}-2\frac{(-q;q^2)_\infty}{(q;q)_\infty^3}(-q^4,-q^{12},q^{16};q^{16})_\infty \qquad (1.16)$$

$$\psi(q)=-\sum_{n=0}^{\infty}(q;q^2)_n+\frac{1}{2(q^2;q^2)_\infty^2}\sum_{r=-\infty}^{\infty}q^{2r^2+r}(4r+1)(-1)^r \qquad (1.17)$$

Proof.

For (1.14), replace z with z^2 , s and t with $-z$ in (1.9), and then let $z \to 1$.

A similar application of (1.9), again with z replaced with z^2 , s replaced with iz and t replaced with $-iz$ and once again letting $z \to 1$ leads to (1.15).

For (1.16), in (1.9) again replace z with z^2, and then replaces with iz, t with $-iz$, let $z \to \sqrt{q}$ and divide through by $1+q$.

Finally, the transformation at (1.25) follows similarly from (1.9), this time with z replaced with z^2, s replaced with $z/\sqrt{q}$ and t replaced with $-z/\sqrt{q}$ and once again letting $z \to 1$. Note that convergence of the first series on the right of (1.17) is in the Cesaro sense.

As Watson pointed out certain bilateral series related to fifth order mock theta functions, which are essentially the sums of pairs of fifth order mock theta functions, are expressible as theta functions, or combinations of infinite q-products. It seems less well known that the bilateral series associated with two of Ramanujan's third order mock theta functions are also expressible as infinite products. We also give similar statement for Watson's [6] third order mock theta function $v(q)$.

Theorem 1.3. If $|q|<1, then$

$$\phi(q)+\sum_{r=1}^{\infty}(-1;q^2)_r q^r = \sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}=\frac{(-q,-q,q^2;q^2)_\infty}{(q,-q^2;q^2)_\infty} \tag{1.18}$$

$$v(q)+\sum_{r=0}^{\infty}(-q;q^2)_r q^r = \sum_{r=-\infty}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}=2(-q^2,-q^2;q^2)_\infty(q^4;q^4)_\infty; \tag{1.19}$$

$$\Psi(q)+\sum_{r=0}^{\infty}(q;q^2)_r(-1)^r = \sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}=\frac{(-q,-q,q^2;q^2)_\infty}{2(q,-q^2;q^2)_\infty}. \tag{1.20}$$

Proof. From (1.8) (replace q with q^2 , set $b = -z/t$, $a = -z$, and let $c \to \infty$),

$$\sum_{r=-\infty}^{\infty}\frac{(-z/t;q)_r t^r q^{r(r+1)/2}}{(tq;q)_r}=\frac{(-t^2q^2/z-zq,-q/z,q^2;q^2)_\infty}{(tq,-tq/zq)_\infty}.$$

and from (1.8) (with $s = -t$),

$$\sum_{n=-\infty}^{\infty}\frac{z^n q^{n^2}}{(t^2q^2;q^2)_n}=\frac{(-tq/z;q)_\infty}{(-tq,-t^2q/z;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(-z/t;q)_r(t)^r q^{r(r+1)/2}}{(tq;q)_r}$$

Together, these equations imply that

$$\sum_{n=-\infty}^{\infty}\frac{z^n q^{n^2}}{(t^2q^2;q^2)_n}=\frac{(-zq,-q/z,q^2;q^2)_\infty}{(t^2q^2,-t^2q/z;q^2)_\infty}. \tag{1.21}$$

The identity at (1.18) is now immediate upon setting $z = 1$ and $t^2 = -1$, and that at (1.20) results similarly upon setting $z = 1$ and $t^2 = 1/q$. The identity at (1.19) follows :

Mock Theta Function Identities Deriving from Bilateral Basic Hypergeometric Series upon setting $z = q$, $t^2 = -q$, multiplying the resulting product by $1/(1 + q)$, and finally performing some elementary q-product manipulations.

Note that the convergence of the sum added to $\psi(q)$ on the left side of (1.20) is in the Cesaro sense. Note also that comparison of the infinite products on the right sides of (1.26) and (1.28) yields the rather curious identity.

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}=2\sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}, \tag{1.22}$$

where, by the previous comment, convergence of the part of the bilateral series on the right consisting of terms of negative index is again in the Cesaro sense. The summation formulae in the preceding theorem have some interesting implications. Firstly, they allow condition (0) above to be made explicit for some of the third order mock theta functions. We recall the recent result for $f(q)$ in [7].

Theorem 1.4. (Folsom, Ono and Rhoades [7]) If ζ is a primitive even-order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\ldots(1+\zeta^n)^2\zeta^{n+1}. \tag{1.22}$$

Here

$$b(q)=\frac{(q;q)_\infty}{(-q;q)_\infty^2}. \tag{1.23}$$

The infinite product representation of $b(q)$ was not stated in [7], but was stated by Rhoades in [7]. Note that Theorem 1.3 was also proved recently by Zudilin [8].

The following results are immediate upon rearranging the identities in Theorem 1.2, and letting q tend radially to the specified root of unity from within the unit circle, since the other series accompanying each of the mock theta functions in the bilateral sums terminates (the interchange of summation and limit in each of the corresponding series on the right is justified by the absolute convergence of each of these series).

Corollary 1.5.

(i) If ζ is a primitive even-order $4k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\phi(q)-\frac{(q^2,-q,-q;q^2)_\infty}{(-q^2,q;q^2)_\infty}\right)=-2\sum_{n=0}^{\kappa-1}(1+\zeta^2)(1+\zeta^4)\ldots(1+\zeta^{2n})\zeta^{n+1} \tag{1.24}$$

(ii) If ζ is a primitive even-order $4k + 2$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}(v(q)-2(-q^2;q^2)_\infty^2(q^4;q^4)_\infty = -\sum_{n=0}^{k}(1+\zeta)(1+\zeta^3)\ldots(1+\zeta^{2n-1})\zeta^n. \tag{1.25}$$

(iii) If ζ is a primitive odd-order $2k + 1$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\left(\Psi(q)-\frac{(q^2,-q,-q;q^2)}{2(-q^2,q;q^2)_\infty}\right)=-\sum_{n=0}^{k}(1-\zeta)(1-\zeta^3)\ldots(1-\zeta^{2n-1})(-1)^n. \qquad (1.26)$$

Remark: The results in Corollary 1.4 were also proved in [9, 10], using somewhat similar arguments, as were the results in Corollary 1.5 below. The second implication is that they imply some summation formulae for some of the bilateral series appearing in Theorem 1.2.

Corollary 1.6. If |q| < 1, then

$$\sum_{r=-\infty}^{\infty}\frac{q^{r^2+r}(2rq^r+1)}{(1+q^r)^2}=\frac{(q;q^2)_\infty^4(q;q)_\infty^4}{4} \qquad (1.27)$$

$$\sum_{r=-\infty}^{\infty}\frac{q^{2r^2+2r}(r+1)}{(1+q^{2r+1})^2}=\frac{(-q^2;q^2)_\infty^3(q^4;q^4)_\infty}{2(-q;q^2)_\infty}+\frac{(-q^4,-q^{12},q^{16};q^{16})_\infty}{2} \qquad (1.28)$$

Proof. The first identity (1.27) follows from combining the results at (1.15) and (1.18) and then replacing q^2 with q. The identity at (1.28) follows directly from comparing the identities (1.16) and (1.19).

References :

1. Watson, G. N. The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936), 55–80.
2. Gordon, B.; McIntosh, R. J. A survey of classical mock theta functions. Partitions, q-series, and modular forms, 95–144, Dev. Math., 23 (2012), Springer, New York.
3. Hickerson, D. R.; Mortenson, E. T. Hecke-type double sums, Appell-Lerch sums, and mock theta functions, I. Proc. Lond. Math. Soc. (3) 109 (2014), no. 2, 382–422.
4. Gordon, B.; McIntosh, R. J. Some eighth order mock theta functions. J. London Math. Soc. (2) 62 (2000), no. 2, 321-335.
5. Choi, Y-S. The basic bilateral hypergeometric series and the mock theta functions. Ramanujan J. 24 (2011), no. 3, 345–386.
6. Watson, G. N. The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936), 55–80.
7. Folsom, A.; Ono, K.; Rhoades, R.C. Mock theta functions and quantum modular forms. Forum Math. Pi 1 (2013), e2, 27 pp.

8. Zudilin, W. On three theorems of Folsom, Ono and Rhoades. Proc. Amer. Math. Soc. 143 (2015), no. 4, 1471–1476.

9. Bajpai, J.; Kimport, S.; Liang, J.; Ma, D.; Ricci J. Bilateral series and Ramanujan's radial limits. Proc. Amer. Math. Soc. 143 (2015), 479–492.

10. Mortenson, E. T. Ramanujans radial limits and mixed mock modular bilateral q - hypergeometric series. Proceedings of the Edinburgh Mathematical Society, to appear.

****