DOUBLE DIFFERENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

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ABSTRACT

In this research paper, we will discuss about the construction of the sequence spaces, viz.,ℓ_∞, c, c_0 sequence spaces. The algebraic and topological properties of these spaces are also discussed in detail. Further, some basic definitions of particular type of sequence spaces are given in this paper. We will also define new sequence spaces as double difference sequence spaces using the Orlicz function and paranorm. We proved that all the new double difference sequence spaces are complete normed spaces. Some inclusion relations have been proved for new sequence spaces.

Keywords: Orlicz function, Double difference sequence spaces, Linear spaces, Normed Linear sequence spaces, paranormed sequence spaces.

1. INTRODUCTION

In this research paper, ℓ_∞, c and c_0 respectively denotes the spaces of all bounded sequences, convergent sequence sequences and null sequences. We will also define some basic definitions and will be helpful latter to prove some results.

Double difference in a sequence is defined as

$$\Delta^2 x = (\Delta^2 x_n)_{n=1}^{\infty} = \big(\Delta(\Delta x_n)\big)_{n=1}^{\infty} = \big(\Delta(x_n - x_{n+1})\big)_{n=1}^{\infty} = (\Delta x_n - \Delta x_{n+1})_{n=1}^{\infty} = (x_n - 2x_{n+1} + x_{n+2})_{n=1}^{\infty}$$

There are some double difference sequence spaces[5,6,9] and are defined as

$$\ell_\infty(\Delta^2) = \{x = (x_n) : \Delta^2 x \in \ell_\infty\}$$

$$c(\Delta^2) = \{x = (x_n) : \Delta^2 x \in c\}$$

$$c_0(\Delta^2) = \{x = (x_n) : \Delta^2 x \in c_0\}$$

Orlicz function is a function F:$[0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with

$F = 0, F(x) > 0$, for $x > 0$ and $F(x) \to \infty \; as \; x \to \infty$.

An Orlicz function F is said to satisfy Δ_2 −condition, if there exists a constant $k > 0$, such that

$$F(2x) \leq kF(x), \text{ for all values of } x, x \geq 0.$$

Δ_2 −condition is equivalent to $F(\ell x) \leq k\ell F(x)$, for all values of x and for $\ell > 1$. If the convexity of Orlicz function is replaced by $F(x+y) \leq F(x) + F(y)$, then this function is called modulus function, introduced by Nakano. It was further investigated from sequence space point of view by Ruckle and many others.

Now we define the Orlicz sequence space([1], [4]) by using the idea of Orlicz function and is defined as

$$\ell_F = \left\{x \in \omega : \sum_{n=1}^{\infty} F\left(\frac{x_n}{\rho}\right) < \infty, \;\; for\ some\ \rho > 0\right\}$$

The norm of ℓ_F sequence space is

$$||x|| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} F\left(\frac{|x_n|}{\rho}\right) \leq 1\right\}$$

Now define the sequence spaces for an Orlicz function F

$$c_0(\Delta^2, F) = \left\{x = (x_n) : \lim_{n\to\infty} F\left(\frac{|\Delta^2 x_n|}{\rho}\right) = 0, for\ some\ \rho > 0\right\}$$

$$c(\Delta^2, \ell) = \left\{x = (x_n) : \lim_{n\to\infty} F\left(\frac{|\Delta^2 x_n - \ell|}{\rho}\right) = 0,\ \ for\ some\ \rho > 0,\ \ \ell \in \mathbb{C}\right\}$$

$$\ell_\infty(\Delta^2, F) = \left\{x = (x_n) : \sup_{n\geq 0} F\left(\frac{|\Delta^2 x_n|}{\rho}\right) < \infty,\ \ for\ some\ \rho > 0\right\}$$

The norm of these sequence spaces is defined as

$$||x||_{\Delta^2} = \inf\left\{\rho > 0\ : \sup_{n\geq 0} F\left(\frac{|\Delta^2 x_n|}{\rho}\right) \leq 1\right\}$$

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1 Linear space. A space X is said to be linear if

(1) $x + y \in X\ ; \forall\ x, y \in X$

(2) $\alpha x \in X\ ; x \in X, \alpha \in K$

Definition 2.2 Normed linear Space. Let $||\ .\ ||$ is a function from space X to $\mathbb{R}^+$. The space $(X, ||.||)$ is said to be normed linear space over filed K if it holds the following axioms

(1) $||x|| \geq 0\ \ and\ ||x|| = 0\ \Leftrightarrow x = 0$

(2) $||\beta x|| = |\beta|.||x||,\ \ where\ \beta \in K\ \ and\ x \in X$

(3) $||x + y|| \leq ||x|| + ||y||, \qquad \forall\ x, y \in X$

Example 2.1. The set $\mathbb{R}^n$ or $\mathbb{C}^n$ is a normed linear space with norm defined as; $||x|| = [\sum_{n=1}^{n} |x_i|^2]^{\frac{1}{n}}$

Definition 2.3. A Banach space is a complete normed space under the norm $d(x, y) = ||x - y||$

Example 2.2. (1) space ℓ^p is an example of Banach space with norm defined as;

$||x|| = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$

(2) $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Banach space with norm defined as; $||x|| = [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}}$

Definition 2.4 Cauchy Sequence. A sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence if for every positive real number ϵ, there is a positive integer N such that for all natural numbers $m, n > N$, $|x_m - x_n| < \epsilon$.

Definition 2.5 continuous Function. A function $f(x)$ is said to be continuous at a point $x = a$ if it satisfy the following three conditions

(1) $f(a)$ exists.

(2) $\lim_{x\to a} f(a)$ exists.

(3) $\lim_{x\to a} f(x) = f(a)$.

Another definition of continuous function, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta, x \neq a$ implies that $|f(x) - f(a) < \epsilon$.

Definition 2.6 Convex Function. A function $f(x)$ is convex on an interval $[a,b]$ if for any two points x_1 and x_2 in $[a,b]$ and any λ, where $0<\lambda<1$, such that; $f[\lambda x_1+(1-\lambda)x_2]\le \lambda f(x_1)+(1-\lambda)f(x_2)$.

Definition 2.7 Paranormed space. A Paranormed space (X,P) is a vector space X with zero element α together with a function $P:X\to\mathbb{R}^+$ (called a paranorm on X) which satisfy the following axioms

(1) $P(\alpha)=0$.

(2) $P(x)=P(-x)\,;x\in X$.

(3) $P(x_1+x_2)\le P(x_1)+P(x_2)\,;x_1,x_2\in X$.

(4) Scalar multiplication is continuous i.e., if $\gamma_n\to\gamma\ as\ n\to\infty$ and $\{x_n\}$ is a sequence of vectors with $P(x_n-x)\to 0\ as\ n\to\infty$, then $P(\gamma_n x_n-\gamma x)\to 0\ as\ n\to\infty$.

A papanorm is called total if $P(x)=0\ \Rightarrow x=\alpha$.

Definition 2.8. The norm on the double difference sequence space $\ell_\infty(\Delta^2)$ is defined as

$$||\Delta^2 x||_\infty=\sup_{n\in\mathbb{N}}|\Delta^2 x_n|$$

Definition 2.9. Norm on space $c(\Delta^2)$ is defined as

$$||\Delta^2 x||_\infty=\sup_{n\in\mathbb{N}}|\Delta^2 x_n|$$

Definition 2.10. Norm on space $c_0(\Delta^2)$ is defined as

$$||\Delta^2 x||_0=\max_{n\in\mathbb{N}}|\Delta^2 x_n|$$

Example 2.3. The sequence $\ell_\infty(\Delta^2)$ is given as

$$\{x_n\}_{n=1}^\infty=\{(-1)^n\}_{n=1}^\infty=\{-1,1,-1,1,-1,1,\dots\}.$$

Aim: To show that $\{(-1)^n\}_{n=1}^\infty=\{-1,1,-1,1,-1,1,\dots\}\in\ell_\infty(\Delta^2)$.

Let $\{x_n\}_{n=1}^\infty=\{x_1,x_2,x_3,x_4,x_5,\dots\}$

Where, $x_1=-1,x_2=1,x_3=-1,x_4=1,x_5=-1$

$\{\Delta x_n\}_{n=1}^\infty=\{\Delta x_1,\Delta x_2,\Delta x_3,\Delta x_3,\Delta x_4,\dots\}$

$\Rightarrow\ \{\Delta x_n\}_{n=1}^\infty=\{x_1-x_2,x_2-x_3,x_3-x_4,\dots\}$

$\Rightarrow\ \{\Delta x_n\}_{n=1}^\infty=\{-1-1,1-(-1),-1-1,1-(-1),\dots\}$

$\{\Delta x_n\}_{n=1}^\infty=\{-2,2,-2,2,\dots\}$

Also, $\{\Delta^2 x_n\}_{n=1}^\infty=\{\Delta^2 x_1,\Delta^2 x_2,\Delta^2 x_3,\dots\}$

$\{\Delta^2 x_n\}_{n=1}^\infty=\{\Delta x_1-\Delta x_2,\Delta x_2-\Delta x_3,\Delta x_3-\Delta x_4,\dots\}$

$\Rightarrow\ \{\Delta^2 x_n\}_{n=1}^\infty=\{-2-2,2-(-2),-2-2,2-(-2),\dots]$

$\{\Delta^2 x_n\}_{n=1}^\infty=\{-4,4,-4,4,\dots\}$, which is bounded sequence.

Therefore, $\{\Delta^2 x_n\}_{n=1}^\infty\in\ell_\infty$.

Hence, $\{(-1)^n\}_{n=1}^\infty\in\ell_\infty(\Delta^2)$.

Example 2.4. The sequence $c(\Delta^2)$ is discussed as follows:

$$\{x_n\}_{n=1}^\infty=\{1,2,4,7,11,\dots\}$$

Aim: To show that $\{x_n\}_{n=1}^\infty\in c(\Delta^2)$

Let $\{x_n\}_{n=1}^\infty=\{x_1,\ x_2,x_3,x_4,x_5,\dots\}$

Where, $x_1=1,x_2=2,x_3=4,x_4=7,x_5=11$

$\{\Delta x_n\}_{n=1}^\infty=\{\Delta x_1,\Delta x_2,\Delta x_3,\Delta x_4,\dots\}$

$\Rightarrow\ \{\Delta x_n\}_{n=1}^\infty=\{x_1-x_2,x_2-x_3,x_3-x_4,\dots\}$

$\Rightarrow\ \{\Delta x_n\}_{n=1}^\infty=\{1-2,2-4,4-7,7-11,\dots\}$

$\Rightarrow\ \{\Delta x_n\}_{n=1}^\infty=\{-1,-2,-3,-4,\dots\}$

Also, $\{\Delta^2 x_n\}_{n=1}^{\infty} = \{\Delta^2 x_1, \Delta^2 x_2, \Delta^2 x_3, \dots\}$

$\Rightarrow \{\Delta^2 x_n\}_{n=1}^{\infty} = \{\Delta^2 x_1 - \Delta^2 x_2, \Delta^2 x_2 - \Delta^2 x_3, \Delta^2 x_3 - \Delta^2 x_4, \dots\}$

$\Rightarrow \{\Delta^2 x_n\}_{n=1}^{\infty} = \{-1-(-2), -2-(-3), -3-(-4), \dots\}$

$\Rightarrow \{\Delta^2 x_n\}_{n=1}^{\infty} = \{-1+2, -2+3, -3+4, \dots\}$

$\{\Delta^2 x_n\}_{n=1}^{\infty} = \{1,1,1,1, \dots\}$, Which is convergent sequence.

Therefore, $\{\Delta^2 x_n\}_{n=1}^{\infty} \in c$

Hence, $\{x_n\}_{n=1}^{\infty} = \{1,2,4,7,11, \dots\} \in c(\Delta^2)$

3. MAIN RESULTS

Theorem 3.1. The spaces $\ell_\infty(\Delta^2), c(\Delta^2)$ and $c_0(\Delta^2)$ are linear spaces.

Proof. We shall prove that all three double difference sequence spaces are linear spaces as follows:

(1). For $\ell_\infty(\Delta^2)$ *is a linear space.*

Since, $\ell_\infty(\Delta^2)$ $=\{x = (x_n) : \Delta^2 x \in \ell_\infty\}$. Let $x, y \in \ell_\infty(\Delta^2)$. Then $\Delta^2 x, \Delta^2 y \in \ell_\infty$ or $\Delta^2 x_n, \Delta^2 y_n \in \ell_\infty$.

We know that, $\ell_\infty = \{x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} |x_n| < \infty\}$

Therefore, $\sup_{n\in\mathbb{N}} |\Delta^2 x_n| < \infty$ and $\sup_{n\in\mathbb{N}} |\Delta^2 y_n| < \infty$

$\Rightarrow (\sup_{n\in\mathbb{N}} |\Delta^2 x_n| + \sup_{n\in\mathbb{N}} |\Delta^2 y_n|) < \infty$

$\Rightarrow \sup_{n\in\mathbb{N}} |\Delta^2 x_n + \Delta^2 y_n| < \infty$

$\Rightarrow (\Delta^2 x_n + \Delta^2 y_n) \in \ell_\infty$

$\Rightarrow \Delta^2(x_n + y_n) \in \ell_\infty$

$\Rightarrow \Delta^2(x + y) \in \ell_\infty$

$\Rightarrow (x + y) \in \ell_\infty$

Now, let α be a scalar and $x \in \ell_\infty(\Delta^2)$ be an arbitrary sequence.

Therefore, $\Delta^2 x_n \in \ell_\infty$. Then

$\sup_{n\in\mathbb{N}} |\Delta^2 x_n| < \infty$

$\Rightarrow \alpha(\sup_{n\in\mathbb{N}} |\Delta^2 x_n|) < \infty$

$\Rightarrow \alpha\Delta^2 x_n \in \ell_\infty$

$\Rightarrow \alpha x_n \in \ell_\infty(\Delta^2)$

$\Rightarrow \alpha x \in \ell_\infty(\Delta^2)$

Hence, $\ell_\infty(\Delta^2)$ is a linear space.

(**2**) For the sequence space $c(\Delta^2)$ to be linear space.

Since, $c(\Delta^2) = \{x = (x_n) : \Delta^2 x \in c\}$. Let $x, y \in \ell_\infty(\Delta^2)$.

$\Rightarrow \Delta^2 x, \Delta^2 \in c \; or \; \Delta^2 x_n, \Delta^2 y_n \in c.$ Also, $c = \left\{x = (x_n) \in \omega : \lim_{n\to\infty} |x_n - \ell| = 0, for\ some\ \ell \in \mathbb{C}\right\}$

Since, $\lim_{n\to\infty} |\Delta^2 x_n - \ell_1| = 0$ and $\lim_{n\to\infty} |\Delta^2 x_n - \ell_2| = 0$

$\Rightarrow \lim_{n\to\infty} |\Delta^2 x_n - \ell_1| + \lim_{n\to\infty} |\Delta^2 x_n - \ell_2| = 0$

$\Rightarrow \lim_{n\to\infty} \{|\Delta^2 x_n - \ell_1| + |\Delta^2 x_n - \ell_2|\} = 0$

$\Rightarrow \lim_{n\to\infty} \{|\Delta^2 x_n| + |-\ell_1| + |\Delta^2 x_2| + |-\ell_2|\} \le 0$

$or \; \lim_{n\to\infty} \{|\Delta^2 x_n| + |\ell_1| + |\Delta^2 x_2| + |\ell_2|\} \le 0$

$or \; \lim_{n\to\infty} \{|\Delta^2 x_n| + |\Delta^2 y_n| + |\ell_1| + |\ell_2|\} \le 0$

$or \lim_{n\to\infty}\{|(\Delta^2 x_n + \Delta^2 y_n) - (\ell_1 + \ell_2)|\} \le 0$

$\Rightarrow (\Delta^2 x_n + \Delta^2 y_n) \in c$

$\Rightarrow (x_n + y_n) \in c(\Delta^2)$

$\Rightarrow (x + y) \in c(\Delta^2)$.

And let α be a scalar and $x \in c(\Delta^2)$.

$\Rightarrow \Delta^2 x \in c$

$\Rightarrow \lim_{n\to\infty} |\Delta^2 x_n - \ell| = 0$

$\Rightarrow \lim_{n\to\infty} |\Delta^2 x - \ell| = 0$

$\Rightarrow \lim_{n\to\infty} \alpha|\Delta^2 x - \ell| = 0$

$\Rightarrow \lim_{n\to\infty} |\alpha\Delta^2 x - \alpha\ell| = 0$

$\Rightarrow \alpha\Delta^2 x \in c$

$\Rightarrow \alpha x \in c(\Delta^2)$.

Hence, $c(\Delta^2)$ is a linear space.

(3) For the sequence space $c_0(\Delta^2)$ is linear space. $c_0(\Delta^2) = \{x = (x_n) : \Delta^2 \in c_0\}$. Let $x, y \in c_0(\Delta^2)$.

Therefore, $\Delta^2 x, \Delta^2 \in c_0$

Since, $c_0 = \{x = (x_n) \in \omega : \lim_{n\to\infty} x_n = 0$

Therefore, $\lim_{n\to\infty} \Delta^2 x_n = 0$ and $\lim_{n\to\infty} \Delta^2 y_n = 0$

$\Rightarrow \lim_{n\to\infty} \Delta^2 x_n + \lim_{n\to\infty} \Delta^2 y_n = 0$

$\Rightarrow \lim_{n\to\infty} (\Delta^2 x_n + \Delta^2 y_n) = 0$

$\Rightarrow (\Delta^2 x_n + \Delta^2 y_n) \in c_0$

$\Rightarrow (x_n + y_n) \in c_0(\Delta^2)$

$\Rightarrow (x + y) \in c_0(\Delta^2)$

Let α be a scalar and $x \in c_0(\Delta^2)$.

$\Rightarrow \Delta^2 x \in c_0$

$\Rightarrow \lim_{n\to\infty} \Delta^2 x_n = 0$

$\Rightarrow \lim_{n\to\infty} \alpha\Delta^2 x_n = 0$

$\Rightarrow \alpha\Delta^2 x_n \in c_0$

$or \ \alpha\Delta^2 x \in c_0$

$\Rightarrow \alpha x \in c_0(\Delta^2)$.

Hence, $c_0(\Delta^2)$ is a linear space.

Theorem 3.2. $\ell_\infty(\Delta^2, F)$ is a Banach space with norm

$$||x|_{\Delta^2} = \inf\left\{\rho > 0 : \sup_{n\ge 0}(\frac{|\Delta^2 x_n|}{\rho} \le 1\right\}$$

Proof. Let (x^i) be a Cauchy sequence in $\ell_\infty(\Delta^2, F)$. Where, $(x^i) = \left(x_1^i, x_2^i, x_3^i, \dots\right) \in \ell_\infty(\Delta^2, F)$ for each $i \in \mathbb{N}$.

Let, $r, x_0 > 0$ be fixed. Since (x^i) is a Cauchy sequence. Therefore by the definition of Cauchy sequence, for each $\frac{\epsilon}{rx_0} > 0$, there exists a positive integer N such that

$$||x^i - x^j||_{\Delta^2} < \frac{\epsilon}{rx_0} \qquad \forall \quad i, j \ge N \tag{3.1}$$

Now by the definition of norm

$\sup_{n\ge 0} F(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{||x^i - x^j||_{\Delta^2}}) \le 1, \ \forall \ i, j \ge N$

$\Rightarrow \; F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{||x^i - x^j||_{\Delta^2}}\right) \leq 1$

Hence, we can find $r > 0$ with $F\left(\frac{r x_0}{2}\right) \geq 1$ such that

$F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{||x^i - x^j||_{\Delta^2}}\right) \leq 1 \leq F\left(\frac{r x_0}{2}\right)$

$\Rightarrow \; F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{||x^i - x^j||_{\Delta^2}}\right) \leq F\left(\frac{r x_0}{2}\right)$

Since F is non-decreasing function, therefore

$\Rightarrow \; \left(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{||x^i - x^j||_{\Delta^2}}\right) \leq \frac{r x_0}{2}$

$\Rightarrow \; |\Delta^2 x_n^i - \Delta^2 x_n^j| \leq \left(\frac{r x_0}{2}\right)(||x^i - x^j||_{\Delta^2})$

$|\Delta^2 x_n^i - \Delta^2 x_n^j| \leq \left(\frac{r x_0}{2}\right)\left(\frac{\epsilon}{r x_0}\right)$ $\quad Using\ (3.1)$

$\Rightarrow \; |\Delta^2 x_n^i - \Delta^2 x_n^j| < \left(\frac{\epsilon}{2}\right) < \epsilon$

$\Rightarrow \; |\Delta^2 x_n^i - \Delta^2 x_n^j| < \epsilon$

$\Rightarrow \; (\Delta^2 x_n^i)$ is a Cauchy sequence in $\mathbb{R}$.

Therefore, for all $\epsilon (0 < \epsilon < 1)$ there exists a positive integer N such that

$|\Delta^2 x_n^i - \Delta^2 x_n^j| < \epsilon \;\; for\ all\ i, j \geq N.$

Since F is continuous. Therefore by the definition of continuity of a function F, we have

$\sup_{n \geq N} F\left(\frac{|\Delta^2 x_n^i - \lim_{n\to\infty} \Delta^2 x_n^j|}{\rho}\right) \leq 1$

$\Rightarrow \; \sup_{n \geq N}\left(\frac{|\Delta^2 x_n^i - \Delta^2 x}{\rho}\right) \leq 1$

Taking infimum of such $\rho's$ we get

$\inf\left\{\rho > 0 : \sup_{n \geq N} F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x}{\rho}\right) \leq 1\right\} < \epsilon \;\; for\ all\ i > N\ and\ j \to \infty$

Since $(x_i) \in \ell_\infty(\Delta^2, F)$ and F is an Orlicz function which is continuous as well.

Therefore, $x \in \ell_\infty(\Delta^2, F)$. Hence, $\ell_\infty(\Delta^2, F)$ is a Banach space with the norm

$$||x||_{\Delta^2} = \inf\left\{\rho > 0 : \sup_{n \geq 0} F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right\}$$

Theorem 3.3. Let F be an Orlicz function which satisfies Δ_2 −condition, then

(1) $c_0(\Delta^2) \subset c_0(\Delta^2, F)$

(2) $c(\Delta^2) \subset c(\Delta^2, F)$

(3) $\ell_\infty(\Delta^2) \subset \ell_\infty(\Delta^2, F)$.

Proof. (1) Let $x \in c_0(\Delta^2)$.

$\Rightarrow \; \lim_{n\to\infty} \Delta^2 x_n = 0 \;\; or\ \Delta^2 x_n \to 0, \;\; as\ n \to \infty$

$\Rightarrow \; \left(\frac{\Delta^2 x_n}{\rho}\right) \to 0, \;\; as\ n \to \infty$

$\Rightarrow \; F\left(\frac{|\Delta^2 x_n|}{\rho}\right) \to 0, \;\; as\ n \to \infty$

$\Rightarrow \; \lim_{n\to\infty} F\left(\frac{|\Delta^2 x_n|}{\rho}\right) = 0 \quad \Rightarrow \quad x \in c_0(\Delta^2, F)$

Hence, $c_0(\Delta^2) \subset c_0(\Delta^2, F)$.

(2) Let $x \in c(\Delta^2)$.

$\Rightarrow \; \Delta^2 x_n \to \ell \quad as\ n \to \infty$

$\Rightarrow \ \Delta^2 x_n - \ell \to 0 \quad \Rightarrow \ \left(\frac{\Delta^2 x_n - \ell}{\rho}\right) \to 0$

Therefore, $F\left(\frac{|\Delta^2 x_n - \ell|}{\rho}\right) \to F(0) \ \Rightarrow \ F\left(\frac{|\Delta^2 x_n - \ell|}{\rho}\right) \to 0$

Then, $F(0) = 0 \ \Rightarrow \ \lim_{n\to\infty} F\left(\frac{|\Delta^2 x_n - \ell|}{\rho}\right) = 0$

$\Rightarrow \ x \in c(\Delta^2, F)$. Hence, $c(\Delta^2) \subset c(\Delta^2, F)$.

(3) Let $x \in \ell_\infty(\Delta^2)$.

$\Rightarrow \ |\Delta^2 x_n| \le N \ \Rightarrow \ \left(\frac{|\Delta^2 x_n|}{\rho}\right) \le \left(\frac{N}{\rho}\right)$

Then by the Δ_2 −condition of Orlicz function F, We have

$F\left(\frac{|\Delta^2 x_n|}{\rho}\right) \le F\left(\frac{N}{N}\right) \le k\ell F(N)$. Where $\ell = \frac{1}{\rho}$

$\Rightarrow \ \sup F\left(\frac{|\Delta^2 x_n|}{\rho}\right) < \infty \quad \Rightarrow \ x \in \ell(\Delta^2, F)$

Hence, $\ell_\infty(\Delta^2) \subset \ell_\infty(\Delta^2)$.

4. PARANORMED DOUBLE DIFFERENCE SEQUENCE SPACES

Now we define sequence space for an Orlicz function F in Paranormed sequence spaces as $c_0(p), c(p), \ell_\infty(p)$([7], [6]). Let $x = (x_n)$ be a sequence of positive real numbers.

$$c_0(\Delta^2, F, p) = \left\{x = (x_n) : \lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_k|}{\rho}\right)\right)^{p_n} = 0\right\}$$

$$c(\Delta^2, F, p) = \left\{x = (x_n) : \lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_k - \ell|}{\rho}\right)\right)^{p_n} = 0\right\}$$

$$\ell_\infty(\Delta^2, F, p) = \left\{x = (x_n) : \sup_{n\ge 0}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n} < \infty\right\}$$

It is to be noted that $c_0(\Delta^2, F, p) = c_0(\Delta^2, F), c(\Delta^2, F, p) = c(\Delta^2, F)$, if p_k is constant.

The norm G of these paranormed sequence space is

$$G(x) = inf\left\{\rho^{\frac{p_k}{H}} : \left\{\sup_{n\ge 0}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n}\right\} \le 1\right\}$$

Where, $H = max\left(1, \sup_{n\ge 0} p_n\right)$.

Theorem 4.1. $\ell_\infty(\Delta^2, F, p)$ is complete paranormed space with norm G

$G(x) = inf\left\{\rho^{\frac{p_k}{H}} : \left\{\sup_{n\ge 0}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n}\right\}^{\frac{1}{H}} \le 1\right\}$, Where $H = max\left\{1, \sup_{n\ge 0} p_n\right\}$

Proof. Let (x_i) be any Cauchy sequence in the space $\ell_\infty(\Delta^2, F, p)$ and $r, x_0 > 0$ be fixed. For each $\frac{\epsilon}{r x_0} > 0$, there exists a positive integer N such that

$$G(x^i - x^j) < \frac{\epsilon}{r x_0}, \quad for\ all\ i, j \ge N \qquad (4.1)$$

Now using definition of paranorm, we get

$\left\{\sup_{n\ge 0}\left(F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{G(x^i - x^j)}\right)\right)^{p_n}\right\}^{\frac{1}{H}} \le 1, \quad for\ all\ i, j \ge N$

$\Rightarrow \ \sup_{n\ge 0}\left(F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x_n^j|}{G(x^i - x^j)}\right)\right)^{p_n} \le 1, for\ all\ i, j \ge N$

$$\Rightarrow \; F\left(\frac{\left|\Delta^2 x_n^i - \Delta^2 x_n^j\right|}{G(x^i - x^j)}\right) \leq 1$$

For $r > 0$ with $F\left(\frac{rx_0}{2}\right) \geq 1$, we have

$$\Rightarrow \; F\left(\frac{\left|\Delta^2 x_n^i - \Delta^2 x_n^j\right|}{G(x^i - x^j)}\right) \leq F(\frac{rx_0}{2})$$

$$\Rightarrow \; \left(\frac{\left|\Delta^2 x_n^i - \Delta^2 x_n^j\right|}{H(x^i - x^j)}\right) \leq \left(\frac{rx_0}{2}\right)$$

$$\Rightarrow \; \left|\Delta^2 x_n^i - \Delta^2 x_n^j\right| \leq G(x^i - x^j)\left(\frac{rx_0}{2}\right)$$

$$\Rightarrow \; \left|\Delta^2 x_n^i - \Delta^2 x_n^j\right| \leq \left(\frac{rx_0}{2}\right)\left(\frac{\epsilon}{rx_0}\right) = \left(\frac{\epsilon}{2}\right) \qquad Using\ (4.1)$$

$$\Rightarrow \; \left|\Delta^2 x_n^i - \Delta^2 x_n^j\right| \leq \frac{\epsilon}{2} < \epsilon$$

$$\Rightarrow \; |\Delta^2 x_n^i - \Delta^2 x_n^j < \epsilon$$

Hence, $(\Delta^2 x^i)$ is a Cauchy sequence in $\mathbb{R}$.

Now using the continuity of F, we have

$$\left\{\sup_{n\geq N}\left(F\left(\frac{|\Delta^2 x_n^i - \lim_{j\to\infty} \Delta^2 x_n^j|}{\rho}\right)\right)^{p_n}\right\}^{\frac{1}{H}} \leq 1$$

$$\Rightarrow \; \left\{\sup_{n\geq N}\left(F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x}{\rho}\right)\right)^{p_n}\right\}^{\frac{1}{H}} \leq 1$$

Taking infimum of $\rho's$, we get

$$inf\left\{\rho^{\frac{p_n}{H}} : \left\{\sup_{n\geq N}\left(F\left(\frac{|\Delta^2 x_n^i - \Delta^2 x}{\rho}\right)\right)^{p_n}\right\}^{\frac{1}{H}} \leq 1\right\} < \epsilon \qquad for\ all\ i,j \geq N\ \ and\ j \to \infty.$$

Since, $(x^i) \in \ell_\infty(\Delta^2, F, p)$ and F is continuous. Therefore, $x \in \ell_\infty(\Delta^2, F, p)$.

Hence, $\ell_\infty(\Delta^2, F, p)$ is a complete paranormed space with norm

$$G(x) = inf\left\{\rho^{\frac{p_k}{H}} : \left\{\sup_{n\geq 0}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n}\right\}^{\frac{1}{H}} \leq 1\right\}, \text{ Where } H = max\left\{1, \sup_{n\geq 0} p_n\right\}$$

Theorem 4.2. Let $0 < p_n \leq q_n < \infty$ for each n, then $c_0(\Delta^2, F, p) \subseteq c_0(\Delta^2, F, q)$.

Proof. Let $x \in c_0(\Delta^2, F, p)$.There exists $\rho > 0$ such that

$$\lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n}{\rho}\right)\right)^{p_n} = 0$$

For sufficiently large k, $F\left(\frac{|\Delta^2 x_n|}{\rho}\right) \leq 1$

Since F is non decreasing function, therefore

$$\lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{q_n} \leq \lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n} = 0$$

$$\Rightarrow \; \lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{q_n} = 0 \;\Rightarrow\; x \in c_0(\Delta^2, F, q)$$

Hence, $c_0(\Delta^2, F, p) \subseteq c_0(\Delta^2, F, q)$.

Theorem 4.3. For the sequence space $c_0(\Delta^2, F)$ and $c_0(\Delta^2, F, q)$, we have

(1) Let $0 < \inf p_n \leq 1$.Then $c_0(\Delta^2, F, p) \subseteq c_0(\Delta^2, F)$
(2) Let $1 \leq p_n \leq \sup p_n < \infty$. Then $c_0(\Delta^2, F) \subseteq c_0(\Delta^2, F, p)$.

Proof. We prove the results as follows:

(1) Let $x \in c_0(\Delta^2, F, p)$, then $\lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n} = 0$

Since, $0 < \inf p_n \le p_n \le 1$.

$$\lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right) \le \lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n} = 0 \quad \Rightarrow \quad x \in c_0(\Delta^2, F).$$

Therefore, $c_0(\Delta^2, F, p) \subseteq c_0(\Delta^2, F)$.

(2) Let $p_n \ge 1$ for each n and $\sup_{n\in\mathbb{N}} p_n < \infty$. Let $x \in c_0(\Delta^2, F)$, then for each $\epsilon(0 < \epsilon < 1)$ there exists a positive integer N such that $F\left(\frac{|\Delta^2 x_n|}{\rho}\right) \le \epsilon, \ \forall\, n \ge N$.

Since, $1 \le p_n \le \sup p_n < \infty$, we have

$$\lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right)^{p_n} \le \lim_{n\to\infty}\left(F\left(\frac{|\Delta^2 x_n|}{\rho}\right)\right) \le \epsilon < 1.$$

$\Rightarrow \ x \in c_0(\Delta^2, F, p)$. Hence, $c_0(\Delta^2, F) \subseteq c_0(\Delta^2, F, p)$.

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