AN ANALYSIS ON THE INTEGRAL MODULUS OF CONTINUITY OF SINE SERIES OF ORDER K BELONGING TO CLASS S AND OF 1ST ORDER BELONGING TO CLASS R

ANJU KHASA, RESEARCH SCHOLAR, DEPARTMENT OF MATHEMATICS, KALINGA UNIVERSITY, RAIPUR.

1. INTRODUCTION

Let F(x) be a function of period 2π in $L_p(1 \le p < \infty)$. Then the integral modulus of continuity of order k of F in L_p is defined by

$$\omega_p^k(h; F) = \sup_{0<|t|\le h} \left\| \Delta_t^k F(x) \right\|_{L_p},$$

where

$$\Delta_t^k F(x) = \sum_{\alpha=0}^{k} (-1)^{k-\alpha} \binom{k}{\alpha} F(x+\alpha t)$$

and $\|.\|_{L_p}$ denotes the norm in L_p.

Concerning the integral modulus of continuity of order 1 of a sine series whose coefficients from a quasi-convex null sequence, Izumi [20] and Taljakovskii [40] have obtained some interesting estimates. The class of quasi-convex null sequence has further been extended by Teljakovskii [45] in the following form :

Let

(1.1) $$\sum_{k=1}^{\infty} a_k \sin kx$$

be a sine series satisfying $a_k = o(1)$, $k \to \infty$. If there exists a sequence $\langle A_k \rangle$ such that

(1.2) $A_k \downarrow 0,\ k \to \infty,$

(1.3) $$\sum_{k=0}^{\infty} A_k < \infty,$$

(1.4) $|a_k - a_{k+1}| = |\Delta a_k| \le A_k$ for all k,

then we say that (1.1) belongs to the class S.

Setting $A_k = \sum_{m=k}^{\infty} \left|\Delta^2 a_m\right|$, we observe that every quasi-convex null sequence satisfies the condition S.

Let g(x) be the sum of the sine series (1.1) belonging to the class S. Teljakovskii [45] showed that the condition

$$(1.5) \qquad \sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$$

is sufficient for the integration of the series (1.1) belonging to the class S.

The aim of this chapter is to find an estimate for the integral modulus of continuity of order k of the series (1.1) belonging to the class S.

2. Results :- We establish the following theorem :

Theorem :- If (1.1) belongs to the class S and (1.5) holds, then

$$\omega_1^k\left(\frac{1}{n}; g\right) \le B_k n^{-k} \log n \sum_{v=1}^{n} (v+1)^{k+1} \Delta A_v$$

$$+B_k \sum_{v=n+1}^{\infty} (v+1)\left(1+\log\frac{v}{n}\right)\Delta A_v,$$

where B_k is a constant depending upon k and not necessarily the same at each occurance.

Letting $A_v = \sum_{m=v}^{\infty} \left|\Delta^2 a_m\right|$, the case k = 1 of our theorem yields

Corollary. If $<a_k>$ is a quasi-convex null sequence satisfying (1.5), then

$$\omega_1\left(\frac{1}{n}; g\right) \le Bn^{-1} \log n \sum_{v=1}^{n} (v+1)^2 \left|\Delta^2 a_v\right|,$$

$$+B \sum_{v=n+1}^{\infty} (v+1)\left(1+\log\frac{v}{n}\right)\left|\Delta^2 a_v\right|.$$

This result corresponds to a theorem of Izumi [20] as stated in Teljakovskii [40].

Proof of the Theorem : Under the assumed hypothesis, g is integrable. Since the symmetry of the function implies

$$\left|\Delta_t^k g(-x)\right| - \left|\Delta_{-t}^k g(x)\right|,$$

therefore

$$\int_{-\pi}^{\pi}\left|\Delta_t^k g(x)\right| dx = \int_0^{\pi}\left|\Delta_{-t}^k g(x)\right| dx + \int_0^{\pi}\left|\Delta_t^k g(x)\right| dx.$$

Hence, to prove the theorem, it is sufficient to estimate

$$\int_0^{\pi}\left|\Delta_{\pm t}^k g(x)\right| dx, \qquad \text{for } 0 < t \le \frac{\pi}{n}.$$

We write

(2.1) $$\int_0^{\pi}\left|\Delta_{\pm t}^k g(x)\right| dx = \int_0^{(k+1)\frac{\pi}{n}} + \int_{(k+1)\frac{\pi}{n}}^{\pi}$$

$$= I_1 + I_2, \text{ say.}$$

We first estimate I_1. Denoting by $D_v(x)$ the kernel conjugate to the Dirichlet kernel, the use of partial summation yields

$$g(x) = \sum_{v=1}^{\infty}\Delta a_v D_v(x)$$

$$= \sum_{v=1}^{\infty} A_v \frac{\Delta a_v}{A_v} D_v(x)$$

$$= \sum_{v=1}^{\infty}\Delta A_v \sum_{i=o}^{v}\frac{\Delta a_i}{A_i} D_i(x).$$

Then

$$I_1 \le \sum_{v=1}^{n}\left[\Delta A_v \int_0^{(k+1)\frac{\pi}{n}}\sum_{i=0}^{v}\left|\Delta_{\pm t}^k D_i(x)\right| dx\right] + \int_0^{(k+1)\frac{\pi}{n}}\left|\Delta_{\pm t}^k \sum_{v=n+1}^{\infty}\Delta A_v \sum_{i=0}^{v} D_i(x)\right| dx$$

$$= I_{11} + I_{12}, \text{ say.}$$

If $D_i^{(k)}(x)$ denotes the k^{th} derivative of $D_i(x)$, then to estimate I_{11} we use the equality (Aljancic [5], Ram [27])

$$\left|D_i^{(k)}(x)\right| = \begin{cases} B_k i^{k+1}, & 0 \le x \le \pi \\ B_k i^k x^{-1}, & 0 < x \le \pi \end{cases} \quad (k = 1, 2, \ldots)$$

and obtain

$$I_{11} \le B_k t^k \sum_{v=1}^{n} \Delta A_v \int_0^{(k+1)\frac{\pi}{n}} \left(\sum_{i=0}^{v} \left| D_i^{(k)}(x \pm \theta_i t) \right| \right) dx \qquad (0 < \theta_i < k)$$

$$\le B_k n^{-k} \sum_{v=1}^{n} \Delta A_v (v+1)^{k+1}.$$

To estimate I_{12}, we use the inequality (Timan [47])

$$\frac{1}{\pi} \int_0^{c/n} |D_v(x)|\, dx \le \frac{2}{\pi} \log \frac{v}{n} + o(1), \qquad c > 0, \quad v \ge n$$

and obtain

$$I_{12} \le B_k \left(\sum_{v=n+1}^{\infty} \Delta A_v \sum_{i=2}^{v} \left[\log \frac{i}{n} + o(1) \right] \right)$$

$$= B_k \left(\sum_{v=n+1}^{\infty} \Delta A_v \left[(v+1) \log \frac{v}{n} + (v+1) \right] \right).$$

It follows therefore that

(2.3) $$I_1 \le B_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+1} \Delta A_v$$

$$+ B_k \left[\sum_{v=n+1}^{\infty} (v+1) \left(1 + \log \frac{v}{n} \right) \Delta A_v \right].$$

To estimate I_2, we have

$$I_2 = \int_{(k+1)\frac{\pi}{n}}^{\pi} \left| \Delta_{\pm t}^k g(x) \right| dx$$

$$\le \int_{(k+1)\frac{\pi}{n}}^{\pi} \left| \sum_{v=1}^{n} \Delta a_v \; \Delta_{\pm t}^k D_v(x) \right| dx + \int_{(k+1)\frac{\pi}{n}}^{\pi} \left| \Delta_{\pm t}^k \sum_{v=n+1}^{\infty} \Delta a_v \; D_v(x) \right| dx$$

$= I_{21} + I_{22}$, say.

We now write

$$I_{21} \le \sum_{m=1}^{n-1} \int_{(k+1)\frac{\pi}{m+1}}^{(k+1)\frac{\pi}{m}} \left| \sum_{v=1}^{n} \Delta a_v \Delta_{\pm t}^k D_v(x) \right| dx$$

By virtue of $t \le \frac{\pi}{n}$ and $x \ge (k+1)\frac{\pi}{(m+1)}$, it follows that

$$x - kt \ge \frac{k+1}{m+1}\pi - \frac{k}{n}\pi = \frac{\pi}{m+1} + k\pi\left(\frac{1}{m+1} - \frac{1}{n}\right) \ge \frac{\pi}{m+1}.$$

Therefore in the sub interval $\left[(k+1)\frac{\pi}{(m+1)}, (k+1)\frac{\pi}{m}\right]$.

Using (2.2), we have

$$\left|\sum_{v=1}^{n}\Delta a_v \Delta_{\pm t}^{k} D_v(x)\right| \le B_k t^k \sum_{v=1}^{n}\left|A_v \frac{\Delta a_v}{A_v}\right| \max_{x-kt\le\xi\le x+kt}\left|D_v^{(k)}(\xi)\right|$$

$$\le B_k t^k \sum_{v=1}^{m} v^{k+1}\left|A_v \frac{\Delta a_v}{A_v}\right| + \frac{B_k t^k}{x-kt}\sum_{v=m+1}^{n} v^k \left|A_v \frac{\Delta a_v}{A_v}\right|$$

$$\le B_k t^k \sum_{v=1}^{m} v^{k+1} A_v + B_k t^k m \sum_{v=m+1}^{n} v^k A_v.$$

But

$$\sum_{v=1}^{m} v^{k+1} A_v = \sum_{v=1}^{m}\Delta A_v \sum_{i=0}^{v} i^{k+1} + A_{m+1}\sum_{i=0}^{m} i^{k+1}$$

$$\le \sum_{v=1}^{m}(v+1)^{k+2}\Delta A_v + m^{k+2}A_{m+1},$$

and

$$\sum_{v=m+1}^{n} v^k A_v = \sum_{v=m+1}^{n}\Delta A_v \sum_{i=0}^{v} i^k + A_{n+1}\sum_{i=0}^{k} i^k - A_{m+1}\sum_{i=0}^{m} i^k$$

$$\le \sum_{v=m+1}^{n}(v+1)^{k+1}\Delta A_v + n^{k+1}A_{n+1}.$$

Therefore

$$I_{21} \le B_k n^{-k}\left[\sum_{m=1}^{n-1} m^{-2}\left(\sum_{v=1}^{m}(v+1)^{k+2}\Delta A_v + m^{k+2}A_{m+1}\right)\right]$$

$$+B_k n^{-k}\left[\sum_{m=1}^{n-1} m^{-1}\left(\sum_{v=m+1}^{n}(v+1)^{k+1}\Delta A_v + n^{k+1}A_{n+1}\right)\right]$$

$$= B_k n^{-k}\left[\sum_{m=1}^{n-1} m^{-2}\sum_{v=1}^{m}(v+1)^{k+2}\Delta A_v + \sum_{m=1}^{n-1} m^k A_{m+1}\right.$$

$$+\sum_{m=1}^{n-1} m^{-1} \sum_{v=m+1}^{n} (v+1)^{k+1} \Delta A_v + \sum_{m=1}^{n-1} m^{-1} n^{k+1} A_{n+1} \Bigg]$$

The first term in the square bracket is

$$\sum_{v=1}^{n-1} (v+1)^{k+2} \Delta A_v \left(\sum_{m=v}^{n-1} m^{-2} \right) \le \sum_{v=1}^{n-1} (v+1)^{k+2} \Delta A_v \left(\sum_{m=v}^{\infty} m^{-2} \right)$$

$$\le B_k \sum_{v=1}^{n-1} (v+1)^{k+1} \Delta A_v,$$

the second term is

$$\sum_{m=1}^{n-1} m^k A_{m+1} = \sum_{m=1}^{n-1} \Delta A_{m+1} \sum_{i=0}^{m} i^k + A_n \sum_{i=0}^{n} i^k$$

$$\le \sum_{m=1}^{n-1} m^{k+1} \Delta A_{m+1} + n^{k+1} A_n,$$

and the third term is

$$\sum_{m=1}^{n-1} m^{-1} \sum_{v=m+1}^{n} (v+1)^{k+1} \Delta A_v = \sum_{v=1}^{n-1} (v+1)^{k+1} \Delta A_v \sum_{m=1}^{v-1} m^{-1}$$

$$\le B_k \sum_{v=2}^{n-1} (v+1)^{k+1} \Delta A_v \log v$$

$$\le B_k \log n \sum_{v=1}^{n-1} (v+1)^{k+1} \Delta A_v.$$

Therefore

$$I_{21} \le B_k n^{-k} \log n \sum_{v=1}^{n} (v+1)^{k+1} \Delta A_v.$$

Lastly, making use of Abel's transformation and Fomin's Lemma (Teljakovskii [42], Lemma 1), we have

$$I_{22} \le \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_{(k+1)\frac{\pi}{n} \pm \alpha t}^{\pi \pm \alpha t} \left| \sum_{v=n+1}^{\infty} \Delta a_v D_v(x) \right| dx$$

$$\le B_k \int_{\frac{\pi}{n}}^{\pi \pm k\frac{\pi}{n}} \left| \sum_{v=n+1}^{\infty} \Delta a_v D_v(x) \right| dx$$

$$\le B_k \int_{\frac{\pi}{v}}^{(k+1)\frac{\pi}{n}} \left| \sum_{v=n+1}^{\infty} A_v \frac{\Delta a_v}{A_v} D_v(x) \right| dx$$

$$= B_k \int_{\frac{\pi}{v}}^{(k+1)\frac{\pi}{n}} \left[\left| \sum_{v=n+1}^{\infty} \Delta A_v \sum_{i=0}^{v} \alpha_i D_i(x) + A_{n+1} \sum_{i=0}^{n} \alpha_i D_i(x) \right| \right] dx,$$

$$\left(\alpha_i = \frac{\Delta a_i}{A_i} \right)$$

$$\le B_k \left[\sum_{v=n+1}^{\infty} \Delta A_v \int_{\frac{\pi}{v}}^{(k+1)\frac{\pi}{n}} \left| \sum_{i=0}^{v} \alpha_i D_i(x) \right| dx + A_{n+1} \int_{\frac{\pi}{v}}^{(k+1)\frac{\pi}{n}} \left| \sum_{i=0}^{n} \alpha_i D_i(x) \right| dx \right]$$

$$\le B_k \left[\sum_{v=n+1}^{\infty} (v+1) \Delta A_v + (n+1) A_{n+1} \right]$$

$$\le B_k \sum_{v=n+1}^{\infty} (v+1) \Delta A_v .$$

Hence,

(2.4) $$I_2 \le B_k \left[n^{-k} \log n \sum_{v=1}^{n} (v+1)^{k+1} \Delta A_v + \sum_{v=n+1}^{\infty} (v+1) \Delta A_v \right].$$

The assertion of the theorem now follows from (2.1), (2.3) and (2.4).

ON THE INTEGRAL MODULUS OF CONTINUITY OF SINE SERIES OF 1ST ORDER BELONGING TO CLASS R

1. Introduction

Let F be a function of period 2π in L_p ($1 \le p < \infty$). Then the integral modulus of continuity of the first order of F is defined by

$$\omega_p(h, F) = \sup_{0<|t|\le h} \| F(x+t) - F(x) \|_p ,$$

where $\| . \|_p$ denotes the norm in L_p.

Let g(x) denote the sum of the sine series $\sum_{v=1}^{\infty} b_v \sin vx$. Throughout this chapter, the letter A will denote a constant having different values in different contexts.

Concerning integral modulus of continuity of sine series Aljancic and Tomic [6] proved the following theorem :

Theorem A : If the function b(x) satisfies the conditions

(i) $0 < b(x) \downarrow 0$ as $x \to \infty$

(ii) b(x) is convex,

(iii) $\int_0^1 b(t)\,dt = o\left[x\, b(x)\right]$

(iv) $\int_x^\infty t^{-1} b(t)\,dt = o\left[b(x)\right]$,

then $g(x) \in L(0, \pi)$ and

$$\omega_1\left(g; \frac{\pi}{n}\right) = o\left(b_n\right).$$

Later on Aljancic and Tomic [7] proved the following similar result:

Theorem B : Let $< b_n >$ be a convex sequence such that

(i) $b_n \downarrow 0$,

(ii) $\sum_{v=1}^{n} b_v = o\left(n\, b_n\right)$

(iii) $\sum_{v=n+1}^{\infty} \frac{b_v}{v} = o\left(b_n\right).$

Then

$$\omega_1\left(g; \frac{\pi}{n}\right) = o\left(b_n\right).$$

Omitting the conditions (ii) and (iii) in Theorem B. Izumi and Izumi [20] obtained the following result :

Theorem C : Let $<b_n>$ be a convex sequence tending to zero. Then

$$\omega_1\left(g; \frac{1}{m}\right) = o\left(m^{-1} \sum_{n=1}^{m} b_n\right),$$

Assuming only the quasi-convexity of $<b_n>$, they also obtained, in the same paper, the following estimate for the intergral modulus of continuity of g :

Theorem D : If the sequence $<b_v>$ is quasi-convex, that is

$$\sum_{v=1}^{\infty}(v+1)\left|\Delta^2 b_v\right|<\infty,$$

where $\Delta^2 b_v = \Delta b_v - \Delta b_{v+1} = b_v - 2b_{v+1} + b_{v+2}$, then

$$\omega_1\left(\frac{1}{n}, g\right) \le \frac{A}{n}\sum_{v=1}^{n} v^2 \,|\,\Delta^2 b_{v-1}\,| + A\sum_{v=n+1}^{\infty} v\left(1+\log\frac{v}{n}\right)|\,\Delta^2 b_{v-1}\,|.$$

(This is a corrected version of the result as given in Teljakovskii [40]).

The above theorem was further improved by Teljakovskii [40] in the form of the following form :

Theorem E : Let <b_v> be a quasi-convex null sequence satisfying $\sum_{v=1}^{\infty}\frac{|b_v|}{v}<\infty$. Then the integral modulus of continuity of g satisfies the relation

$$\omega_1\left(\frac{1}{n}, g\right) \le \frac{A}{n}\sum_{v=1}^{n} v^2 \,|\,\Delta^2 b_{v-1}\,| + A\sum_{v=n+1}^{\infty} v\,|\,\Delta^2 b_{v-1}\,| + A\sum_{v=n}^{\infty}\frac{|b_v|}{v}.$$

Concerning the behaviour of the sine series, Kano [21] proved the following result :

Theorem F : If <b_v> is a null sequence such that

(1.1) $$\sum_{v=1}^{\infty} v^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|<\infty,$$

then

(1.2) $$\sum_{v=1}^{\infty} b_v \sin vx$$

is a Fourier series, or equivalently, it represents an integrable function g.

The aim of this chapter is to estimate $\omega_1(h, g)$ under the condition (1.1).

2. Lemma :- The following lemma will be used in the proof of our theorem.

Lemma :- Let $0<t\le\frac{1}{n}(n=1,2,\ldots)$. If $K_v(x)$ denotes the Fejer kernel, then

(2.1) $$\int_0^{\pi}\left|K_v'(x+t)-K_v'(x-t)\right|dx \le \begin{cases} Atv^2, & v=1,2,\ldots,n \\ Av, & v=1,2,\ldots\ldots \end{cases}$$

Proof of the Lemma : We have

$$\int_0^{\pi}\left|K_v'(x+t)-K_v'(x+t)\right|dx \le At\left(\int_0^{2/v}+\int_{2/v}^{\pi}\right)\left|K_V''(x+\theta_v t)\right|dx$$

$$= At(J_1 + J_2), \text{ say,}$$

with – 1 <θ_v< 1. But we know that

$$|K_v^{//}(x)| \le \begin{cases} Av^3, & 0 \le x \le \pi \\ A\,vx^{-2}, & 0 < x \le \pi \end{cases}.$$

Therefore $J_1 \le Av^2$ and

$$J_2 \le Av \int_{2/v}^{\pi} \frac{1}{(x+\theta_v t)^2} dx$$

$$\le Av \int_{2/v}^{\pi} \frac{1}{(x-t)^2} dx$$

By virtue of $t \le \frac{1}{n}$ and v = 1, 2, ..., n. $x \ge \frac{2}{v}$; we have $x \ge \frac{2}{n} \ge 2t$. This yields

$$\frac{1}{x-t} \le \frac{2}{x} \qquad (x \ge 2t)$$

Thus, we have

$$J_2 \le Av \int_{2/v}^{\pi} \frac{dx}{x^2} \le Av \int_{2/v}^{\infty} \frac{dx}{x^2}$$

$$\le Av^2.$$

This proves first part of the inequality (2.1).

To prove the second part of (2.1), we use Zygumund's Theorem [11, p. 458] and have

$$\int_0^{\pi} |K_v^{/}(x+t) - K_v^{/}(x-t)| dx \le \int_t^{n+t} |K_v^{/}(x)| dx + \int_{-t}^{\pi-t} |K_v^{/}(x)| dx$$

$$\le A \int_{-\pi}^{\pi} |K_v^{/}(x)| dx$$

$$\le Av \int_{-\pi}^{\pi} |K_v(x)| dx$$

$$= A\,\pi\,v.$$

3. Result :- In the present chapter, we prove the following result :

Theorem :- Let <b_v> be a null sequence satisfying (1.1). Then

$$\omega_1\left(\frac{1}{n}, g\right) \le \frac{A}{n}\sum_{v=1}^{n}(v+1)^3\left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A\sum_{v=n+1}^{\infty}(v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

Proof of the Theorem : Theorem F implies that g is integrable. Let

$$S_n(x) = \sum_{v=1}^{n} b_v \sin vx .$$

Using Abel's transformation twice, we obtain

$$S_n(x) = -\frac{d}{dx}\sum_{v=1}^{n}\frac{b_v}{v}\cos vx$$

$$= -\left[\sum_{v=1}^{n-1}\Delta\left(\frac{b_v}{v}\right)\left(D_v'(x)-\frac{1}{2}\right)+\frac{b_n}{n}\left(D_n'(x)-\frac{1}{2}\right)\right]$$

$$= -\left[\sum_{v=1}^{n-2}(v+1)\Delta^2\left(\frac{b_v}{v}\right)K_v'(x)+n\Delta\left(\frac{b_{n-1}}{n-1}\right)K_{n-1}'(x)\right]$$

$$+\frac{1}{2}\sum_{v=1}^{n-1}\Delta\left(\frac{b_v}{v}\right)-\frac{b_n}{n}D_n'(x)+\frac{1}{2}\frac{b_n}{n},$$

where $D_v(x)$ and $K_v(x)$ denote Dirichlet kernel and Fejer kernel respectively. Then [21, 159]

$$g(x) = \lim_{n\to\infty} S_n(x) = -\sum_{v=1}^{\infty}(v+1)\Delta^2\left(\frac{b_v}{v}\right)K_v'(x).$$

To prove our theorem, it is sufficient to estimate

$$\int_0^{\pi}\left|g\left(x+\frac{1}{n}\right)-g\left(x-\frac{1}{n}\right)\right|dx .$$

We write

$$\int_0^{\pi}\left|g\left(x+\frac{1}{n}\right)-g\left(x-\frac{1}{n}\right)\right|dx = \int_0^{\pi}\left|\sum_{v=1}^{\infty}(v+1)\Delta^2\left(\frac{b_v}{v}\right)\left[K_v'\left(x+\frac{1}{n}\right)-K_v'\left(x-\frac{1}{n}\right)\right]\right|dx$$

$$\int_0^{2/n}+\int_{2/n}^{\pi} = I_1 + I_2, \text{ say.}$$

Then

$$I_2 \le \left(\sum_{v=1}^{n}+\sum_{v=n+1}^{\infty}\right)\left[(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_{2/n}^{\pi}\left|K_v'\left(x+\frac{1}{n}\right)-K_v'\left(x-\frac{1}{n}\right)\right|dx\right]$$

$= 1_{21} + I_{22}$, say.

The first part of the inequality (2.1) implies

$$I_{21} = \sum_{v=1}^{n}(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_{2/n}^{\pi}\left|K_v^{/}\left(x+\frac{1}{n}\right)-K_v^{/}\left(x-\frac{1}{n}\right)\right|dx$$

$$\leq \frac{A}{n}\sum_{v=1}^{n}(v+1)^3\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

The second part of the inequality (2.1) yields

$$I_{22} = \sum_{v=n+1}^{\infty}(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_{2/n}^{\pi}\left|K_v^{/}\left(x+\frac{1}{n}\right)-K_v^{/}\left(x-\frac{1}{n}\right)\right|dx$$

$$\leq A\sum_{v=n+1}^{\infty}(v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

Therefore

(3.1) $$I_2 \leq \frac{A}{n}\sum_{v=1}^{n}(v+1)^3\left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A\sum_{v=n+1}^{\infty}(v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

We now estimate I_1. We have

$$I_1 \leq \int_0^{2/n}\left|\sum_{v=1}^{\infty}(v+1)\Delta^2\left(\frac{b_v}{v}\right)\left[K_v^{/}\left(x+\frac{1}{n}\right)-K_v^{/}\left(x-\frac{1}{n}\right)\right]\right|dx$$

$$\leq\left(\sum_{v=1}^{n}+\sum_{v=n+1}^{\infty}\right)\left[(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_0^{2/n}\left|K_v^{/}\left(x+\frac{1}{n}\right)-K_v^{/}\left(x-\frac{1}{n}\right)\right|dx\right]$$

$= 1_{11} + I_{12}$, say.

Now, the use of first part of (2.1) gives

$$I_{11} = \sum_{v=1}^{n}(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_0^{2/n}\left|K_v^{/}\left(x+\frac{1}{n}\right)-K_v^{/}\left(x-\frac{1}{n}\right)\right|dx$$

$$\leq \frac{A}{n}\sum_{v=1}^{n}(v+1)^3\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

Similarly the use of second part of (2.1) yields

$$I_{12} = \sum_{v=n+1}^{\infty}(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_0^{2/n}\left|K_v^{/}\left(x+\frac{1}{n}\right)-K_v^{/}\left(x-\frac{1}{n}\right)\right|dx$$

$$\leq A\sum_{v=n+1}^{\infty}(v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

We have therefore

(3.2) $$I_1 \le \frac{A}{n}\sum_{v=1}^{n}(v+1)^3\left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A\sum_{v=n+1}^{\infty}(v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

Combining (3.1) and (3.2), we obtain

$$\int_0^{\pi}\left|g\left(x+\frac{1}{n}\right)-g\left(x-\frac{1}{n}\right)\right|dx \le \frac{A}{n}\sum_{v=1}^{n}(v+1)^3\left|\Delta^2\left(\frac{b_v}{v}\right)\right|$$

$$+A\sum_{v=n+1}^{\infty}(v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

This completes the proof of the theorem.

CHAPTER – VIII

ON THE ESTIMATION OF MODULUS OF CONTINUITY OF HIGHER ORDER TRIGONOMETRIC SERIES

1. Introduction

Let F be a function of period 2π in L_p $(1 \le p < \infty)$. Then the integral modulus of continuity of the order k of F in L_p is defined by

$$\omega_p^k(h\,;F) = \sup_{0<|t|\le h}\left\|\Delta_t^k F(x)\right\|_{L_p},$$

where

$$\Delta_t^k F(x) = \sum_{\alpha=0}^{k}(-1)^{k-\alpha}\binom{k}{\alpha}F(x+\alpha t)$$

and $\|\cdot\|_{L_p}$ denotes the norm in L_p.

Let g(x) denote the sum of the sine series $\sum_{v=1}^{\infty} b_v \sin vx$. Throughout this chapter the letter A with or without subscripts denotes a constant having different values in different contexts and depending upon the subscripts.

In this chapter, we obtain an estimate for $\omega_1^k(h;\, g)$ under the conditions

(1.1) $b_v \to 0,\ v \to \infty$

(1.2) $$\sum_{v=1}^{\infty} v^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right| < \infty.$$

The results obtained are the generalization of the result presented in Chapter VII.

2.Lemma : The following lemma is the generalization of the lemma proved in Chapter VII.

Lemma,Let $0 < t \le \frac{1}{n}$ (n = 1, 2, ...) and let k be a natural number. If $k_v(x)$ denotes the Fejer kernel, then

(2.1) $$\int_0^{\pi} \left|\Delta^k_{\pm t} K'_v(x)\right| dx \le \begin{cases} A_k t^k v^{k+1}, & v = 1, 2, \ldots, n \\ A_k v, & v = 1, 2, \ldots.. \end{cases}$$

Proof of the Lemma :- We have

$$\int_0^{\pi} \left|\Delta^k_{\pm t} K'_v(x)\right| dx \le A_k t^k \left(\int_0^{\frac{k+1}{v}} + \int_{\frac{k+1}{v}}^{\pi}\right) \left|K_v^{(k+1)}(x \pm \theta_v t)\right| dx$$

$$= A_k t^k (J_1 + J_2), \text{ say}$$

with $0 < \theta_v < k$. Now because of

$$\left|K_v^{(k+1)}(x)\right| \le \begin{cases} A_k v^{k+2}, & 0 \le x \le \pi \\ A_k v^k x^{-2}, & 0 < x \le \pi \end{cases} (k = 1, 2, \ldots.),$$

we have $J_1 \le A_k v^{k+1}$ and

$$J_2 \le A_k v^k \int_{\frac{k+1}{v}}^{\pi} \frac{1}{(x \pm \theta_v t)^2} dx \le A_k v^k \int_{\frac{k+1}{v}}^{\pi} \frac{1}{(x - kt)^2}.$$

By virtue of $t \le \frac{1}{n}$ and v = 1, 2, ..., n, $x \ge \frac{k+1}{v}$; we have $x \ge \frac{k+1}{n} \ge (k+1)t$. This yields

$$\frac{1}{x - kt} \le \frac{k+1}{x} (x \ge (x+1)t).$$

Thus, we obtain

$$J_2 \le A_k v^k \int_{\frac{k+1}{v}}^{\pi} \frac{dx}{x^2} \le A_k v^k \int_{\frac{k+1}{v}}^{\infty} \frac{dx}{x^2} \le A_k . v^{k+1}.$$

This proves first part of the inequality (2.1). To prove the second part of (2.1), we use Zygmund's Theorem [11, p. 458] and have

$$\int_0^\pi \left|\Delta_{\pm t}^k K_v^/(x)\right| dx \le \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_0^\pi \left|K_v^/(x \pm \alpha t)\right| dx$$

$$\le \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_{-\pi}^{\pi} \left|K_v^/(x)\right| dx$$

$$\le A_k v \int_{-\pi}^{\pi} \left|K_v(x)\right| dx$$

$$= \pi A_k v.$$

3. **Results** :- We establish the following theorem :

Theorem : Let <b_v> be a null sequence satisfying (1.1). Then

$$\omega_1^k\left(\frac{1}{n}; g\right) \le \frac{A_k}{n^k} \sum_{v=1}^{n} (v+1)^{k+2} \left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

The case k = 1 of this theorem yields the theorem proved in Chapter VII.

Proof of the theorem : Theorem F of Chapter VII implies that g is integrale.

Let

$$S_n(x) = \sum_{v=1}^{n} b_v \sin vx .$$

Two applications of Abel's transformation yield

$$S_n(x) = -\frac{d}{dx} \sum_{v=1}^{n} \frac{b_v}{v} \cos vx$$

$$= -\left[\sum_{v=1}^{n-1} \Delta\left(\frac{b_v}{v}\right)\left(D_v^/(x) - \frac{1}{2}\right) + \frac{b_n}{n}\left(D_n^/(x) - \frac{1}{2}\right)\right]$$

$$= -\left[\sum_{v=1}^{n-2} (v+1)\Delta^2\left(\frac{b_v}{v}\right) K_v^/(x) + n\Delta\left(\frac{b_{n-1}}{n-1}\right) K_{n-1}^/(x)\right]$$

$$+ \frac{1}{2} \sum_{v=1}^{n-1} \Delta\left(\frac{b_v}{v}\right) - \frac{b_n}{n} D_n^/(x) + \frac{1}{2}\frac{b_n}{n},$$

where $D_v(x)$ and $K_v(x)$ denote Dirichlet kernel and Fejer kernel respectively. Then [21, p. 159]

$$g(x) = \lim_{n\to\infty} S_n(x) = -\sum_{v=1}^{\infty} (v+1)\Delta^2\left(\frac{b_v}{v}\right) K_v^/(x).$$

Since the symmetry of the function g implies

$$\left|\Delta_t^k g(-x)\right| = \left|\Delta_t^k g(x)\right|,$$

therefore

$$\int_{-\pi}^{\pi}\left|\Delta_t^k g(x)\right|dx = \int_0^{\pi}\left|\Delta_{-t}^k\ g(x)\right|dx + \int_0^{\pi}\left|\Delta_t^k\ g(x)\right|dx.$$

Hence, to prove our theorem, it is sufficient to estimate

$$\int_0^{\pi}\left|\Delta_{\pm t}^k\ g(x)\right|dx \qquad \text{for } 0 < t \le \frac{1}{n}.$$

We write

$$\int_0^{\pi}\left|\Delta_{\pm t}^k g(x)\right|dx = \int_0^{\pi}\left|\Delta_{\pm t}^k \sum_{v=1}^{\infty}(v+1)\Delta^2\left(\frac{b_v}{v}\right)K_v'(x)\right|dx$$

$$= \int_0^{(k+1)/n} + \int_{(k+1)/n}^{\pi}$$

$$= I_1 + I_2.$$

We first estimate I_2. We have

$$I_2 \le \left(\sum_{v=1}^{n} + \sum_{v=n+1}^{\infty}\right)\left[(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_{\frac{k+1}{n}}^{\pi}\left|\Delta_{\pm t}^k K_v'(x)\right|dx\right]$$

$$= I_{21} + I_{22}, \text{ say.}$$

Now, by first part of the inequality (2.1), we have

$$I_{21} \le \sum_{v=1}^{n}(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_{\frac{k+1}{n}}^{\pi}\left|\Delta_{\pm t}^k K_v'(x)\right|dx$$

$$\le A_k n^{-k}\sum_{v=1}^{n}(v+1)^{k+2}\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

The second part of the inequality (2.1) implies

$$I_{22} = \sum_{v=n+1}^{\infty}(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right|\int_{\frac{k+1}{n}}^{\pi}\left|\Delta_{\pm t}^k K_v'(x)\right|dx$$

$$\le A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

Thus

(3.1) $$I_2 \le A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

To estimate I_1. We have

$$I_1 \le \int_0^{\frac{k+1}{n}} \left|\Delta_{\pm t}^k \left(\sum_{v=1}^{\infty} (v+1)\Delta^2\left(\frac{b_v}{v}\right) K_v^{/}(x)\right)\right| dx$$

$$\le \left(\sum_{v=1}^{n} + \sum_{v=n+1}^{\infty}\right)\left[(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right| \int_0^{\frac{k+1}{n}} \left|\Delta_{\pm t}^k K_v^{/}(x)\right| dx\right]$$

$= 1_{11} + I_{12}$, say.

Now, usingfirst part of (2.1), we have

$$I_{11} = \sum_{v=1}^{n} (v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right| \int_0^{\frac{k+1}{n}} \left|\Delta_{\pm t}^k K_v^{/}(x)\right| dx$$

$$\le A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left|\Delta^2\left(\frac{b_v}{v}\right)\right|$$

and using the second part of (2.1), we have

$$I_{12} = \sum_{v=n+1}^{\infty} (v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right| \int_0^{\frac{k+1}{n}} \left|\Delta_{\pm t}^k K_v^{/}(x)\right| dx$$

$$\le A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

We have therefore

(3.2) $$I_1 \le A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

Combining (3.1) and (3.2), it follows that

$$\int_0^{\pi} \left|\Delta_{\pm t}^k g(x)\right| dx \le A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left|\Delta^2\left(\frac{b_v}{v}\right)\right|$$

$$+A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left| \Delta^2 \left(\frac{b_v}{v} \right) \right|.$$

This completes the proof of the Theorem.

REFERENCES:

1. Integrability of trigonometric series. The estimation of the integral modulus of continuity, *Mat. Sbormik*, 91(133) (1973), No. 4, 557-573.
2. A sufficient condition of Sidon for the integrability of trigonometric series, *Mat. Zametki*, 14 (1973), 317-328.
3. Some estimates for trigonometric series with quasi-convex coefficients, *Mat. Sbornik*, 63(105) (1964), 426-444.
4. On a problem concerning convergence of Fourier series in metric L, *Mat. Zametki*, 1(1967), 91-98.
5. Concerning a sufficient condition of Sidon for the integrability of trigonometric series, *Mat. Zametki*, 14(1973), 317-328.
6. Convergence of Fourier series with quasi-montonic coefficient in metric L, *Trudy Mat. Inst. ANSSSR*, 134(1975), 310-313.
7. Theory of approximation of function of real variables, *Hindustan Publishing Corporation*, India, 1966.
8. On the Fourier series of bounded functions, *Proc. London Math. Soc.*, (2), 12(1913), 41-70.
9. A remark on the integral modulus of continuity, *Revista Univ.Nac. Tucuman*, 7(1950), 259-269.
10. Trigonometric series, 2nd Edition, *Cambridge University Press, London*, 1959.
11. On the integral modulus of continuity of Fourier series, *Bull. Acad. Royale Belgicue*, Cl. Sci. 58(1972/73), 337-343.
12. 27. On the integral modulus of continuity of Fourier series, *Journal D'Analyse Mathematique*, 28(1975), 78-85.
13. 28. Convergence of certain cosine sums in the metric space L, *Proc. Amer. Math. Soc.*, 66(1977), 251-255.
14. 29. On integral modulus of continuity of Fourier series, *Kyungpook Math. J.*, 18 (1978), 251-255.
15. 30. A sufficient condition for the integrability of Rees-Stanojevic sum, *Kyungpook Mathematical Journal*, 19 (1979), 257-260.
16. 31. Integrability of Rees-Stanojevic sums, *Acta Sci. Math.* 42(1980), 153-155.