Construction of Four Level Second Order Slope Rotatable Designs

B.Guravaiah[1], R.Md.Shareef[2], O.V.Rajasekharam[3], G.V.S.R.Anjaneyulu[4]
Department of Statistics
Acharya Nagarjuna University. Guntur-522510(A.P).

Abstract:

Hader and Park (1978) introduced Slope Rotatable Central Composite Designs (SRCCD). Park and Kim (1992) suggested a measure of slope rotatability for second order response surface designs. Jang and Park (1993) suggested a measure and graphical method for evaluating slope rotatability in response surface designs. All these are SOSRD with five or three levels for each factor. Anjaneyulu et al (2005) constructed some new second order rotatable designs. As SOSRD with four levels for each of the factor are sometimes necessary, there is a need to investigate methods for constructing such designs. In this paper, construction of four level second order slope rotatable designs using μ-resolvable balance incomplete block designs is suggested with an example.

Keywords: Second Order Slope Rotatable Design; Balanced Incomplete Block Design; Response Surface Methodology.

1. Introduction

The study of Box and Wilson (1951) which introduces the notion of composite designs seems to be the fundamental work in RSM. The introduction of the "star portion" to argument of two level factorial array was done to allow for efficient estimation of Quadratic terms in second order model. The ideas of Box and Wilson (1951) were estimated by box (1954) and Box and Wilson (1951) were extended by Box (1954) and Box and Youle (1995). To attain the rotatability it requires that the variance of a predicted value remain constant at points that are equidistant from the design center. RSM has applications in chemical, Physical, engineering industrial and biological fields and the use of RSM has accelerated. Possible areas of application degree polynomials in two or three variables are frequently seen and in computer experiments. Draper and Herzberg (1971) explained lack of fit of response surfaces Myres (1971) discussed various concepts of RSM in detail.

Box and Hunter (1957), Das and Narasimham (1962), Raghavarao (1963), Gupta and Das (1975), Nigam (1977) and several others suggested various methods for construction of second order rotatable design (SORD).

The study of rotatable design mainly emphasized by on the estimation of differences and its precision. Estimation differences in response at two different points in the factor space will often be a great importance. It difference in response at two different points close together, estimation of local slope (rate of change) of the response is of the interest. Estimation of slopes occurs frequently in practical situation. For instance, there are cases in which we want to estimate the rate of reaction in chemical experiment, rate of change in the yield of crop to various fertilizer doses, etc.

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2. Rotatable Designs

Let v factor affect the yield and suppose the yield Y satisfies the functional relation

$$Y = f\left(x_1, x_2, \ldots, x_v\right) + e$$

Where $x_1, x_2, \ldots, x_v$ are the levels of the 'v' factor used for getting that response. We assume that 'f' can be represented adequately in a small region of interest by polynomial of degree d.

The estimates of the coefficients in the polynomial f1 can be obtained by the method of least squares. It was observed by Box and Hunter (1957) that, instead of considering the variances of individual coefficients, the accuracy of the estimated response at a point $(x_1, x_2, \ldots, x_v)$ should provide a criterion of the selection of design.

Definition:- A 'v'- dimensional design of order 'd' is said to be a rotatable design if the variance of the estimated response at the point$(x_1, x_2, \ldots, x_v)$ is function of the square of the distance of the point from a suitable origin, so that variances of all estimated responses at points equidistant from the origin are the same, By using the properties of the spherical distribution Box and Hunter(1957) given conditions for the N-design points to form a second order rotatable design.

3. Conditions for SORD:

A Second order response surface design $D = \left(\left(X_{iu}\right)\right)$ for fitting

$$y_u(x_{1u}, x_{2u}, \ldots, x_{vu}) = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum\sum_{i<j}^{v} b_{ij} x_{iu} x_{ju} + e_u$$

Where x_{iu} denotes level of i^{th} factor (i=1,2,3…..v) in the u_{th} run (u=1,2,3..n) the experiment , e_u's are uncorrelated random errors with mean zero and variance σ^2 is said to be a second order slope rotatable design.

If the variance of the estime of first order partial derivative $\left(\partial y / \partial x_i \right)$ with respect to each of independent variables (x_i) in only a function of distance $d^2 = \left(\Sigma x_i^2 iu\right)$ of point $(x_{1u}, x_{2u}, ..., x_{uv})$ from the origin of the design. Such a spherical variance function for the estimation slopes in the second order surfaces is achieved. If the design points satisfy the following conditions [C.F Hader and Park (1978)]

Simple symmetry conditions: -

A.
$$\sum x_i = 0, \qquad \sum x_i x_j = 0,$$
$$\sum x_i x_j x_k = 0, \qquad \sum x_i x_j x_k x_l = 0,$$
$$\sum x_i^2 x_j = 0, \qquad \sum x_i^3 = 0,$$
$$\sum x_i^3 x_j = 0, \text{ etc., for } i \neq j \neq k \neq l.$$

B. (i) $\sum x_i^2 = \text{constant} = N\lambda_2$.

(ii) $\sum x_i^4 = \text{constant} = CN\lambda_4$.

C. $\sum x_i^2 x^2{}_j = \text{constant} = N\lambda_4$.

D. Non-Singular condition

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(c+v-1)}$$

E. Slope rotatability condition

$$V\left(\hat{b}_{ij}\right) = \tfrac{1}{4} v(\hat{b}_{ij})$$

This implies,

$$\lambda_4\left[v(5-c)-(c-3)^2\right] + \lambda_2^2\left[v(c-5)+4\right] = 0 \qquad \text{….. (1)}$$

μ-Resolvable Balance Incomplete Block Designs: The balance incomplete block design with parameters v,b,r,k,λ and μ said to be μ-resolvable if the blocks(b=mt) can be separate into t –sets of m-blocks each such that each set contains every treatment exactly μ=times **(**r= μ t). Furthermore, a μ-resolvable BIBD said to be affine μ-resolvable if any two blocks belonging to different sets contain q_1 treatments in common. Whereas any two blocks belong two different sets contain q_2 treatments in common.

In this case, it holds that

$$B = v + t - 1,\; q_1 = (\mu-1)k/((m-1) = k+\mu-r),\; q_2 = (\mu-k)/(m = k^2/v)$$

Let N ($=n_{ij}$) be the incident matrix of a μ-resolvable BIBD with parameters (v,b,r,k,λ) such that n_{ij}= α the J[th] treatment occurs i[th] block and n_{ij}= β**.** Otherwise Evidently N is b*(v-1) array of symbols (α, β) then following Das and Narasimham (1962), these will generate b2^p combinations of symbols and (α and β)On multiplying the b rows with combination of 2^p factorial where the levels of 2^p factorial are ± 1 and 2^p is the no of combinations in a resolution V fraction of 2^{v-1}.

The above procedure gives a set of D of N(=b.2^p), (v-1) dimensional design points using the design points we can construct four levels SOSRD in (v-1) factor as given in the following theorem.

Theorem 3.0:

The Above b.2^P design points D will form a four level (v-1) factor SOSRD, if the equation given below admits a positive real solution

$$\left(\alpha^2/\beta^2\right)^4\left[b\{(v-1)\lambda(5\lambda-r)-(r-3\lambda)^2\}+r^2\{(v-1)(r-5\lambda)+4\mu\}\right]$$
$$+(\alpha^2/\beta^2)[b(r-\lambda)\{2(v-1)(10\lambda-r)+12(r-3\lambda)\}+2r(b-r)\{(v-1)(r-5\lambda+4\lambda\}$$
$$+r^2(r-\lambda)(18-10v)]+(\alpha^2/\beta^2)[b\{(v-1)\{(5\lambda-r)(b-2r+\lambda)$$
$$+20(r-\lambda)^2+\lambda(4b-9r+5\lambda)\}-36(r-\lambda)^2-2(r-3\lambda)(5r-2b-3\lambda)\}$$
$$+(b-r)^2\{(v-1)(r-5\lambda)+4\lambda\}+r^2\{(v-1)(9r-4b-5\lambda)$$
$$+4(b-2r+\lambda)\}+2r(b-r)(r-\lambda)(18-10v)]+(\alpha^2/\beta^2)[b((r-\lambda)\{2(v-1)(9b-19r+10\lambda)$$
$$+12(5t-2b-3\lambda)\}+(b-r)^2(r-\lambda)(18-10v)$$
$$+2r(b-r)\{(v-1)(9r-4b-5\lambda)+4(b-2r+\lambda)\}+b[(v-1)(4b-9r-5\lambda)(b-2r+\lambda)$$
$$-(5r-2b-3\lambda)^2]+(b-r)^2[(v-1)(9r-4b-5\lambda)+4(b-2r+\lambda)=0$$

..(2)

Proof: - For the design points generated from the conditions (A), (B) and (C) of (1) are true. Conditions (B) and (C) of (1) are true as follows

2.(i) $\sum x_{iu}^2 = 2^p\left[r\alpha^2+(b-r)\beta^2\right]=N\lambda_2$

(ii) $\sum x_{iu}^4 = 2^p\left[r\alpha^4+(b-r)\beta^4\right]=CN\lambda_4$

3. $\sum x_{iu}^2 y_{iu}^2 = 2^p\left[\lambda\alpha^4+((b-2r+\lambda)\beta^4\right]+2(r-\lambda)\alpha^2\beta^2 = N\lambda_4$ …(3)

From 2(ii) and (C) of (3) we get 'c' given by

$$c=\frac{2^p\left[r\alpha^4+(b-r)\beta^4\right]}{2^p\left[\lambda\alpha^4+((b-2r+\lambda)\beta^4\right]+2(r-\lambda)\alpha^2\beta^2} \qquad ....(4)$$

Substituting for λ_2,λ_4 and C in (E) of (1) we get equation (2), which on simplification leads to fourth degree equation in t, where = $\left(\alpha^2/\beta^2\right)$,However the design need to be satisfy of the non-singularly condition(D) of (1).

Example (1) : - We illustrate the above method with the construction of four level SOSRD for 8-factor with the help of $\mu-$resolvable BIBD (v=9, b=18, r=8, k=4, $\lambda=3$)will give a four level. SOSRD in N=288 design points for 8- factor

Hence above 2 and 3 of (3) are

2. (i) $\sum x_{iu}^2 = 128\alpha^2 + 160\beta^2 = N\lambda_2$

(ii) $\sum x_{iu}^4 = 128\alpha^4 + 160\beta^4 = CN\lambda_4$

3. $\sum x_{iu}^2 y_{iu}^2 = 48\alpha^2 + 80\beta^2 + 160\alpha^2\beta^2 = N\lambda_4$

From 2(ii) and 3 of (4) we get 'c' as

$$c = \frac{8\alpha^4 + 10\beta^4}{3\alpha^4 + 5\beta^4 + 10\alpha^2\beta^2} \quad \text{...(5)}$$

Substituting for λ_2, λ_4 and c in of (E) of (1) and on simplification we get the bi quadratic equation in t (Viz.,)

$190t^4+520t^3-1260t^2+200t-21250 = 0$ .....(6)

Where t= $\left(\alpha^2 / \beta^2\right)$ solving (6) we get t=3.089971

I.e.. $\left(\alpha / \beta\right)$=1.757831

(B) Can be obtained from scaling condition. It can be verified that non-singularity condition (D) of (1) is satisfied.

Hence the consider design in a four level SOSRD

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