πgη-CLOSED SETS AND SOME RELATED TOPICS

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Abstract: In this paper, we introduce a new class of sets called πgη-closed sets in topological spaces and investigate the relationship with other existing generalized closed sets. Moreover, we also introduce and studied the concepts of πgη-continuous and πgη-irresolute functions via πgη-closed sets which are defined by me. We obtain some properties about πgη-closed sets, πgη-continuous and πgη-irresolute functions, and πgη-compactness.

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1. Introduction

Many investigations related to generalized closed sets and generalized continuous functions have been published in various forms of closed sets and continuous functions. In 1937, Stone [**15**] introduced the notion of regular open sets. In 1965, Njastad [**13**] introduced the concept of α-open sets. In 1968, the notion of π-open sets were introduced by Zaitsev [**18**] which are weaker form of regular open sets in topological spaces. In 1969, Singal and Mathur [**14**] introduced and studied the concept of nearly compact spaces. In 1970, Levine [**10**] initiated the study of so called generalized closed (briefly g-closed) sets. In 1993, Maki et al. [**12**] introduced the concept of α-T_0 and α-T_0 spaces. In 1994, Maki et al. [**11**] introduced the notion of αg-closed sets. In 2000, Dontchev and Noiri [**5**] introduced the notion of πg-closed sets. In 2007, Arockiarani and Janaki [**1**] introduced the notion of πgα-closed sets in topological spaces. In 2009, Janaki [**8**] Studied πgα-closed sets in topology. In 2019, Subbulakshmi, Sumathi, Indirani [**16**] introduced and investigated the notion of η-open sets. Recently, Kumar and Sharma [**9**] introduced and investigated the notion of η-T_k (k = 0, 1, 2) and η-R_k (k = 0, 12) axioms in topological spaces.

2. Preliminaries

Throughout this paper, spaces (X, ℑ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by Cl(A) and Int(A) respectively. A subset A is said to be **regular open** [15] (resp. **regular closed [15]**) if A = Int(Cl(A)) (resp. A = Cl(Int(A)). The finite union of regular open sets is said to be **π-open [18]**. The complement of a π-open set is said to be **π-closed [18]**.

Definition 2.1. A subset A of a topological space (X, ℑ) is said to be
(i) **α-open** [**13**] if A ⊂ Int(Cl(Int(A))).
(ii) **η-open** [**16**] if A ⊂ In(Cl(Int(A))) ∪ Cl(Int(A)).
(iii) **η-closed** [**16**] if A ⊃ Cl(Int(Cl(A))) ∪ Int(Cl(A)).

The complement of a **α-open** set is called **α-closed**. The intersection of all α-closed (resp. η-closed) sets containing A, is called **α-closure** (resp. **η-closure**) of A, and is denoted by **α-Cl(A)** (resp. **η-Cl(A))**. The **η-interior** of A, denoted by **η-Int(A)** is defined as union of all η-open sets contained in A. We denote the family of all η-open (resp. η-closed) sets of a topological space by **η-O(X)** (resp. **η-C(X)**).

Definition 2.2. A subset A of a space (X, ℑ) is said to be
(1) **generalized closed** (briefly **g-closed**) [**10**] if Cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(2) **πg-closed** [5] if Cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(3) **α-generalized closed** (briefly **αg-closed**) [**11**] if α-Cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(4) **$\pi g\alpha$-closed** [**1**] if α-Cl$(A) \subset U$ whenever $A \subset U$ and U is π-open in X.
(5) **generalized η-closed** (briefly **gη-closed**) **[17]** if **η**-Cl$(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.
(6) **g-open** (resp**. πg-open**, **αg-open, $\pi g\alpha$-open, gη-open**) set if the complement of A is g-closed (resp. πg-closed, αg-closed, $\pi g\alpha$-closed, gη-closed).

Lemma 2.3. For any subsets A and B of a space $(X, \Im)$ the following hold:
(a) η-Cl$(A) = A \cup [\text{Cl}(\text{Int}(\text{Cl}(A))) \cap \text{Int}(\text{Cl}(A))]$,
(b) η-Cl$(X - A) = X - \eta$-Int(A),
(c) $x \in \eta$-Cl(A) if and only if $A \cap U \neq \phi$ for every $U \in \eta$-O(X, x),
(d) $A \in \eta$-C(X) if and only if $A = \eta$-Cl(A).

3. $\pi g\eta$-closed Sets

Definition 3.1. A subset A of a space $(X, \Im)$ is said to be **$\pi g\eta$-closed** if **η**-Cl$(A) \subset U$ whenever $A \subset U$ and U is π-open in X. The family of all $\pi g\eta$-closed subsets of X will be denoted by $\pi g\eta$-C(X).

Theorem 3.2. Every closed set is $\pi g\eta$-closed.
Proof. Let A be a closed set in X. Let U be a π-open set in X such that $A \subset U$. Since A is closed, that is, Cl$(A) = A$, Cl$(A) \subset U$. But we have η-Cl$(A) \subset$ Cl$(A) \subset U$. Therefore η-Cl$(A) \subset U$. Hence A is $\pi g\eta$-closed in X.

Theorem 3.3. For a topological space X the followings hold:
(1) Every g-closed set is $\pi g\eta$-closed.
(2) Every πg -closed set is $\pi g\eta$-closed.
(3) Every α-closed set is $\pi g\eta$-closed.
(4) Every αg-closed set is $\pi g\eta$-closed.
(5) Every $\pi g\alpha$-closed set is $\pi g\eta$-closed.
Proof.
(1) Let A be a g-closed set in X. Let U be a π-open set in X such that $A \subset U$. Since every π-open set is open and since A is g-closed, that is, Cl$(A) \subset U$. But we have η-Cl$(A) \subset$ Cl$(A) \subset U$. Therefore η-Cl$(A) \subset U$. Hence A is $\pi g\eta$-closed in X.

(2) Let A be a πg-closed set in X. Let U be a π-open set in X such that $A \subset U$. Since A is πg-closed, that is, Cl$(A) \subset U$. But we have η-Cl$(A) \subset$ Cl$(A) \subset U$. Therefore η-Cl$(A) \subset U$. Hence A is $\pi g\eta$-closed in X.
(3) Let A be a α-closed set in X. Let U be a π-open set in X such that $A \subset U$. Since A is α-closed, that is, α-Cl$(A) = A$, α-Cl$(A) \subset U$. But we have η-Cl$(A) \subset \alpha$-Cl$(A) \subset U$. Therefore η-Cl$(A) \subset U$. Hence A is $\pi g\eta$-closed in X.

(4) Let A be a αg-closed set in X. Let U be a π-open set in X such that $A \subset U$. Since every π-open set is open and since A is αg-closed, that is, α-Cl$(A) \subset U$. But we have η-Cl$(A) \subset \alpha$-Cl$(A) \subset U$. Therefore η-Cl$(A) \subset U$. Hence A is $\pi g\eta$-closed in X.

(5) Let A be a $\pi g\alpha$-closed set in X. Let U be a π-open set in X such that $A \subset U$. Since A is $\pi g\alpha$-closed, that is, α-Cl$(A) \subset U$. But we have η-Cl$(A) \subset \alpha$-Cl$(A) \subset U$. Therefore η-Cl$(A) \subset U$. Hence A is $\pi g\eta$-closed in X.

Remark 3.4. From the above definitions, theorems and known results the relationship between $\pi g\eta$-closed sets and some other existing generalized closed sets are implemented in the following Figure:

closed	$\Rightarrow$	g-closed	$\Rightarrow$	πg-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
α-closed	$\Rightarrow$	αg-closed	$\Rightarrow$	$\pi g\alpha$-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
η-closed	$\Rightarrow$	gη-closed	$\Rightarrow$	$\pi g\eta$-closed

Where none of the implications is reversible as can be seen from the following examples:

Example 3.5. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Then A = {a, b, c} and B = {a, b, d} are πg-closed as well as πgη-closed but not closed.

Example 3.6. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {d}, {a, d}, {c, d}, {a, c, d}, X}. Then A = {c} is πgα-closed as well as πgη-closed. But it is neither closed nor g-closed. It is not πg-closed.

Example 3.7. Let X = {a, b, c, d} and $\Im$ = {ϕ, {c}, {d}, {c, d}, {b, c, d}, X}. Then A = {b} is g-closed, αg-closed, gη-closed, πgα-closed, πgη-closed. But it is closed.

Example 3.8. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. Then A = {a, b} is πgα-closed as well as πgη-closed but not closed. But it is neither closed nor αg-closed.

Example 3.9. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {c}, {a, c}, X}. Then A = {c} is η-closed as well as πgη-closed but not α-closed.

Example 3.10. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b, c}, X}. Then A = {a, b} is gη-closed as well as πgη-closed but not closed.

Theorem 3.11. For πgη-closed sets of a space (X, $\Im$) the following properties hold:
(a) Every finite union of πgη-closed sets is always a πgη-closed set.
(b) Even a countable union of πgη-closed sets need not be a πgη-closed set.
(c) Even a finite intersection of πgη-closed sets may fail to be a πgη-closed set.
Proof.
(a) Let A and B be any two πgη-closed sets. Therefore η-Cl(A) $\subset$ U and η-Cl(B) $\subset$ U whenever A $\subset$ U, B $\subset$ U and U is π-open. Let A $\cup$ B $\subset$ U where U is π-open.
Since, η-Cl(A $\cup$ B) $\subset$ η-Cl(A) $\cup$ η-Cl(B) $\subset$ U, we have A $\cup$ B is πgη-closed.

(b) Let R be the real line with the usual topology. Every singleton is πgη-closed. However, A = {1 / i : i = 2, 3, …….} is not πgη-closed, since A $\subset$ (0, 1) which is π-open but η-Cl(A) $\not\subset$ (0, 1).

(c) Let X = {a, b, c, d} and let $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Let A = {a, b, c} and B = {a, b, d} are πgη-closed sets. But A $\cap$ B = {a, b} $\subset$ {a, b} which is π-open. η-Cl(A $\cap$ B) $\not\subset$ {a, b}. Hence A $\cap$ B is not πgη-closed.

Theorem 3.12: If A is πgη-closed and B is any set A $\subset$ B $\subset$ η-Cl(A) then B is πgη-closed.
Proof: Since A is πgη-closed, η-Cl(A) $\subset$ U whenever A $\subset$ U and U is π-open. Let B $\subset$ U and U is π-open. Since B $\subset$ η-Cl(A), η-Cl(B) $\subset$ η-Cl(A) $\subset$ U. Hence B is πgη-closed.

Theorem 3.13. Let A be a πgη-closed set in X. Then η-Cl(A) – A does not contain any nonempty π-closed set.
Proof. Let F be a nonempty π-closed set such that F $\subset$ η-Cl(A) – A. Then F $\subset$ η-Cl(A) $\cap$ (X – A) $\subset$ (X – A) implies A $\subset$ X – F where X – F is π-open. Therefore η-Cl(A) $\subset$ X – F implies F $\subset$ (η-Cl(A)^c. Now F $\subset$ η-Cl(A) $\cap$ (η-Cl(A))^c implies F is empty.

Reverse implication does not hold.

Example 3.14. Let X = {a, b, c, d, e} and let $\Im$ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X}. Let A = {c} then η-Cl(A) = {c, d, e}, η-Cl(A) – A = {d, e} does not contain any nonempty regular closed set but A is not πgη-closed set.

Corollary 3.15. Let A be πgη-closed. A is η-closed iff η-Cl(A) – A is π-closed.

Proof. Let A be η-closed set then A = η-Cl(A) implies η-Cl(A) – A = ϕ which is π-closed. Conversely, if η-Cl(A) – A is π-closed then A is η-closed.

Theorem 3.16. If A is π-open and πgη-closed. Then A is η-closed and hence clopen.
Proof. Let A be regular open. Since A is πgη-closed, η-Cl(A) ⊂ A implies A is η-closed. Hence A is closed. (Since every π-open η-closed set is closed). Therefore A is clopen.

Definition 3.17. Let (X, ℑ) be a topological space, A ⊂ X and x ∈ X. Then x is said to be a **η-limit point** of A iff every η-open set containing x contains a point of A different from x, and the set of all η-limit points of A is said to be the η- derived set of A and is denoted by $D_\eta(A)$.

Usual derived set of A is denoted by D(A).

The proof of the following result is analogous to the well known ones.

Lemma 3.18. Let (X, ℑ) be a topological space and A ⊂ X. Then η-Cl(A) = A ∪ $D_\eta(A)$.

Theorem 3.19. Let A and B be πgη-closed sets in (X, ℑ) such that Cl(A) = η-Cl(A) and Cl(B) = η-Cl(B). Then A ∪ B is πgη-closed.
Proof. Let A ∪ B ⊂ U and U is π-open in (X, ℑ). Then η-Cl(A) ⊂ U and η-Cl(B) ⊂ U. Now, Cl(A ∪ B) = Cl(A) ∪ Cl(B) = η-Cl(A) ∪ η-Cl(B) ⊂ U. But η-Cl(A ∪ B) ⊂ Cl(A ∪ B). So, η-Cl(A ∪ B) ⊂ U and hence A ∪ B is πgη-closed.

From the fact that $D_\eta(A)$ ⊂ D(A) and **Lemma 3.18** we have the following,

Remark 3.20. For any subset A ⊂ X such that D(A) ⊂ $D_\eta(A)$. Then Cl(A) = η-Cl(A).

Theorem 3.21. For a space X, the following are equivalent:
(a) X is extremally disconnected,
(b) Every subset of X is πgη-closed,
(c) The topology on X generated by πgη-closed sets is the discrete ones.
Proof. (a) ⇒ (b).
Assume that X is extremally disconnected. Let A ⊂ U where U is π-open in X. Since U is π-open, it is the finite union of regular open sets and X is extremally disconnected, U is finite union of clopen sets and hence U is clopen. Therefore η-cl(A) ⊂ cl(A) ⊂ cl(U) ⊂ U implies A is πgη-closed.

(b) ⇒ (a).
Let A be a regular open set of X. Since A is πgη-closed by **Theorem 3.16**, A is clopen. Hence X is extremally disconnected.

(b) ⇔ (c) is obvious.

4. πgη-open sets

Definition 4.1. Let (X, ℑ) be a topological space. A subset A of X is called **π-generalized η-open** (briefly **πgη-open**) iff its complement is πgη-closed set. We denote the family of all πgη-open (resp. πgη-closed) sets of a topological space by **πgη-O(X)** (resp. **πgη-C(X))**.

Lemma 4.2. If A be a subset of X, then
(a) η-Cl(X – A) = X – η-Int(A).
(b) η-Int(X – A) = X – η-Cl(A).

Theorem 4.3. A subset A of a space X is πgη-open iff F ⊂ η-Int(A) whenever F is π-closed and F ⊂ A.
Proof. Let F be π-closed set such that F ⊂ A. Since X – A is πgη-closed and X – A ⊂ X – F where F ⊂ η-Int(A). Conversely.

Let $F \subset \eta$-Int(A) where F is π-closed and $F \subset A$. Since $F \subset A$ and $X - F$ is π-open, η-Cl(X – A) = X –η-Int(A) $\subset$ X – F. Therefore A is $\pi g\eta$-open.

Theorem 4.4. If η-Int(A) $\subset B \subset A$ and A $\pi g\eta$-open then B is $\pi g\eta$-open.
Proof: Since η-Int(A) $\subset B \subset A$, by **Theorem 3.12**, η-Cl(X – A) $\supset$ (X – B) implies B is $\pi g\eta$-open.
Remark 4.5. For any $A \subset X$, η-Int(η-Cl(A) – A) = ϕ.

Theorem 4.6. If $A \subset X$ is $\pi g\eta$-closed then η-Cl(A) – A is $\pi g\eta$-open.
Proof. Let A be $\pi g\eta$-closed and F be a π-closed set such that $F \subset \eta$-Cl(A) – A. By **Theorem 3.13**, F = $\varnothing$ implies $F \subset \eta$-Int(η-Cl(A) – A)). By **Theorem 4.3**, η-cl(A) – A is $\pi g\eta$-open.

Converse of the above theorem is not true.

Example 4.7. Let X = {a, b, c} and let $\Im$ = {ϕ, {a}, {b}, {a, b}, X}. Let A = {b}. Then A is not $\pi g\eta$-closed but η-Cl(A) – A = {a, b} is $\pi g\eta$-open.

Definition 4.8. A topological space X is called a **$\pi g\eta$-$T_{1/2}$ space** if every $\pi g\eta$-closed set is η-closed.

Theorem 4.9. Let (X, $\Im$) be a topological space.
(a) η-O(X) $\subset \pi g\eta$-O(X),
(b) A space X is $\pi g\eta$-$T_{1/2}$ iff η-O(X) = $\pi g\eta$-O(X).
Proof. (a) Let A be a η-open set, then X – A is η-closed so X – A is $\pi g\eta$-closed. Thus A is $\pi g\eta$-open. Hence η-O(X) $\subset \pi g\eta$-O(X).

(b) Necessity: Let (X, $\Im$) be $\pi g\eta$-$T_{1/2}$ space. Let A be $\pi g\eta$-open. Then X – A is $\pi g\eta$-closed. By hypothesis, X – A is η-closed. Thus A is η-open. Therefore η-O(X) = $\pi g\eta$-O(X).
Sufficiency: Let η-O(X) = $\pi g\eta$-O(X). Let A be $\pi g\eta$-closed. Then X – A is $\pi g\eta$-open. X – A is η-open. Hence A is η-closed. This implies (X, $\Im$) is $\pi g\eta$-$T_{1/2}$ space.

Lemma 4.10. Let A be a subset of X and $x \in X$. Then $x \in \eta$-Cl(A) iff $V \cap \{x\} \neq \phi$ for every η-open set V containing x.

Theorem 4.11. For a topological space X the following are equivalent:
(a) X is $\pi g\eta$-$T_{1/2}$ space.
(b) Every singleton set is either π-closed or η-open.
Proof. (a) $\Rightarrow$ (b): Let X be a $\pi g\eta$-$T_{1/2}$ space. Let $x \in X$ and assuming that {x} is not π-closed. Then clearly X – {x} is not π-open. Hence X – {x} is trivially a $\pi g\eta$-closed. Since X is $\pi g\eta$-$T_{1/2}$ space, X – {x} is η-closed. Therefore {x} is η-open.

(b) $\Rightarrow$ (a): Assume every singleton set of X is either π-closed or η-open. Let A be a $\pi g\eta$-closed set. Let $x \in \eta$-Cl(A).

Case I: Let {x} be π-closed. Suppose x does not belong to A. Then $x \in \eta$-Cl(A) – A. By **Theorem 3.7**, $x \in A$. Hence η-Cl(A) $\subset$ A.

Case II: Let {x} be η-open. Since $x \in \eta$-Cl(A), we have $A \cap \{x\} \neq \phi$ implies $x \in A$. Therefore η-Cl(A) $\subset$ A. Therefore A is η-closed.

5. $\pi g\eta$-continuous and $\pi g\eta$-irresolute Functions

Definition 5.1. A function $f : X \rightarrow Y$ is called:
(a) **α-continuous** [**12**] (resp. **η-continuous** [**17**]) if $f^{-1}(V)$ is α-closed (resp. η-closed) in X for every closed set V of Y,
(b) **g-continuous** [3] (resp. **αg-continuous** [**11**], **gη-continuous** [**17**]) if $f^{-1}(V)$ is g-closed (resp. αg-closed, gη-closed) in X for every closed set V of Y,

(c) **πg-continuous** [5] (resp. **πgα-continuous** [1], **πgη-continuous**) if $f^{-1}(V)$ is πg-closed (resp. πgα-closed, πgη-closed) in X for every closed set V of Y,

Definition 5.2. A function f : X → Y is called **η-irresolute** [9] (resp. **gη-irresolute** [17], **πgη-irresolute**) if $f^{-1}(V)$ is η-closed (resp. gη-closed, πgη-closed) in X for every η-closed (resp. gη-closed, πgη-closed) set V of Y.

Proposition 5.3. Every πgη-irresolute function is πgη-continuous.

Remark 5.4. From the above definitions, proposition and known results, we have following diagram:

π-continuous				
⇓				
continuous	⇒	g- continuous	⇒	πg- continuous
⇓		⇓		⇓
α- continuous	⇒	αg- continuous	⇒	πgα- continuous
⇓		⇓		⇓
η- continuous	⇒	gη- continuous	⇒	πgη- continuous
⇑		⇑		⇑
η-irresolute	⇒	gη-irresolute	⇒	πgη-irresolute

Where none of the implications is reversible as can be seen from the following examples:

Example 5.5. Let X = {x, y, z}, ℑ = {ϕ, X, {x}}, Y = {a, b} and ρ = {ϕ, Y, {a}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = f(z) = b and f(y) = x. Then f is g-continuous as well as gη-continuous. It is also πgη-continuous but not continuous.

Example 5.6. Let X = {x, y, z}, ℑ = {ϕ, X, {x}, {y}, {x, y}} and f : (X, ℑ) → (X, ℑ) defined as follows: f(x) = f(y) = x and f(z) = z. Then f is π-continuous as well as continuous.

Example 5.7. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {x}, {z}, {x, z}} and ρ = {ϕ, Y, {x}, {y}, {x, y}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = x, f(y) = z and f(z) = y. Then $f^{-1}(\{z\}) = \{y\}$, $f^{-1}(\{x, z\}) = \{x, y\}$, $f^{-1}(\{y, z\}) = \{y, z\}$. Therefore, f is gη-continuous as well as πgη-continuous.

Example 5.8. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {x}, {y, z}} and ρ = {ϕ, Y, {x}, {z}, {x, z}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = z, f(y) = y and f(z) = x. Then $f^{-1}(\{y\}) = \{y\}$ is not closed, α-closed in X. Here the set {y} is closed in Y. Therefore, f is not continuous, α-continuous.

Example 5.9. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {x}, {z}, {x, z}} and ρ = {ϕ, Y, {y, z}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = x, f(y) = z and f(z) = y. Then f is gη-continuous as well as πgη-continuous.

Example 5.10. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {y, z}} and ρ = {ϕ, Y, {x}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = y, f(y) = z and f(z) = x. Then f is gη-continuous as well as πgη-continuous.

Example 5.11. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {x}, {z}, {x, z}} and ρ = {ϕ, Y, {x}, {y}, {x, y}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = x, f(y) = z and f(z) = y. Then $f^{-1}(\{x\}) = \{x\}$, $f^{-1}(\{y\}) = \{z\}$, $f^{-1}(\{z\}) = \{y\}$, $f^{-1}(\{x, z\}) = \{x, y\}$, $f^{-1}(\{y, z\}) = \{y, z\}$. Since inverse image of every gη-open set in Y is gη-open in X. Therefore, f is gη-irresolute as well as gη-continuous. It is also πgη-continuous.

Theorem 5.12. Let f : X → Y be a function.
(a) If f is πgη-irresolute and X is πgη-$T_{1/2}$ space, then f is η-irresolute.
(b) If f is πgη-continuous and X is πgη-$T_{1/2}$ space, then f is η-continuous.
Proof. (a) Let V be η-closed in Y. Since f is πgη-irresolute, $f^{-1}(V)$ is πgη-closed in X. Since X is πgη-$T_{1/2}$ space, $f^{-1}(V)$ is η-closed in X. Hence f is η-irresolute.

(b) Let V be closed in Y. Since f is πgη-continuous, $f^{-1}(V)$ is πgη-closed in X. By assumption, it is η-closed. Therefore f is η-continuous.

Definition 5.13. A function f : X → Y is called **π-irresolute** [**2**] if $f^{-1}(V)$ is π-closed in X for each π-closed set V of Y .

Definition 5.14. A function f : X → Y is called **pre η-closed** if f(V) is η-closed in Y for each η-closed set V of X.

Theorem 5.15. Let f : X → Y be π-irresolute and pre η-closed map. Then f(A) is πgη-closed in Y for every πgη-closed set A of X.
Proof. Let A be πgη-closed set in X. Let f(A) ⊂ V where V is π-open in Y . Then A ⊂ $f^{-1}(V)$ and A is πgη-closed in X implies η-Cl(A) ⊂ $f^{-1}(V)$. Hence η-Cl(f(A)) ⊂ η-Cl(f(η-Cl(A))) = f(η-Cl(A)) ⊂ V. Therefore f(A) is πgη-closed in Y .

Definition 5.16. A function f : X → Y is **π-open map** [**8**] if f(V) is π-open set in Y for every π-open set V of X.

Theorem 5.17. If f : X → Y is η-irresolute and π-open bijection, then f is πgη-irresolute.
Proof. Let V be πgη-closed set in Y . Let $f^{-1}(V)$ ⊂ U where U is π-open in X. Hence V ⊂ f(U) and f(U) is π-open implies η-Cl(V) ⊂ f(U). Since f is η-irresolute, f^{-1}(η-Cl(V)) is η-closed in X. Hence η-Cl($f^{-1}(V)$) ⊂ η-Cl(f^{-1}(η-Cl(V))) = f^{-1}(η-Cl(V)) ⊂ U. Therefore $f^{-1}(V)$ is πgη-closed and thus f is πgη-irresolute.

Theorem 5.18. Let f : X → Y be pre η-closed and πgη-irresolute surjection. If X is πgη-$T_{1/2}$ space, then Y is also a πgη-$T_{1/2}$ space.
Proof. Let F be πgη-closed set in Y. Since f is πgη-irresolute, $f^{-1}(F)$ is πgη-closed in X. Since X is πgη-$T_{1/2}$ space, $f^{-1}(F)$ is η-closed in X and hence f(f^{-1} (F)) = F is η-closed in Y. This shows that Y is πgη-$T_{1/2}$ space.

6. Some Covering Properties

Definition 6.1. A topological space X is said to be:
(a) **nearly compact** [**14**] if every regular open cover of X has a finite subcover.
(b) **countably compact** [**4**] if every open countable cover of X has a finite subcover.
(c) **nearly countably compact** [7] if every countable cover by regular open sets has a finite subcover.
(d) **nearly Lindelof** [**6**] if every cover by regular open sets has a countable subcover.
(e) **πgη-compact** if every πgη-open cover of X has a finite subcover.
(f) **πgη-Lindelof** if every cover by πgη-open sets has a countable subcover.
(g) **countably πgη-compact** if every πgη-open countable cover of X has a finite subcover.

Corollary 6.2. For a topological space X the followings hold:
(a) If X is πgη-Lindelof, then X is Lindelof.
(b) If X is πgη-compact, then X is compact.
(c) If X is countably πgη-compact, then X is countably compact.
(d) If X is πgη-compact, then X is πgη-Lindelof.
(e) If X is πgη-compact, then X is nearly compact.
(f) If X is πgη-compact, then X is nearly Lindelof.
(g) If X is countably πgη-compact, then X is nearly countably compact.

Definition 6.3. A function f : X → Y is called **πgη-open** if f(U) is πgη-closed in Y for each πgη-closed set in X.

Definition 6.4. A function f : X → Y is called **almost πgη-continuous** if $f^{-1}(V)$ is πgη-closed in X for every regular closed set V of Y .

Theorem 6.5. Every πgη-compact subset of a πgη-compact space is πgη-compact space relative to X.
Proof. Straightforward.

Theorem 6.6. Let f : X → Y be a function. If f is πgη-continuous surjection (resp. almost πgη-continuous) and X is πgη-compact space, then Y is compact (resp. nearly compact).
Proof. Straightforward.

Theorem 6.7. Let f : X → Y be a function and A ⊂ X. If f is πgη-irresolute and A is πgη-compact, then f(A) is πgη-compact.
Proof. Straightforward.

Theorem 6.8. Let f : X → Y be a function. If f is πgη-open bijection and Y is πgη-compact, then X is πgη-compact.
Proof. Straightforward.

Remark 6.9. Every πgη-continuous function is almost πgη-continuous function.

Theorem 6.10. Let f : X → Y be an almost πgη-continuous surjection.
(a) If X is πgη-compact, then Y is nearly compact.
(b) If X is πgη-Lindelof, then Y is nearly Lindelof.
(c) If X is countably πgη-compact, then Y is nearly countably compact.
Proof. Straightforward.

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