HYPERGEOMETRIC SOLUTION OF TEMPERATURE DISTRIBUTIONPROBLEM WITH VARIABLE CONDUCTIVITY

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Abstract : In this paper, we consider steady state temperature distribution of the skin and subcutaneous tissue region in cylindrical layers of extremities, incorporating the variation of thermal conductivity with depth in the dermis. Also the metabolic heat generation is taken to be inversely proportional to radius vector there. In this section dependence of blood mass flow rate on age has been taken care by means of the equilibration constant defined by A in dermis. It is assumed that the temperature distribution is uniform along θ and z directions and varies along radial direction. An analytic method has been adopted to solve the resulting differential equations coupled with various boundary conditions. The solution in terms of hyper geometric functions have been obtained in this case.

***Index Terms* - -Heat Transfer; Mathematical Model, thermal conductivity, heat generation, finite element method.**

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I. INTRODUCTION

know that every living body is endowed with the mechanism of thermo-regulation by which body core temperature is maintained within a narrow range, imperative for its survival. The core temperature of human body despite various thermal changes in environment is maintained primarily with the twin phenomena of heat generation and heat transfer. The skin being the surface of interaction between the environment and the body core plays an important role in thermo-regulation. Other processes contributing significantly towards thermo-regulation include perfusion of blood from body core to the peripheral tissues and the tissue metabolic heat generation. Several studies have been conducted to investigate the effect of these processes on temperature distribution in skin and subcutaneous tissues. Cooper and Trezek obtained solution of equation in SST region by taking all parameters as constant[1,2]. Patterson made experimental attempts, to determine temperature profiles in skin and subcutaneous region[4,5]. Saxena solved equation by similarity transformation in SST region[6]. Saxena and Arya used variational finite element method to solve the problem of steady state temperature distribution in three layered skin and subcutaneous region[7]. Saxena, Arya and Bindra obtained unsteady state temperature distribution in human skin and subcutaneous tissue region, by using the variational finite element method and Laplace transform method [9].

II Mathematical Formulation

The bio heat transfer equation[11] takes the following form in steady state when expressed in cylindrical coordinates assuming symmetry along angular direction θ and axial direction z.

$$\frac{1}{r}\frac{d}{dr}K\left(r\frac{dT}{dr}\right)+M(T_b-T)+S=0 \quad \text{.........(1)}$$

Where,

T = tissue temperature

T_b = temperature of blood assume to be same as body core temperature

M = $m_b c_b$

m_b = blood mass perfusion rate

C_b = specific heat of blood

S = metabolic heat generation

K = Thermal conductivity of tissue

If the dependence of blood mass flow rate on age is incorporated in this equation a factor (1-A) is to be introduced in the term corresponding to mass blood flow rate. A is known as equilibrium constant. Its value varies between zero and one depending on age. It is to be noted that blood mass flow rate decreases with age due to constriction of blood vessels. Hence equation (1) takes the form

$$\frac{1}{r}\frac{d}{dr}\left(Kr\frac{dT}{dr}\right) + (1-A)\,M\,(T_b - T) + S = 0 \qquad \text{..... (2)}$$

The boundary conditions are as follows:

$$-K\frac{dT}{dr} = h\,(T - T_a) + LE \qquad \text{.... (3)}$$

Where T_a = atmospheric temperature.

at the skin surface.

and $T_3 = T_b$ ..... (4)

at the inner boundary.

Denoting the thickness of epidermis, dermis and sub dermal part by (a_4-a_3), (a_3-a_2) and (a_2-a_1) respectively, the following functions are assumed for the parameters involved in the equation (2) for each layer.

(i) **Epidermis :**($a3 \leq r \leq a_4$)

$K^{(1)} = k_1$, $M^{(1)} = O$, $S^{(1)} = 0$

(ii) **Dermis :**($a_2 \leq r \leq a_3$)

$K^{(2)} = k_2$ $M^{(2)} = m$,

$S^{(2)} = s\,(T_b - T_2)$

(iii)**Sub dermal part :**($a_1 \leq r \leq a_2$)

$K^{(3)} = k_3$, $M^{(3)} = m$, $S^{(3)} = s\,(T_b - T_3)$

Here k_1, k_2, k_3, m and s are assumed to be constant and a_i (i = 1, 2, 3,4) denote the distance from the centre. The interface and boundary conditions are given by

$$-K_1\frac{dT_1}{dr} = h\,(T_1 - T_a) + LE \qquad \text{at } r = a \text{ ... (5)}$$

$$K^{(1)}\frac{dT_1}{dr} = K^{(2)}\frac{dT_2}{dr} \qquad \text{at } r = a \text{ ... (6)}$$

$T_1 = T_2$ at $r = a_3$ ... (7)

$$K^{(2)}\frac{dT_2}{dr} = K^{(3)}\frac{dT_3}{dr} \qquad \text{at } r = a_2 \text{ ... (8)}$$

$T_2 = T_3$ at $r = a_2$ ... (9)

$T_3 = T_b$ at $r = a_1$ ... (10)

On one-dimensional equations from equation (2) to equation (10) by using $T_i = T_b\,(1-V_i)$ for i =1,2,3, these reduce to the following form :

Outer skin: ($r = a_4$)

$$\frac{dV_1}{dr} = -n_1V_1 + n_2 \qquad \text{.... (11)}$$

(i) Epidermis : ($a_3 \leq r \leq a_4$)

$$\frac{d}{dr}\left(r\frac{dV_1}{dr}\right) = 0 \qquad \text{.... (12)}$$

Interface I: ($r=a_3$)

$$K_1\frac{dV_1}{dr} = K_2\frac{dV_2}{dr} \qquad \text{.... (13)}$$

$V_1 = V_2$ .... (14)

(ii) Dermis : $(a_2 \leq r \leq a_3)$

$$r\frac{d^2V_2}{dr^2}+\frac{dV_2}{dr}-b_2V_2r=0 \quad .... (15)$$

Interface II: $(r=a_2)$

$$K_2\frac{dV_2}{dr}=K_3\frac{dV_3}{dr} \quad .... (16)$$

$V_2 = V_3$ .... (17)

(iii) Sub dermal part: $(a_1 \leq r \leq a_2)$

$$r\frac{d^2V_3}{dr^2}+\frac{dV_3}{dr}-b_3V_3r=0 \quad .... (18)$$

and at inner body core

$V_3 = 0$ at $r = a_1$ .... (19)

where,

$T_i = T_b(1-V_i)$; $i = 1,2,3$

$$b_3=\frac{(m+s)}{k_3}, \qquad b_2=\frac{m(1-A)+s}{k_2}$$

$$T_a=T_b(1-V_a), \qquad n_1=\frac{h}{k_1}$$

$$n_2=\frac{h}{k_1}V_a+\frac{LE}{k_1T_b}$$

III Solutions

Differential equation (12) can be solved directly. Equations (15) and (18) can be transformed into the Bessel's differential equation, by using the transformations

Where i=2,3

Hence the solution for heat flow in the skin and underlying tissue layers of extremities are obtained in the following form :

$$V_1 = C_1 \log r + C_2$$

$$V_2 = C_3 I_0(\sqrt{b_2}\, r) + C_4 K_0(\sqrt{b_2}\, r)$$

$$V_3 = C_5 I_0(\sqrt{b_1}\, r) + C_6 K_0(\sqrt{b_1}\, r)$$

Where I_0 and K_0 are modified Bessels functions of first and second kind respectively.

The values of the constants C_i (i=1 to 6) have been determined by using interface and boundary conditions and are given below:

$$C_1=\frac{a_4n_2}{1+n_1a_4l_{21}}, \quad C_2=\left(\frac{a_4n_2}{1+n_1a_4l_{21}}\right)l_{20}$$

$$C_3=\frac{C_1}{a_3l_{19}},$$

$$C_4=C_3\frac{l_{11}}{l_{12}}$$

$$C_5=\frac{C_3l_9+C_4l_{10}}{l_8}, \quad C_6=-C_5\frac{l_1}{l_2}$$

where

$$l_1 = I_0\left(\sqrt{b_1}\, a_1\right), \qquad l_2 = K_0\left(\sqrt{b_1}\, a_1\right)$$

$$l_3 = I_0\left(\sqrt{b_1}\, a_2\right) - \frac{l_1}{l_2} K_0\left(\sqrt{b_1}\, a_2\right)$$

$$l_4 = I_0\left(\sqrt{b_2}\, a_2\right), \qquad l_5 = K_0\left(\sqrt{b_2}\, a_2\right)$$

$$l_6 = \frac{d}{dr} I_0\left(\sqrt{b_1}\, a_2\right), \qquad l_7 = \frac{d}{dr} K_0\left(\sqrt{b_1}\, a_2\right)$$

$$l_8 = l_6 - \frac{l_1}{l_2} l_7, \qquad l_9 = \frac{d}{dr} I_0\left(\sqrt{b_2}\, a_2\right)$$

$$l_{10} = \frac{d}{dr} K_0\left(\sqrt{b_2}\, a_2\right), \quad l_{11} = l_3 l_9 - l_4 l_8$$

$$l_{12} = l_5 l_8 - l_3 l_{10},$$

$$l_{13} = I_0\left(\sqrt{b_2}\, a_3\right)$$

$$l_{14} = K_0\left(\sqrt{b_2}\, a_3\right),$$

$$l_{15} = \log a_3$$

$$l_{16} = \frac{l_{12}\, l_{13} + l_{11}\, l_{14}}{l_{12}}, \quad l_{17} = \frac{d}{dr} I_0\left(\sqrt{b_2}\, a_3\right)$$

$$l_{18} = \frac{d}{dr} K_0\left(\sqrt{b_2}\, a_3\right), \quad l_{19} = \frac{l_{17}\, l_{12} + l_{11}\, l_{18}}{l_{12}}$$

$$l_{20} = \frac{l_{16} - l_{15}\, a_3\, l_{19}}{a_3\, l_{19}},$$

$$l_{21} = \log a_4 + l_{20}$$

IV Numerical Results

The calculations have been performed for three different age groups and two sets of thicknesses of skin. The values of A (equilibration constant) and S (metabolic heat generation rate) have been taken accordingly from table-I .[3,7]

Table -1

Metabolic Heat Generation and Equilibration Constant

AGE (years) →	**20**	**40**	**60**
S=Metabolic heat generation rate(cal /cm^3-min)	0.02173	0.020925	0.020205
A=Equilibration constant	0.1	0.2	0.3

In addition the following values of physical and physiological constant have been considered [8,9]

h = 0.009 cal/cm^2-min

E=0

L=579 cal/gm.

T_a=15^{o}c

m = 0.003 cal/cm^3-mindegC,

T_b=37^{o}c

Depending upon the structure of skin and underlying tissues we have considered two sets of different thicknesses as follows. The coefficient of thermal conductivity K, also assume different values according to thickness.[7,8]

For Set I

K_1=0.030 K_2=0.045 K_3=0.045
a_1=1cm a_2=2cm a_3=3cm a_4=3.5cm

For Set II
K_1=0.030 K_2=0.060 K_3=0.060
a_1=5.5cm a_2=7.0cm a_3=8.5cm a_4=9.5cm

V Conclusion

Temperature variation is very interesting for different sets of the values of skin thickness and age groups. In general, it has been observed that the variation is sharp and nonlinear near the outer surface and becomes almost linear in the sub dermal regions. This linearity is still extended for larger thicknesses of dermis and sub dermal parts. This variation may be due to the different rates of metabolic heat generation and the value of A for different age groups. The rates of metabolic heat generations and blood mass flow are zero in epidermis. we observe that the efficiency of blood mass flow rate at the young age of twenty to withstand the effect of ambient temperature is better than at the age of forty. Surface temperature at the age of forty falls to $27^{o}C$ as compared to $28.5^{o}C$ at the age of twenty.
Another interesting feature of the results is the interface temperature between epidermis and dermis. Apart from depending on the thickness of epidermis, it also depends on the size of the dermis. For the same thickness of regions the value of this temperature is slightly less for the higher ages. The interface temperature between lower sub dermal part and upper sub dermal part is mainly dominated by the body core temperature. In spite of greater thicknesses of layers, interface temperature between epidermis and dermis is slightly higher in them as compared to the temperature in lesser thicknesses of layers. This indicates the fact that blood perfusion can counter the effect of atmospheric conditions more efficaciously if larger thickness is available.

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