k-Isolate Domination Number of Total Graphs

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Abstract:

A dominating set S of a graph G is said to be a k-isolate dominating set if $\langle S \rangle$ has at least k-isolated vertices. In this paper, k-isolate domination number of Total graph of Path, cycle and comb graphs are found.

Keywords: Dominating set, Isolate dominating set, k-isolate dominating set, k-isolate domination number, Total graph, path, cycle and comb graphs.

1. INTRODUCTION

In a graph $G=(V,E)$, the degree of a vertex v in V is the number of edges incident with v and is denoted by $\deg(v)$. A dominating set for a graph G is a subset S of V such that every vertex in $V-S$ is adjacent to atleast one vertex in S. A dominating set S is such that the sub graph $<S>$ induced by S has at least one isolated vertex is called an isolate dominating set. The concept of isolate domination number is first developed by I.Sahul Hamid and S. Balamurugan [1]:

A dominating set S of a graph G is said to be a k-isolate dominating set if $<S>$ has at least k-isolated vertices [5]. The k-isolate dominating set S is said to be a minimal k -isolate dominating set if proper subset of S is not an isolate dominating set.

The concept of the total graph was introduced by Anderson and Badawi[2]. The total graph $T(G)$ of a graph G is the graph whose the vertex set is $V(G)\cup E(G)$ and the two vertices in the vertex set of $T(G)$ are adjacent in $T(G)$ whenever the vertices in $V(G)$ are either adjacent or the vertices in $V(G)$ are incident with the edges in $E(G)$ in G . The structural properties of total graph are investigated in [3],

In this paper we discussed about the k-isolate domination number of total graph of path, cycle and comb graph.

2. Preliminary Results

***Theorem* 2.1[5]:** For the path P_n we have $\gamma_{ki}\left(P_n\right)=\begin{cases}\left\lceil \frac{n}{3}\right\rceil & , k\leq\left\lceil \frac{n}{3}\right\rceil \\ k & , \left\lceil \frac{n}{3}\right\rceil < k \leq \left\lceil \frac{n}{2}\right\rceil \\ \text{Does not exists} & , k>\left\lceil \frac{n}{2}\right\rceil \end{cases}$

Theorem2.2[5]: Let C_n be a cycle with n vertices $\left(n\geq 3\right)$, then

$$\gamma_{ki}\left(C_n\right)=\begin{cases}\left\lceil \frac{n}{3}\right\rceil & , k\leq\left\lceil \frac{n}{3}\right\rceil \\ k & , \left\lfloor \frac{n}{3}\right\rfloor < k < \left\lceil \frac{n+1}{2}\right\rceil \\ \text{does not exists,} & k\geq\left\lceil \frac{n+1}{2}\right\rceil \end{cases}$$

Theorem 2.3[4]: $\gamma_0(T(P_n))=\left\lceil \frac{2n-1}{5}\right\rceil, \textit{ for } n\geq 2.$

Theorem 2.4[4]: $\gamma_0(T(C_n))=\left\lceil \frac{2n}{5}\right\rceil, \textit{ for } n\geq 3.$

3. Main Results

Theorem 3.1: For the total graph of path P_n ,

$$\gamma_{ki}(T(P_n))=\begin{cases}\left\lceil \frac{2n-1}{5}\right\rceil, & \textit{for } k\leq\left\lceil \frac{2n-1}{5}\right\rceil \\ k & , \textit{for } \left\lceil \frac{2n-1}{5}\right\rceil < k \leq \left\lceil \frac{2n-1}{3}\right\rceil \\ \textit{does not exist}, & \textit{for } k>\left\lceil \frac{2n-1}{3}\right\rceil \end{cases}$$

Proof:

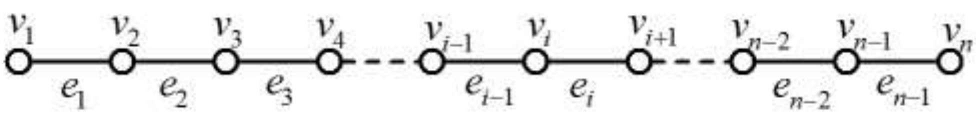

Figure: Path graph

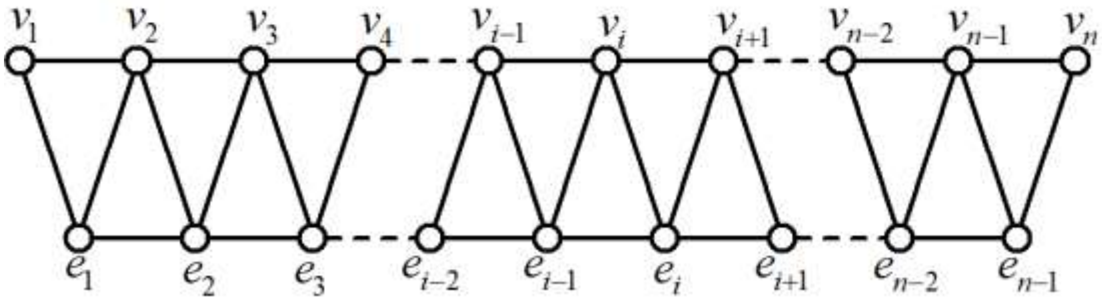

Figure: Total graph of Path

Case (i): $k \le \left\lceil \frac{2n-1}{5} \right\rceil$

Refer Theorem [2.3]

Case(ii): $\left\lceil \frac{2n-1}{5} \right\rceil < k \le \left\lceil \frac{2n-1}{3} \right\rceil$

Let $k = \left\lceil \frac{2n-1}{5} \right\rceil$

From Case (i), the minimal k-isolate dominating set is $\{v_2, e_4, v_7, e_9, \ldots, v_i \ or \ e_j\}$ where $i = n \ or \ n-1$ and $j = n-1$. To obtain the minimal $k+1$-isolate dominating set, replace the vertex v_2 by v_1, v_3. By adding the vertices v_1, v_3, we will get minimal $k+1$-isolate dominating set as $\{v_1, v_3, e_4, v_7, e_9, \ldots, v_i \ or \ e_j\}$. Continue the above process by replacing the vertex v_m by v_{m-1}, v_{m+1} (m=7, 12,…,i) .On continuing the process if $i = n$ we should not replace the vertex v_i as it is the last vertex. Also if the last vertex is e_j, then we should remove that vertex.. Finally we get the minimal $\left(\left\lceil \frac{2n-1}{3} \right\rceil - 1 \right)$- isolate dominating set as $\{v_1, v_3, e_4, v_6, v_8, e_9, \ldots, v_i \ or \ e_j\}$ where $i = n \ or \ n-1$ and $j = n-1$. To get the minimal $\left\lceil \frac{2n-1}{3} \right\rceil$-isolate dominating set, consider the path in the total graph of path $T(P_n)$ as $v_1, e_1, v_2, e_2, v_3, e_3, \ldots, e_{n-1}, v_n$ containing $2n-1$ vertices. Since the vertex v_1 dominates e_1, v_2 and itself, select the vertex v_1 from the first three vertices. Similarly, as e_2 dominates itself and the vertices v_3, e_3, select the vertex e_2. Hence by case (i) of Theorem(2.1) we have minimal $\left\lceil \frac{2n-1}{3} \right\rceil$-isolate dominating set as $\{v_1, e_2, v_4, e_5, \ldots, v_{n-2}, e_{n-1} \ or \ v_{n-2}, e_{n-1}, v_n \ or \ v_{n-1}\}$ according as $n \equiv 0 (\mathrm{mod}\, 3) \ or \ n \equiv 1 (\mathrm{mod}\, 3) \ or \ n \equiv 2 (\mathrm{mod}\, 3)$. Hence $\gamma_{ki}(T(P_n)) = k$ when $\left\lceil \frac{2n-1}{5} \right\rceil < k \le \left\lceil \frac{2n-1}{3} \right\rceil$

Case(iii): $k > \left\lceil \frac{2n-1}{3} \right\rceil$

Let $k = \left\lceil \frac{2n-1}{3} \right\rceil$

Clearly in this total graph of path, there are atmost k-isolated vertices. Hence it does not exists when $k > \left\lceil \frac{2n-1}{3} \right\rceil$

Theorem 3.2: For the total graph of cycle, $\gamma_{ki}(T(C_n)) = \begin{cases} \left\lceil \frac{2n}{5} \right\rceil, \ for\ k \le \left\lfloor \frac{2n}{5} \right\rfloor \\ k \qquad , for\ \left\lfloor \frac{2n}{5} \right\rfloor < k \le \left\lfloor \frac{2n}{3} \right\rfloor \\ does\ not\ exist,\ for\ k > \left\lfloor \frac{2n}{3} \right\rfloor \end{cases}$

Proof: Let the vertices of cycle be $v_1, v_2, v_3, ..., v_n$ and the edges be In total graph $e_1, e_2, e_3, ..., e_n$. In total graph of cycle graph, the number of vertices be $2n$ namely $v_1, v_2, ..., v_n, e_1, e_2, ..., e_n$.

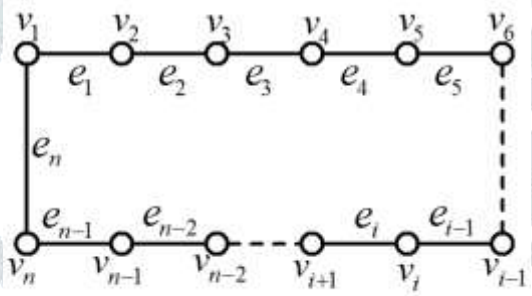

Figure: Cycle graph

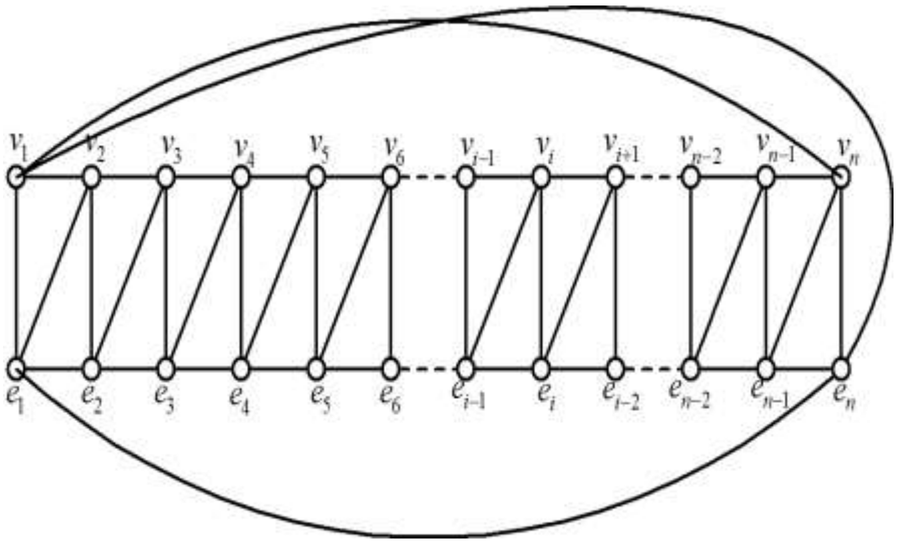

Figure: Total graph of Cycle

Case (i): $k \le \left\lfloor \frac{2n}{5} \right\rfloor$

Refer Theorem[2.4]

Case (ii): $\left\lfloor \frac{2n}{5} \right\rfloor < k \le \left\lfloor \frac{2n}{3} \right\rfloor$

\Let $k = \left\lfloor \frac{2n}{5} \right\rfloor$

By case (i) the minimal k- isolate dominating set is $\{v_2, e_4, v_7, e_9, ..., v_i \ or \ e_j\}$ where $i = n, j = n \ or \ n-1$. To obtain $k+1$-isolate dominating set, fix the vertex v_2 and replace the vertex e_4 by the vertices e_3 and v_5, which satisfies the isolate domination condition and the minimal $k+1$-isolate dominating set is $\{v_2, e_3, v_5, v_7, e_9, ..., v_i \ or \ e_j\}$. Also to obtain $k+2$-isolate dominating set, replace the next vertex e_9 by e_8 and v_{10} and the corresponding set is $\{v_2, e_3, v_5, v_7, e_8, v_{10}, ..., v_i \ or \ e_j\}$. Continuing in this way, by replacing the vertex e_r by e_{r-1} and v_{r+1} $(r = 4, 9, 14, ..., j-1)$, we get the minimal $\left(\left\lfloor \frac{2n}{3} \right\rfloor - 1 \right)$-isolate dominating set as $\{v_2, e_3, v_5, v_7, e_8, v_{10}, ..., e_{r-1}, v_{r+1}, ..., v_i \ or \ e_j\} (r = 4, 9, 14, ... j-1)$ Also in $T(C_n)$ we have a cycle $v_1, e_1, v_2, e_2, v_3, ..., v_n, e_n, v_1$ containing $2n$ vertices.

By case (i) of Theorem[2.2], we have minimal $\left\lfloor \frac{2n}{3} \right\rfloor$- isolate dominating set $\{v_1, e_2, v_4, e_5, v_7, e_8, ..., v_i \ or \ e_j\} (i, j = n-1 \ when \ n \equiv 0, 2 (\bmod 3) \ and \ j = n-2 \ when \ n \equiv 1 (\bmod 3))$ Case(iii): $k > \left\lfloor \frac{2n}{3} \right\rfloor$

Let $k = \left\lfloor \frac{2n}{3} \right\rfloor$

Since each vertex is of degree 4, the minimal k-isolate dominating set does not exist when $k > \left\lfloor \frac{2n}{3} \right\rfloor$.

Comb Graph: Let P_n be a graph with n vertices. Comb is a graph obtained by joining a single pendant edge to each vertex of a path containing $2n$ vertices and $2n-1$ edges. It is denoted by $P_n \odot K_1$.

Theorem 3.3: Let $P_n \odot K_1$ be the comb graph and $T(P_n \odot K_1)$ be the total graph of comb graph. Then

$$\gamma_{ki}[T(P_n \odot K_1)] = \begin{cases} n & ,k \le n \\ k & ,n < k \le n + \left\lceil \dfrac{n-1}{2} \right\rceil \\ does\ not\ exist, & k > n + \left\lceil \dfrac{n-1}{2} \right\rceil \end{cases}$$

Proof:

Let the number of vertices and edges of comb graph $P_n \odot K_1$ be $2n$ and $2n-1$ namely, $u_1,u_2,...,u_n,v_1,v_2,...,v_n$ and $e_1,e_2,...,e_{2n-1}$. The cardinality of the vertices of total graph of comb graph is $n+n+2n-1=4n-1$ vertices namely $u_1,u_2,...,u_n,v_1,v_2,...,v_n,e_1,e_2,...,e_{2n-1}$. Let us denote $v_1,v_2,...,v_n$ as the corresponding vertices of u_i and $d(v_i)=2$, the degree of the end vertices of path namely u_1,u_n in $T(P_n \odot K_1)$ is 4 and the remaining vertices $u_2,u_3,...,u_{n-1}$ be 6. Also $d(e_i)=3(i=1,2n-1)$; $d(e_i)=4(i=3,5,...,2n-3)$ $d(e_2)=d(e_{2n-2})=5$; $d(e_i)=6(i=4,6,...,2n-4)$.

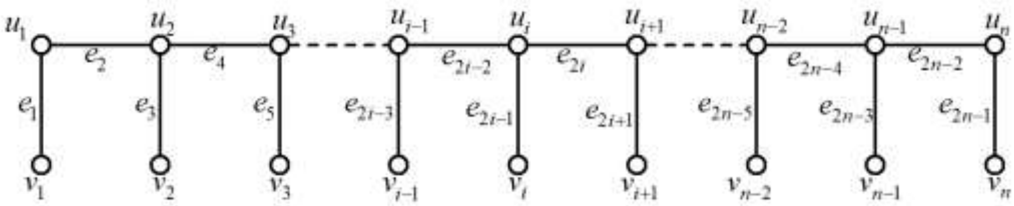

Figure: Comb graph

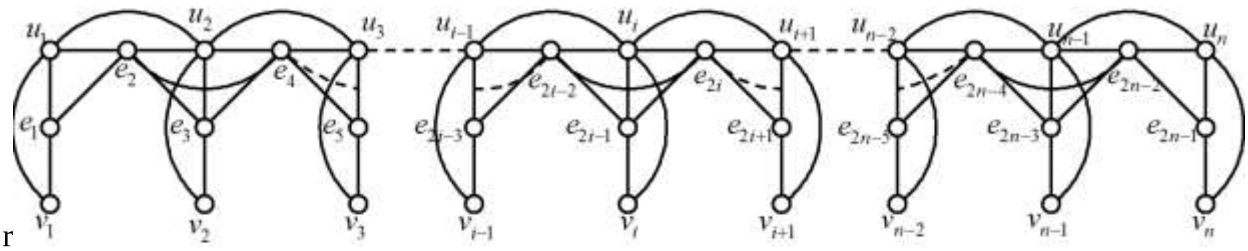

Figure: Total graph of Comb graph

Case(i): $k \le n$

Since the vertices $e_i(i=1,3,5,...,2n-1)$ dominates all the pendant vertices $v_i\ (1 \le i \le n)$, $u_i(1 \le i \le n)$ vertices and the remaining $e_i(i=2,4,6,...,2n-2)$ vertices, the minimal n- isolate dominating set is $\{e_1,e_3,e_5,\ldots,e_{2n-1}\}$ containing n vertices. Hence $\gamma_{ki}[T(P_n \odot K_1)]=n$ when $k \le n$.

Case (ii): $n < k \leq n + \left\lceil \frac{n-1}{2} \right\rceil$

Let $k = n$.

By case (i), the minimal k- isolate dominating set is $\{e_1, e_3, e_5, \ldots, e_{2n-1}\}$. To obtain the $(k+1)$- isolate dominating set, the vertices e_1, e_3 in the above set is superseded by the other three vertices e_2, v_1, v_2. Hence the set $\{e_2, v_1, v_2, e_5, \ldots, e_{2n-1}\}$ satisfies the minimal $(k+1)$- isolate dominating set containing $k+1$ vertices. In this way at each step exclude two vertices e_{i-1}, e_{i+1} $(i = 2, 6, 10, \ldots, 2n-4, if\ n\ is\ odd)$ or $(i = 2, 6, 10, \ldots, 2n-4, if\ n\ is\ even)$ and include their corresponding vertices v_{i-1}, v_{i+1} and a vertex e_i. Eventually, if n is odd, there is no pair for the vertex e_{2n-1}, then displace e_{2n-1} by its corresponding vertex v_n. Hence $\gamma_{ki}[T(P_n \square K_1)] = k$ when $n < k \leq n + \left\lceil \frac{n-1}{2} \right\rceil$.

Case (iii): $k > n + \left\lceil \frac{n-1}{2} \right\rceil$

Let $k = n + \left\lceil \frac{n-1}{2} \right\rceil$

By case (ii), the minimal k- isolate dominating set is $\{e_2, v_1, v_2, e_6, v_3, v_4, e_{10}, v_5, v_6, \ldots, v_n\}$. If we consider $k+1$ vertices, the minimal $(k+1)$- isolate dominating set does not exist. Hence minimal k- isolate dominating set does not exist when $k > n + \left\lceil \frac{n-1}{2} \right\rceil$

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