BAYESIAN ESTIMATION OF FUNCTION OF UNKNOWN PARAMETER OF SOME PARTICULAR CASES OF MODIFIED POWER SERIES DISTRIBUTION

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ABSTRACT

This paper deals with the Bayesian estimation of a function of the unknown parameter θ of the Generalized Negative Binomial Distribution (GNBD) and Generalized Logarithmic Series Distribution (GLSD).These distributions are particular cases of Modified Power Series distribution (MPSD).The prior distribution for the unknown parameter θ varies from distribution to distribution, depending upon the range of θ.On the part of loss functions, the Squared Error Loss Function (SELF) and two different forms of Weighted Squared Error Loss Function (WSELF) has been considered.

Keywords: Modified Power Series distribution, Generalized Negative Binomial Distribution (GNBD), Generalized Logarithmic Series Distribution (GLSD), Bayes Estimator, Squared Error Loss Function (SELF), Weighted Squared Error Loss Function (WSELF).

1. Introduction: A discrete random variable X is said to have Modified Power Series distribution, if its probability mass function (p. m .f.) $p_\theta(x) = P(X = x)$ is given by,

$$p_\theta(x) = \begin{cases} \frac{a(x)\{g(\theta)\}^x}{f(\theta)}, \text{if } x \in S, \theta \in A \\ 0, \text{Otherwise.} \end{cases} \quad (1)$$

Where, θ is unknown parameter of the distribution, $A \subseteq \mathcal{R}$ (the set of real numbers), $a(x) > 0$, S is a subset of the set of non-negative integers, $g(\theta) > 0$ and $f(\theta)$ is a function of θ such that $\sum_{x \in S} p_\theta(x) = f(\theta)$

As mentioned by Gupta (1974) the p. m .f. given by (1) covers a wide range of discrete distributions. When $g(\theta) = \theta$, (1) coincides with the class of discrete distributions as given by Roy and Mitra (1957).

Gupta (1977), has obtained MVUE of $\phi(\theta) = \theta^r, r \geq 1$.For values of $r < 1$,no unbiased estimator of $\phi(\theta)$ exists and hence no MVUE of $\phi(\theta)$ exists. This is a serious limitation of this classical estimator. Singh (2021) obtained Bayes Estimator of $\phi(\theta) = \theta^r, r \in (-\infty, \infty)$.This is an advantage of Bayesian approach over the classical approach as the range of estimation is increased

In this paper, Bayes Estimator of $\psi(\theta) = \theta^r(1-\theta)^s, r, s \in (-\infty, \infty)$. have been obtained for two distributions which are particular cases of the Modified Power Series distribution specified by the p. m. f. (1). This paper is an extension of the recent work by Singh (2021),as when $s = 0$,we get estimate of $\theta^r, r \in (-\infty, \infty)$.

On the part of loss functions, the usual Squared Error Loss Function (SELF)and two different forms of the Weighted Squared Error Loss Function (WSELF) have been taken.

2. Notations and results used:

Let $X_1, X_2, X_3, \ldots X_N$ be a random sample of size N from the p .m. f given by (1).

Then,

$$T_N = \sum_{i=1}^{N} X_i \qquad (2)$$

We shall use the following result as given by Abramowitz and Stegun (1964):

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \qquad (3)$$

$$\Gamma(x) b^{-x} = \int_0^\infty u^{x-1} e^{-bu} du \qquad (4)$$

$$\frac{\Gamma(b-a)\Gamma(a)M(a,b,z)}{\Gamma(b)} = \int_0^1 u^{a-1}(1-u)^{b-a-1} e^{-zu} du \qquad (5)$$

Where, $M(a, b, z)$ is the Confluent Hypergeometric Function and has a series representation given by,

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \qquad (6)$$

Where, $(a)_0 = 1$ amd

$$(a)_n = \prod_{i=1}^{n} (a + i - 1) \qquad (7)$$

For observed value $t_N = \sum_{i=1}^{N} x_i$ of the statistic $T_N = \sum_{i=1}^{N} X_i$, the likelihood function, denoted by $L(\theta)$, is given by,

$$L(\theta) = k\{g(\theta)\}^{t_N}\{f(\theta)\}^{-N} \qquad (8)$$

Where, k is function of $x_1, x_2, x_3, \ldots x_N$ and does not contain θ.

Let $\pi(\theta)$ be the prior probability density function of θ,then the posterior posterior probability density function of θ ,denoted by $\pi(\theta / t_N)$,is given by,

$$\pi(\theta / t_N) = \frac{L(\theta)\pi(\theta)}{\int_A L(\theta)\pi(\theta) d\theta} \qquad (9)$$

Under the Squared Error Loss Function (SELF), $L(\psi(\theta), d) = (\psi(\theta) - d)^2$,the Bayes Estimate of $\psi(\theta)$,denoted by $\widehat{\psi}_B$ is given by,

$$\widehat{\psi}_B = \int_A \psi(\theta)\pi(\theta / t_N) d\theta \qquad (10)$$

Similarly, under the Weighted Squared Error Loss Function (WSELF), $L(\psi(\theta), d) = W(\theta)(\psi(\theta) - d)^2$, where, $W(\theta)$ is a function of θ, the Bayes Estimate of $\psi(\theta)$,denoted by $\widehat{\psi}_W$ is given by,

$$\widehat{\psi}_W = \frac{\int_A W(\theta)\psi(\theta)\pi(\theta / t_N) d\theta}{\int_A W(\theta)\pi(\theta / t_N) d\theta} \qquad (11)$$

We have taken two different forms of $W(\theta)$,as given below:

(i). $W(\theta) = \theta^{-2}$.The Bayes Estimate of $\psi(\theta)$,denoted by $\widehat{\psi}_M$, is known as the Minimum Expected Loss (MELO) Estimate and is given by,

$$\widehat{\psi}_M = \frac{\int_A \theta^{-2}\psi(\theta)\pi(\theta / t_N) d\theta}{\int_A \theta^{-2}\pi(\theta / t_N) d\theta} \qquad (12)$$

This loss function was used by Tummala and Sathe (1978) for estimating reliability of certain life time distributions and by Zellner (1979) for estimating functions of parameters in econometric models.

(ii). $W(\theta) = \theta^{-2} e^{-a\theta}$.The Bayes Estimate of $\psi(\theta)$,denoted by $\widehat{\psi}_E$, is known as the ExponentiallyWeighted Minimum Expected Loss (EWMELO) Estimate and is given by,

$$\widehat{\psi}_E = \frac{\int_A \theta^{-2} e^{-a\theta} \psi(\theta) \pi(\theta / t_N) d\theta}{\int_A \theta^{-2} e^{-a\theta} \pi(\theta / t_N) d\theta} \qquad (13)$$

This type of loss function was used by the author (1997) for the first time in his work for D.Phil.

SELF and two forms of WSELF were used by Singh, the author, (1999) in the study of reliability of a multicomponent system and (2010) in Bayesian Estimation of the mean and distribution function of Maxwell's distribution. Recently, the author again used these loss functions in Bayesian estimation for the MPSD (2021) and for estimating Loss and Risk Functions of a continuous distribution (2021).

Now, we shall some special cases of the p. m. f. given by (1) and obtain corresponding Bayes Estimate of $\psi(\theta) = \theta^r (1-\theta)^s, r, s \in (-\infty, \infty)$ in two different cases.

3.GENERALIZED NEGATIVE BINOMIAL DISTRIBUTION (GNBD)

We shall consider two cases for the Generalized Negative binomial distribution

Case I:

If we take $a(x) = \frac{n\Gamma(n+\beta x)}{\Gamma(x+1)\Gamma(n+\beta x - x+1)}$, $g(\theta) = \theta(1-\theta)^{\beta-1}$, $f(\theta) = (1-\theta)^{-n}$,

$S = \{0,1,2 \dots \infty\}$, $A = (0,1)$, $\beta \geq 0$, $\theta\beta \in (-1,1)$, n being a positive integer, the corresponding discrete random variable X is said to have Generalized Negative Binomial distribution.

In this case,

$$L(\theta) = c\theta^{t_N} (1-\theta)^{t_N(\beta-1)+nN} \qquad (14)$$

Where, c is a constant and does not involve θ

Since, in this case, $0 < \theta < 1$, we have taken two different prior distributions, namely, $\pi_1(\theta)$ and $\pi_2(\theta)$ as given below:

$$\pi_1(\theta) = \begin{cases} \frac{\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)}, \text{if } p > 0, q > 0, 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \qquad (15)$$

And,

$$\pi_2(\theta) = \begin{cases} \frac{e^{-b\theta}\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)M(p,p+q,-b)}, \text{if } p > 0, q > 0, 0 < \theta < 1, b \geq 0 \\ 0, \text{Otherwise.} \end{cases} \qquad (16)$$

Where,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (17)$$

The posterior p. d .f. of θ , corresponding to the prior $\pi_1(\theta)$, denoted by $\pi_1(\theta / t_N)$, is given by,

$$\pi_1(\theta / t_N) = \begin{cases} \frac{\theta^{t_N+p-1}(1-\theta)^{t_N(\beta-1)+nN+q-1}}{B(t_N+p,,(\beta-1)t_N+nN+q)}, \text{if } p > 0, q > 0, 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (18)$$

Similarly, posterior p. d .f. of θ , corresponding to the prior $\pi_2(\theta)$, denoted by $\pi_2(\theta / t_N)$,is given by

$$\pi_2(\theta / t_N) = \begin{cases} \frac{e^{-b\theta}\theta^{t_N+p-1}(1-\theta)^{t_N(\beta-1)+nN+q-1}}{K}, \text{if } p > 0, q > 0, 0 < \theta < 1, b \geq 0 \\ 0, \text{Otherwise.} \end{cases} \quad (19)$$

Where,

$$K = B(t_N + p, (\beta - 1)t_N + nN + q)M(t_N + p, p + q + \beta t_N + nN, -b) \quad (20)$$

Under the SELF and corresponding to the posterior distribution given by (18), Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{1B}$ is given by,

$$\widehat{\psi}_{1B} = \frac{B(t_N+p+r,(\beta-1)t_N+nN+q+s)}{B(t_N+p,,(\beta-1)t_N+nN+q)} \quad (21)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (18), the MELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{1M}$,is given by,

$$\widehat{\psi}_{1M} = \frac{B(t_N+p+r-2,,(\beta-1)t_N+nN+q+s)}{B(t_N+p-2,,(\beta-1)t_N+nN+q)} \quad (22)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta}$ and corresponding to the posterior distribution given by (18), the EWMELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{1E}$ is given by,

$$\widehat{\psi}_{1E} = \frac{B(t_N+p+r-2,,(\beta-1)t_N+nN+q+s)M_2}{B(t_N+p-2,,(\beta-1)t_N+nN+q)M_1} \quad (23)$$

Where,

$$M_1 = M(t_N + p - 2,, p + q + \beta t_N + nN - 2, -a) \quad (24)$$

$$M_2 = M(t_N + p + r - 2, p + q + \beta t_N + nN + r + s - 2, -a) \quad (25)$$

On the other hand, under the SELF and corresponding to the posterior distribution given by (19), Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{2B}$,is given by,

$$\widehat{\psi}_{2B} = \frac{B(t_N+p+r,,(\beta-1)t_N+nN+q+s)M_4}{B(t_N+p,,(\beta-1)t_N+nN+q)M_3} \quad (26)$$

Where,

$$M_3 = M(t_N + p, p + q + \beta t_N + nN, -b) \quad (27)$$

$$M_4 = M(t_N + p + r, p + q + \beta t_N + nN + r + s, -b) \quad (28)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (19), the MELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{2M}$ is given by,

$$\widehat{\psi}_{2M} = \frac{B(t_N+p+r-2,,(\beta-1)t_N+nN+q+s)M_6}{B(t_N+p-2,,(\beta-1)t_N+nN+q)M_5} \quad (29)$$

Where,

$$M_5 = M(t_N + p - 2, p + q + \beta t_N + nN - 2, -b) \quad (30)$$

$M_6 = M(t_N + p + r - 2, p + q + \beta t_N + nN + r + s - 2, -b)$ (31)

Finally, under the WSELF, when $W(\theta) = \theta^{-2} e^{-a\theta}$ and corresponding to the posterior distribution given by (19),the EWMELO Estimate of $\psi(\theta) = \theta^r (1-\theta)^s$,denoted by $\widehat{\psi}_{2E}$ is given by,

$$\widehat{\psi}_{2E} = \frac{B(t_N+p+r-2,,(\beta-1)t_N+nN+q+s)M_8}{B(t_N+p-2,,(\beta-1)t_N+nN+q)M_7} \quad (32)$$

Where,

$M_7 = M(t_N + p - 2, p + q + \beta t_N + nN - 2, -(a + b))$ (33)

$M_8 = M(t_N + p + r - 2, p + q + \beta t_N + nN + r + s - 2, -(a + b))$ (34)

Remark (1): For $s = 0$,we get Bayes estimator of $\phi(\theta) = \theta^r, r \in (-\infty, \infty)$ as derived recently by the author,(2021), while ,for r= 0 ,we get Bayes estimator of $(1-\theta)^s, s \in (-\infty, \infty)$

SPECIAL CASE: Since, for $\beta = 1$,the GNBD coincides with the Negative Binomial Distribution (NBD), all results as derived above give, Bayes Estimate of $\psi(\theta)$ for the NBD when $\beta = 1$. Additionally ,when $\beta = 1$ and n =1,we get Bayes Estimate of $\psi(\theta)$ for the Geometric distribution.

In this case the probability mass function of Geometric distribution is given by

$$p_\theta(x) = \begin{cases} (1-\theta)\theta^x, \text{if } x = 0,1,2, \dots; 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (35)$$

Case II:

If we take $a(x) = \frac{n\Gamma(n+\beta x)}{\Gamma(x+1)\Gamma(n+\beta x-x+1)}$, $g(\theta) = (1-\theta)\theta^{\beta-1}$, $f(\theta) = \theta^{-n}$,

$S = \{0,1,2 \dots \infty\}$, $A = (0,1)$, $\beta \geq 0, (1-\theta)\beta \in (-1,1)$,n being a positive integer. We get another form of the Generalized Negative Binomial distribution. This form is not given in Gupta (1974) Since, in this case,$0 < \theta < 1$,we have taken two different prior distributions, namely, $\pi_1(\theta)$ and $\pi_2(\theta)$ as given in (15) and (16).

In this case,

$$L(\theta) = c(1-\theta)^{t_N} \theta^{t_N(\beta-1)+nN} \quad (36)$$

Where, c is a constant and does not involve θ

The posterior p. d .f. of θ , corresponding to the prior $\pi_1(\theta)$, denoted by $\pi_{11}(\theta / t_N)$,is given by,

$$\pi_{11}(\theta / t_N) = \begin{cases} \frac{(1-\theta)^{t_N+q-1}\theta^{t_N(\beta-1)+nN+p-1}}{B((\beta-1)t_N+nN+p, t_N+q)}, \text{if } p > 0, q > 0, 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (37)$$

Similarly, posterior p. d .f. of θ , corresponding to the prior $\pi_2(\theta)$, denoted by $\pi_{22}(\theta / t_N)$,is given by

$$\pi_{22}(\theta / t_N) = \begin{cases} \frac{e^{-b\theta}(1-\theta)^{t_N+q-1}\theta^{t_N(\beta-1)+nN+p-1}}{C}, \text{if } p > 0, q > 0, 0 < \theta < 1, b \geq 0 \\ 0, \text{Otherwise.} \end{cases} \quad (38)$$

Where,

$$C = B\big((\beta-1)t_N + nN + p, t_N + q\big)M(p + (\beta-1)t_N + nN, \beta t_N + nN + p + q, -b) \quad (39)$$

Under the SELF and corresponding to the posterior distribution given by (37), Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{11B}$ is given by,

$$\widehat{\psi}_{11B} = \frac{B((\beta-1)t_N+nN+p+r,t_N+q+s)}{B((\beta-1)t_N+nN+p,t_N+q)} \quad (40)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (37), the MELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{1M}$,is given by,

$$\widehat{\psi}_{11M} = \frac{B((\beta-1)t_N+nN+p+r-2,t_N+q+s)}{B((\beta-1)t_N+nN+p,t_N+q)} \quad (41)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta}$ and corresponding to the posterior distribution given by (37), the EWMELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{11E}$ is given by,

$$\widehat{\psi}_{11E} = \frac{B((\beta-1)t_N+nN+p+r-2,t_N+q+s)M_{10}}{B((\beta-1)t_N+nN+p,t_N+q)M_9} \quad (42)$$

Where,

$$M_9 = M((\beta-1)t_N + nN + p - 2, p + q + \beta t_N + nN - 2, -a) \quad (43)$$

$$M_{10} = M((\beta-1)t_N + p + nN + r - 2, p + q + \beta t_N + nN + r + s - 2, -a) \quad (44)$$

On the other hand, under the SELF and corresponding to the posterior distribution given by (38), Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{22B}$,is given by,

$$\widehat{\psi}_{22B} = \frac{B((\beta-1)t_N+nN+p+r,t_N++q+s)M_{12}}{B((\beta-1)t_N+nN+p,t_N+q)M_{11}} \quad (45)$$

Where,

$$M_{11} = M((\beta-1)t_N + nN + p\,, p + q + \beta t_N + nN - b) \quad (46)$$

$$M_{12} = M((\beta-1)t_N + p + nN + r, p + q + \beta t_N + nN + r + s, -b) \quad (47)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (38), the MELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{22M}$ is given by,

$$\widehat{\psi}_{22M} = \frac{B((\beta-1)t_N+p+nN+r-2,t_N++q+s)M_{14}}{B((\beta-1)t_N+nN+p,t_N+q)M_{13}} \quad (48)$$

Where,

$$M_{13} = M((\beta-1)t_N + p + nN - 2, p + q + \beta t_N + nN - 2, -b) \quad (49)$$

$$M_{14} = M((\beta-1)t_N + p + nN + r - 2, p + q + \beta t_N + nN + r + s - 2, -b) \quad (50)$$

Finally, under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta}$ and corresponding to the posterior distribution given by (38),the EWMELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{22E}$ is given by,

$$\widehat{\psi}_{22E} = \frac{B((\beta-1)t_N+p+nN+r-2,t_N++q+s)M_{16}}{B((\beta-1)t_N+nN+p,t_N+q)M_{15}} \quad (51)$$

Where,

$$M_{15} = M((\beta-1)t_N + p + nN - 2, p + q + \beta t_N + nN - 2, -(a+b)) \quad (52)$$

$$M_{16} = M((\beta-1)t_N + p + nN + r - 2, p + q + \beta t_N + nN + r + s - 2\,, -(a+b)) \quad (53)$$

Remark (1): For $s=0$,we get Bayes estimator of $\phi(\theta)=\theta^{r}, r\in(-\infty,\infty)$ while ,for $r=0$,we get Bayes estimator of $(1-\theta)^{s}, s\in(-\infty,\infty)$

SPECIAL CASE: Since, for $\beta=1$,the GNBD coincides with the Negative Binomial Distribution (NBD), all results as derived above give, Bayes Estimate of $\psi(\theta)$ for the NBD when $\beta=1$. Additionally ,when $\beta=1$ and n =1,we get Bayes Estimate of $\psi(\theta)$ for the Geometric distribution.

In this case the probability mass function of Geometric distribution is given by

$$p_{\theta}(x)=\begin{cases}\theta(1-\theta)^{x}, \text{if } x=0,1,2,\ldots; 0<\theta<1\\ 0, \text{Otherwise.}\end{cases} \quad (54)$$

4.GENERALIZED LOGARITHMIC SERIES DISTRIBUTION (GLSD)

We shall consider two cases for the Generalized Negative binomial distribution

Case I:

If we take $a(x)=\frac{\Gamma(\beta x)}{\Gamma(x+1)\Gamma(\beta x-x+1)}$, $g(\theta)=\theta(1-\theta)^{\beta-1}$, $f(\theta)=-\ln(1-\theta)$,

$S=\{1,2\ldots\infty\}$, $A=(0,1)$,$\beta\geq 1$, $\theta\beta\in(0,1)$,n being a positive integer, the corresponding discrete random variable X is said to have Generalized Logarithmic Series distribution.

In this case,

$$L(\theta)=c(1-\theta)^{t_N}\theta^{t_N(\beta-1)}(-\ln(1-\theta))^{-N} \quad (55)$$

Where, c is a constant and does not involve θ

Since in this case, $0<\theta<1$,we take $\pi_3(\theta)$ as the p. d. f. of Negative Log Gamma distribution given by

$$\pi_3(\theta)=\begin{cases}\frac{(k+1)^{N+1}(1-\theta)^{k}\{-\ln(1-\theta)\}^{N}}{\Gamma(N+1)}, \text{if } k>0, 0<\theta<1\\ 0, \text{Otherwise.}\end{cases} \quad (56)$$

Where, N, a positive integer is same as the size of the random sample.

The posterior p. d .f. of θ ,denoted by $\pi_3(\theta/t_N)$, is given by

$$\pi_3(\theta/t_N)=\begin{cases}\frac{\theta^{t_N}(1-\theta)^{t_N(\beta-1)+k}}{B(t_N+1,(\beta-1)t_N+k+1)}, \text{if } k>0,, 0<\theta<1\\ 0, \text{Otherwise.}\end{cases} \quad (57)$$

Under the SELF and corresponding to the posterior distribution given by (57), Bayes Estimate of $\psi(\theta)=\theta^{r}(1-\theta)^{s}$,denoted by $\widehat{\psi}_{3B}$ is given by,

$$\widehat{\psi}_{3B}=\frac{B(t_N+r+1,(\beta-1)t_N+k+s+1)}{B(t_N+1,(\beta-1)t_N+k+1)} \quad (58)$$

Similarly, under the WSELF, when $W(\theta)=\theta^{-2}$ and corresponding to the posterior distribution given by (57), the MELO Estimate of $\psi(\theta)=\theta^{r}(1-\theta)^{s}$,denoted by $\widehat{\psi}_{3M}$ is given by,

$$\widehat{\psi}_{3M}=\frac{B(t_N+r-1,(\beta-1)t_N+k+s+1)}{B(t_N-1,(\beta-1)t_N+k+1)} \quad (59)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta}$ and corresponding to the posterior distribution given by (57), the EWMELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\hat{\psi}_{3E}$ is given by,

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$$\hat{\psi}_{3E} = \frac{B(t_N+r-1,(\beta-1)t_N+k+s+1)M_{18}}{B(t_N-1,(\beta-1)t_N+k+1)M_{17}} \quad (60)$$

Where,

$$M_{17} = M(t_N - 1, \beta t_N + k, -a) \quad (61)$$

$$M_{18} = M(t_N + r - 1, \beta t_N + r + k + s, -a) \quad (62)$$

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SPECIAL CASE: Since, for $\beta = 1$,the GLSD coincides with the Logarithmic Series Distribution (LSD), all results as derived above give, Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$ for the LSD when $\beta = 1$.

In this case the probability mass function of Logarithmic Series Distribution (LSD) is given by

$$p_\theta(x) = \begin{cases} \dfrac{\theta^x}{x(-\ln(1-\theta))}, \text{if } x = ,1,2,\ldots.; 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (63)$$

Case II:

If we take $a(x) = \frac{\Gamma(\beta x)}{\Gamma(x+1)\Gamma(\beta x - x + 1)}$, $g(\theta) = (1-\theta)\theta^{\beta-1}$, $f(\theta) = -\ln\theta$,

$S = \{1,2 \ldots \infty\}$, $A = (0,1)$, $\beta \geq 1$, $(1-\theta)\beta \in (0,1)$,n being a positive integer. We get another form of the Generalized Logarithmic Series distribution. This form is not given in Gupta (1974)

In this case,

$$L(\theta) = c(1-\theta)^{t_N}\theta^{t_N(\beta-1)}(-\ln\theta)^{-N} \quad (64)$$

Where, c is a constant and does not involve θ

Since in this case, $0 < \theta < 1$,we take $\pi_{31}(\theta)$ as the p. d. f. of Negative Log Gamma distribution given by

$$\pi_{31}(\theta) = \begin{cases} \dfrac{(k+1)^{N+1}\theta^k\{-\ln\theta\}^N}{\Gamma(N+1)}, \text{if } k > 0, ,0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (65)$$

Where, N, a positive integer is same as the size of the random sample.

The posterior p. d .f. of θ ,denoted by $\pi_{31}(\theta / t_N)$, is given by

$$\pi_{31}(\theta / t_N) = \begin{cases} \dfrac{(1-\theta)^{t_N}\theta^{t_N(\beta-1)+k}}{B(,(\beta-1)t_N+k+1,t_N+1)}, \text{if } k > 0, ,0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (66)$$

Under the SELF and corresponding to the posterior distribution given by (66), Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\hat{\psi}_{3B}$ is given by,

$$\hat{\psi}_{31B} = \frac{B(t_N+s+1,(\beta-1)t_N+k+r+1)}{B(t_N+1,(\beta-1)t_N+k+1)} \quad (67)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (66), the MELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{31M}$ is given by,

$$\widehat{\psi}_{31M} = \frac{B(t_N+s+1,(\beta-1)t_N+k+r-1)}{B(t_N+1,(\beta-1)t_N+k-1)} \quad (68)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta}$ and corresponding to the posterior distribution given by (66), the EWMELO Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$,denoted by $\widehat{\psi}_{31E}$ is given by,

,

$$\widehat{\psi}_{31E} = \frac{B(t_N+s+1,(\beta-1)t_N+k+r-1)M_{20}}{B(t_N+1,(\beta-1)t_N+k-1)M_{19}} \quad (69)$$

Where,

$$M_{19} = M((\beta-1)t_N + k - 1, \beta t_N + k, -a) \quad (70)$$

$$M_{20} = M((\beta-1)t_N + k + r - 1, \beta t_N + r + k + s, -a) \quad (71)$$

.

SPECIAL CASE: Since, for $\beta = 1$,the GLSD coincides with the Logarithmic Series Distribution (LSD), all results as derived above give, Bayes Estimate of $\psi(\theta) = \theta^r(1-\theta)^s$ for the LSD when $\beta = 1$.

In this case the probability mass function of Logarithmic Series Distribution (LSD) is given by

$$p_\theta(x) = \begin{cases} \frac{(1-\theta)^x}{x(-\ln\theta)}, \text{if } x = ,1,2, \dots; 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (72)$$

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