AN INTRODUCTION OF SUMUDU TRANSFORM

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ABSTRACT

In this paper, we see the definition, some basic properties and fundamental properties of Sumudu transform, relationships between Laplace and Sumudu transforms and Existence of Sumudu transform.

KEYWORDS

Sumudu Transform, Gamma Function, Laplace Transform.

INTRODUCTION

The Sumudu transform is introduced by Watugula. Sumudu transform may be used to solve problems without resorting to a new frequency domain .Due to its simple formulation and consequent special and useful properties, the Sumudu transform has already shown much promise. It is revealed here in and elsewhere that it can help to solve intricate problems in engineering mathematics and applied sciences. However, despite the potential presented by this new operator, only few theoretical investigations have appeared in the literature, over a fifteen-year period. Most of the available transform theory books, if not all, do not refer to the Sumudu transform. Even in relatively recent well known comprehensive handbooks, such as Debnath and Poularikas, no mention of the Sumudu transform can be found.

SUMUDU TRANSFORM

Watugala introduced a new transform and named as Sumudu transform which is defined by the following formula

$$F(u) = \mathcal{S}[f(t);u] = \frac{1}{u}\int_0^\infty e^{-\left(\frac{1}{u}\right)} f(t)dt, u \in (-\tau_1, \tau_2)$$

BASIC SUMUDU TRANSFORM PROPERTIES

Sumudu transform for $f \in A$:

$$G(u) = \mathcal{S}[f(t)] = \int_0^\infty f(ut)e^{-t}\, dt, u \in (-\tau_1, \tau_2)$$

Duality with Laplace transforms:

$$G(u) = \frac{F(1/u)}{u}\ , F(s) = \frac{G(1/s)}{s}$$

Linearity Property:

$$\mathcal{S}\big[af(t) + bg(t) = a\mathcal{S}[f(t)] + b\mathcal{S}[g(t)]\big]$$

Sumudu Transform of Function Derivatives:

$$G_1(u) = \mathcal{S}[f'(t)] = \frac{G(u)-f(0)}{u} = \frac{G(u)}{u} - \frac{f(0)}{u}$$

$$G_2(u) = \mathcal{S}[f''(t)] = \frac{G(u)-f(0)}{u^2} = \frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(o)}{u}$$

$$G_n(u) = \mathcal{S}[f^n(t)] = \frac{G(u)}{u^n} - \frac{f(0)}{u^n} - \cdots - \frac{f^{n-1}(o)}{u}$$

Sumudu transform of integral of a function:

$$\mathcal{S}\left[\int_0^1 f(\tau)d\tau\right] = uG(u)$$

SUMUDU TRANSFORM FUNDAMENTAL PROPERTIES

THE DISCRETE SUMUDU TRANSFORM

Over the set of functions

$$A = \left\{\left(f(t)\middle|\exists M, \tau_{1,}\tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, if\ t \in (-1)^j \times [0,\infty)\right)\right\}, \qquad (1)$$

the Sumudu transform is defined by

$$G(u) = \mathcal{S}[f(t)] = \int_0^\infty f(ut)e^{-t}\,dt, u \in (-\tau_1, \tau_2) \qquad (2)$$

Among others, the Sumudu transform was shown to have units preserving properties and hence may be used to solve problems without resorting to the frequency domain. As will be seen below, this is one of many strength points for this new transform, especially with respect to applications in problems with

physical dimensions. In fact, the Sumudu transform which is itself linear, preserves linear functions, and hence in particular does not change units (see for instance Watugala or Belgacem et al).Theoretically, this point may perhaps best be illustrated as an implication of this more global result.

THEOREM:1

The Sumudu transform amplifies the coefficients of the power series function,

$$f(t) = \sum_{n=0}^\infty a_n t^n \qquad (1.1)$$

by sending it to the power series function,

$$G(u) = \sum_{n=0}^\infty n!\, a_n u^n \qquad (1.2)$$

PROOF :

Let $f(t)$ be in A. If $f(t) = \sum_{n=0}^\infty a_n t^n$ in some interval $I \subset$R, then by Taylor's function expansion theorem,

$$f(t) = \sum_{k=0}^\infty \frac{f^{(n)}(o)}{n!} t^n \qquad (1.3)$$

Therefore, by (2), and that of the gamma function Γ , we have

$$\mathcal{S}[f(t)] = \int_0^\infty \sum_{k=0}^\infty \frac{f^{(n)}(o)}{n!}(ut)^n e^{-t}\,dt$$

$$= \sum_{k=0}^\infty \frac{f^{(n)}(o)}{n!} u^n \int_0^\infty t^n\, e^{-t}dt$$

$$= \sum_{k=0}^{\infty} \frac{f^{(n)}(o)}{n!} u^n \, \Gamma(\mathrm{n}+1)$$

$$= \sum_{n=0}^{\infty} f^{(n)}(0) u^n \qquad (1.4)$$

Consequently, it is perhaps worth noting that since

$$\mathcal{S}[(1+t)^m] = \mathcal{S} \sum_{n=0}^{m} C_n^m t^n$$

$$= \mathcal{S} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} u^n$$

$$\mathcal{S}[(1+t)^m] = \sum_{n=0}^{m} \frac{m!}{(m-n)!} u^n$$

$$= \sum_{n=0}^{m} P_n^m u^n \qquad (1.5)$$

the Sumudu transform sends combinations, C_n^m into permutations, P_n^m, and hence may seem to incur more order into discrete systems.

Also, a requirement that $\mathcal{S}[f(t)]$ converges, in an interval containing u=0, is provided by the following conditions when satisfied, namely, that

$$(i) f^{(n)}(0) \to 0 \; as \; n \to \infty,$$

$$(ii) \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0)}{f^{(n)}(0)} u \right| < 1 \qquad (1.6)$$

This means that the convergence radius r of $\mathcal{S}[f(t)]$ depends on the sequence

$f^{(n)}(0)$, since

$$r = \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0)}{f^{(n)}(0)} \right| \qquad (1.7)$$

Clearly, the Sumudu transform may be used as a signal processing or a detection tool,especially in situations where the original signal has a decreasing power tail .

However, care must be taken, especially if the power series is not highly decaying. This next example may instructively illustrate the stated concern. For instance, consider the function

$$f(t) = \begin{cases} In\,(t+1) & if\ t \in (-1,1] \\ o & otherwise \end{cases} \quad (1.8)$$

Since $f(t) = \sum_{n=1}^{\infty}(-1)^{n-1}\, t^n/_n$ the expect for u=0

$$\mathcal{S}[f(t)] = \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\,u^n \quad (1.9)$$

Diverges throught , because its convergens radius

$$r = \lim_{n\to\infty}\left|\frac{(-1)^{n-1}(n-1)!}{(-1)^n n!}\right|$$

$$\lim_{n\to\infty}\frac{1}{n} = 0 \quad (1.10)$$

RELATION BETWEEN SUMUDU AND LAPLACE TRANSFORM

In our study, we use the following convolution notation: double convolution between two continuous functions F (x, y) and G(x, y) given by

$$F_1(x,y) ** F_2(x,y) = \int_0^y \int_0^x F_1(x-\theta_1, y-\theta_2)F_2(\theta_1,\theta_2)d\theta_1 d\theta_2$$

The single Sumudu transform is defined over the set of the functions

$$A = \{(f(t)|\exists M, \tau_{1,}\tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, if\ t \in (-1)^j \times [0,\infty))\}$$

by $G(u) = \mathcal{S}[f(t)] = \int_0^\infty f(ut)e^{-t}\,dt, u \in (-\tau_1, \tau_2)$

A sufficient condition for the existence of the Sumudu transform of a function f is of exponential order, that is, there exist real constants

M > 0, K_1, and K_2 , such that $|f(t,x)| \le Me^{\frac{t}{K_1}+\frac{x}{K_2}}$

EXISTENCE OF THE SUMUDU TRANSFORM

THEOREM:2

If f is of exponential order, then its Sumudu transform $\mathcal{S}[f(t,x)] = F(v,u)$exists and is given by

$$F(v,u) = \int_0^\infty \int_0^\infty e^{-\frac{t}{v}-\frac{x}{u}}f(t,x)\,dtdx$$

where $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}, \frac{1}{v} = \frac{1}{\mu} + \frac{i}{\xi}$

The defining integral for F exists at points

$\frac{1}{u} + \frac{1}{v} = \frac{1}{\eta} + \frac{1}{\mu} + \frac{i}{\tau} + \frac{i}{\xi}$ in the right half plane $\frac{1}{\eta} + \frac{1}{\mu} > \frac{1}{K_1} + \frac{1}{K_2}$.

PROOF:

Using $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ *and* $\frac{1}{v} = \frac{1}{\mu} + \frac{i}{\xi}$

We can express $F(v, u)$ as

$$F(v,u) = \int_0^\infty \int_0^\infty f(t,x) cos(\frac{t}{\tau} + \frac{x}{\xi}) e^{-\frac{t}{\eta} - \frac{x}{\mu}} dtdx$$

$$-i \int_0^\infty \int_0^\infty f(t,x) sin(\frac{t}{\tau} + \frac{x}{\xi}) e^{-\frac{t}{\eta} - \frac{x}{\mu}} dtdx$$

Then for values of $\frac{1}{\eta} + \frac{1}{\mu} > \frac{1}{K_1} + \frac{1}{K_2}$ we have

$$\int_0^\infty \int_0^\infty |f(t,x)| \left|cos(\frac{t}{\tau} + \frac{x}{\xi})\right| e^{-\frac{t}{\eta} - \frac{x}{\mu}} dtdx \le M \int_0^\infty \int_0^\infty e^{\left(\frac{1}{K_1} - \frac{1}{\eta}\right)t + \left(\frac{1}{K_2} - \frac{1}{\mu}\right)x} dtdx$$

$$\le M \left(\frac{\eta K_1}{\eta - K_1}\right) \left(\frac{\eta K_2}{\mu - K_2}\right)$$

and

$$\int_0^\infty \int_0^\infty |f(t,x)| \left|sin(\frac{t}{\tau} + \frac{x}{\xi})\right| e^{-\frac{t}{\eta} - \frac{x}{\mu}} dtdx \le M \int_0^\infty \int_0^\infty e^{\left(\frac{1}{K_1} - \frac{1}{\eta}\right)t + \left(\frac{1}{K_2} - \frac{1}{\mu}\right)x} dtdx$$

$$\le M \left(\frac{\eta K_1}{\eta - K_1}\right) \left(\frac{\eta K_2}{\mu - K_2}\right)$$

which imply that the integrals defining the real and imaginary parts of F exist for value of $Re\left(\frac{1}{u} + \frac{1}{\mu}\right) > \frac{1}{K_1} + \frac{1}{K_2}$, and this completes the proof.

Thus, we note that for a function f, the sufficient conditions for the existence of the Sumudu transform are to be piecewise continuous and of exponential order.

We also note that the double Sumudu transform of function f(t, x) is defind by

$$F(v,u) = \mathcal{S}_2[f(t,x);(v,u)] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{t}{v}+\frac{x}{u}\right)} f(t,x)\,dtdx$$

where, s_2 indicates double Sumudu transform and $f(t,x)$ is a function which can be expressed as a convergent infinite series. Now, it is well known that the derivative of convolution for two functions f and g is given by

$$\frac{d}{dx}(f*g)(x) = \frac{d}{dx}f(x)*g(x) or\ f(x)*\frac{d}{dx}g(x)$$

and it can be easily proved that Sumudu transform is

$$\mathcal{S}\left[\frac{d}{dx}(f*g)(x);v\right] = u\mathcal{S}\left[\frac{d}{dx}f(x);u\right]\mathcal{S}[g(x);u] \text{ or}$$

$$= u\mathcal{S}[f(x);u]\mathcal{S}\left[\frac{d}{dx}g(x);u\right].$$

The double Sumudu and double Laplace transforms have strong relationships that may be expressed either as

$$(I)\ uvF(u,v) = £_2\left(f(x,y);\left(\frac{1}{u},\frac{1}{v}\right)\right)$$

Or $(II)\ psF(p,s) = £_2\left(f(x,y);\left(\frac{1}{p},\frac{1}{s}\right)\right)$

where $£_2$ represents the operation of double Laplace transform. In particular, the double Sumudu and double Laplace transforms interchange the image of sin(x + t) and cos(x + t). It turns out that

$$s_2[\sin(x+t)] = £_2[\cos(x+t) = \frac{u+v}{(1+u)^2(1+v)^2}$$

And

$$s_2[\cos(x+t)] = £_2[\sin(x+t) = \frac{1}{(1+u)^2(1+v)^2}$$

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