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G-Inverse of Lower Triangular Block Operator Matrix

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Abstract: Inside this paper, we probe the depictions of Drazin spectrum $\sigma_d(M_C)$ and Generalized inverse and generalized Drazin inverse of lower triangular operator matrix on Banach space.

Keywords: Operator Matrices, Drazin spectrum, single-valued extension property, Generalized inverse, Drazin inverse.

I. Introduction

An operator $\mathcal{T}_1 \in L(X)$ is said to be a Drazin invertible if there exists a positive integer k and an operator $S_1 \in L(X)$ such that ${\mathcal{T}_1}^k S_1 \mathcal{T}_1 = {\mathcal{T}_1}^k,\ S_1\mathcal{T}_1 S_1 = S_1\ and\ \mathcal{T}_1 S_1 = S_1\mathcal{T}_1$ The Drazin spectrum is defined by σD(T) = {λ ∈ C : T − λI is not Drazin invertible}.

The Drazin invertible spectrum is define by $\sigma_d(\mathcal{T}_1) = \{\lambda \epsilon\ C: \mathcal{T}_1 - \lambda I\ is\ not\ Drazin\ invertible\ operator\}$

The Drazin invertible operator is defined by an operator $\mathcal{T}_1 \in L(X)$ is said to be Drazin invertible if $\mathcal{T}_1$ is both left and Right Drazin invertible.

It is well known that T is Drazin invertible if and only if T is of finite ascent and descent, which is also equivalent to the fact that T = R ⊕ N where R is invertible and N nilpotent (see [16, Corollary 2.2]). Clearly, $\mathcal{T}_1$ is Drazin invertible if and only if ${\mathcal{T}_1}^*$is Drazin invertible. A bounded linear operator $\mathcal{T}_1$ ∈ L(X) is said to have the single-valued extension property (SVEP, for short) at λ ∈ C if for every open neighborhood $\cup_\lambda\ of\ \lambda$, the constant function f ≡ 0 is the only analytic solution of the equation

$(\mathcal{T}_1 - \mu)f(\mu) = 0\ \ for\ all\ \mu\ \epsilon\ \cup_\lambda$

We use $S_1(\mathcal{T}_1)$ to denote the open set where $\mathcal{T}_1$ fails to have the SVEP and we say that $\mathcal{T}_1$ has the SVEP if $S_1(\mathcal{T}_1)$is the empty set, [12]. It is easy to see that $(\mathcal{T}_1)$ has the SVEP at every point λ ∈ iso σ(T), where iso σ(T) denotes the set of all isolated points of σ(T). Note that (see [12])

$S_1(\mathcal{T}_1) \subseteq \sigma_p(\mathcal{T}_1)\ and\ \sigma(\mathcal{T}_1) = S_1(\mathcal{T}_1) \cup\ \sigma_s(\mathcal{T}_1)$

Also, it follows from [15] if T is of finite ascent and descent then $\mathcal{T}_1$ and have the SVEP. Hence $S_1(\mathcal{T}_1) \cup S_1({\mathcal{T}_1}^*) \subseteq \sigma_d(\mathcal{T}_1)$

For $\mathcal{T}_1 \in$ L(X), $\mathcal{T}_2 \in$ L (Y) and C ∈ L (Y, X) we denote by M_C the operator defined on X ⊕ Y

by $M_C = \begin{bmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_3 & \mathcal{T}_2 \end{bmatrix}$

In [11] it is proved that $\sigma(M_C) \cup [S_1(\mathcal{T}_1{}^*) \cap S_1(\mathcal{T}_2)] = \sigma(\mathcal{T}_1) \cup \sigma(\mathcal{T}_2)$. Numerus mathematicians were interested by the defect set $[\sigma_*(\mathcal{T}_1) \cup \sigma_*(\mathcal{T}_2)] \setminus \sigma_*(M_C)$

See for instance [11, 13, 14] for the spectrum and the essential spectrum,[19] for the Weyl spectrum, [10] for the Browder spectrum and [9, 10] for the essential approximate point spectrum and the Browder essential approximate point spectrum. See also the references therein. For the Drazin spectrum, Campbell and Meyer [7] were the first studied the Drazin invertibility of 2 × 2 lower triangular operator matrices M_C where $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are n × n complex matrices. They proved that $\sigma_d(M_C) \subseteq \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)$

D. S. Djordjevi´c and P. S. Stanimirovi´c generalized the inclusion (1.3) to arbitrary Banach spaces [8]. Inclusion (1.3) may be strict.

The generalized inverse (for short G-Inverse) and generalized Drazin inverse (for short GD-Inverse). Presume T_n is a given lower triangular block matrix and X_n is an arbitrary upper triangular block matrix. The generalized Drazin inverse of a 2 × 2 block operator matrix $\mathcal{T} = \begin{pmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_2 & \mathcal{T}_3 \end{pmatrix}$. Let X and K be separable, infinite dimensional, complex Banach spaces. Denote by $B(X,K)$ the set of all bounded linear operators from X into K. For an operator $\mathcal{T} \in B(X,K), R(A), N(A)$ denote the range, the null space and the adjoint of A, respectively. For $\mathcal{T} \in B(X,K)$, if there exists $\mathcal{T}^+ \in B(X,K)$ satisfying the following four operator equation, $\mathcal{T}\mathcal{T}^+\mathcal{T} = \mathcal{T}, \mathcal{T}^+\mathcal{T}\mathcal{T}^+ = \mathcal{T}^+, \mathcal{T}\mathcal{T}^+ = (\mathcal{T}\mathcal{T}^+)^*, \mathcal{T}^+\mathcal{T} = (\mathcal{T}^+\mathcal{T})^*$, then $\mathcal{T}^+$ is called the G-Inverse of T. It is well known that has the G-inverse if and only if R ($\mathcal{T}$) is closed and the G-inverse of $\mathcal{T}$ is unique (see [16, 20, 24]).

1. Main results and its proof

Theorem 1.1

For $\mathcal{T}_1 \in L(X)$, $\mathcal{T}_2 \in L(Y)$, and $\mathcal{T}_3 \in$ L (Y, X) we have

$$\sigma_d(M_C) \cup [S_1(\mathcal{T}_1{}^*) \cap S_1(\mathcal{T}_2)] = \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)$$

Proof

Since the inclusion $\sigma_d(M_C) \cup [S_1(\mathcal{T}_1{}^*) \cap S_1(\mathcal{T}_2)] \subseteq \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)$ always holds, it suffices to prove the reverse inclusion. Let $\lambda \in \sigma_d(\mathcal{T}_1) \cup \sigma_d(\mathcal{T}_2)/\sigma_d(M_C)$.Without loss of generality, we can assume that $\lambda = 0$. Then M_C is of finite ascent and descent. Hence from [9, Lemma 2.1] we have A is of finite ascent and B is of finite descent. Also, by duality $\mathcal{T}_1{}^*$ is of finite descent and $\mathcal{T}_2{}^*$is of finite ascent. For the sake of contradiction assume that

$0 \notin S_1(\mathcal{T}_1{}^*) \cap S_1(\mathcal{T}_2)$.

Case 1. $0 \notin S_1(\mathcal{T}_1{}^*)$ Since M_C is Drazin invertible, then there exists $\in > 0$ such that for every λ, $0 < |\lambda| < \in, M_C - \lambda$ is invertible. Hence $\mathcal{T}_1 - \lambda$ is right invertible. Thus $0 \notin acc\sigma_{ap}(\mathcal{T}_1) = acc\sigma_s(\mathcal{T}_1{}^*)$. $0 \notin \sigma(\mathcal{T}_1{}^*)$ *then* $\mathcal{T}_1{}^*$ is Drazin invertible and so $\mathcal{T}_1$ is. Now if $0 \in \sigma(\mathcal{T}_1{}^*)$, since $\sigma(\mathcal{T}_1{}^*) = S_1(\mathcal{T}_1{}^*) \cup \sigma_s(\mathcal{T}_1{}^*)$ Then 0 is an isolated point of $\sigma(\mathcal{T}_1{}^*)$. Now $\mathcal{T}_1{}^*$ is of finite decent and $0 \in iso\sigma(\mathcal{T}_1{}^*)$. Hence it follows from [18, Theorem 10.5]

$\mathcal{T}_1{}^*$ is Drazin invertible. Thus $\mathcal{T}_1$ is Drazin invertible. Since M_C is Drazin invertible it follows from [21, lemma 2.7] that $\mathcal{T}_2$ is also Drazin invertible which contradiction our assumption.

Case 2. $0 \notin S_1(\mathcal{T}_2{}^*)$, the proof goes similarly.

Theorem 1.2

Let $\mathcal{T}_1 \in B(X), \mathcal{T}_2 \in B(K), \mathcal{T}_3 \in B(K,X)$ and $\mathcal{T}_2$ be invertible. Then 2 by 2 block operator valued matrix $\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_2 & \mathcal{T}_3 \end{bmatrix}$ is G invertible if and only if R $(\mathcal{T}_1)$ is closed and

$$\begin{bmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_2 & \mathcal{T}_3 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1^{+} - \mathcal{T}_1^{+}\mathcal{T}_2\Delta\mathcal{T}_2^{*}(I - \mathcal{T}_1\mathcal{T}_2^{+}) & -\mathcal{T}_1^{-1}\mathcal{T}_2\Delta\mathcal{T}_3^{*} \\ \Delta\mathcal{T}_2^{*}(I - \mathcal{T}_1\mathcal{T}_1^{+}) & \Delta\mathcal{T}_3^{*} \end{bmatrix}$$

Proof

Since $\begin{bmatrix} \mathcal{T}_1^* & \mathcal{T}_2^* \\ 0 & \mathcal{T}_3^* \end{bmatrix} \begin{bmatrix} I & \mathcal{T}_2^*(\mathcal{T}_1^*\mathcal{T}_3)^{-1} \\ 0 & -(\mathcal{T}_3^*)^{-1} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1^* & 0 \\ 0 & I \end{bmatrix}$

R $(\mathcal{T}^*)$is closed if and only if R$(\mathcal{T}_1^*)$. This shows that $\mathcal{T}$ is invertible if and only if R$(\mathcal{T})$ is closed.
In this case $\mathcal{T}$ has the form

$$\begin{bmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_2 & \mathcal{T}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{T}_{11} & 0 \\ \mathcal{T}_{22} & \mathcal{T}_{21} & \mathcal{T}_3 \end{bmatrix} \begin{bmatrix} N(\mathcal{T}_1) \\ R(\mathcal{T}_1^*) \\ K \end{bmatrix} \to \begin{bmatrix} N(\mathcal{T}_1^*) \\ R(\mathcal{T}_1) \\ K \end{bmatrix}$$

Where $\mathcal{T}_{11}$ as an operator from R$(\mathcal{T}_1^*)$ on to R $(\mathcal{T})$ is invertible. Now N $= \begin{bmatrix} 0 \\ \mathcal{T}_{22} \end{bmatrix}$,

M $= \begin{bmatrix} \mathcal{T}_{11} & 0 \\ \mathcal{T}_{21} & \mathcal{T}_3 \end{bmatrix}$

and $\Delta = (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{T}_3^*(I - \mathcal{T}_1\mathcal{T}_1^*)\mathcal{T}_3)^{-1}$
$= (\mathcal{T}_2^*\mathcal{T}_2 + \mathcal{T}_3^*\mathcal{T}_3)^{-1}$

It is easy to check that

$$\begin{bmatrix} \mathcal{T}_1 & 0 \\ \mathcal{T}_2 & \mathcal{T}_3 \end{bmatrix}^{+} = \begin{bmatrix} 0 & N \\ 0 & M \end{bmatrix}^{*} \left[\begin{bmatrix} 0 & N \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & N \\ 0 & M \end{bmatrix}^{*} \right]^{-1}$$

$$= \begin{bmatrix} 0 & (N^*N + M^*M)^{-1}N^* \\ 0 & (N^*N + M^*M)^{-1}M^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -\mathcal{T}_{11}^*\mathcal{T}_{21}\Delta\,\mathcal{T}_{22}^* & \mathcal{T}_{11}^{\ -1} & 0 \\ \Delta\mathcal{T}_{22}^* & -\mathcal{T}_1^{-1}\mathcal{T}_{21}\Delta\mathcal{T}_3 & \Delta\mathcal{T}_3^* \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{T}_1^{-1} - \mathcal{T}_1^*\mathcal{T}_2\Delta\mathcal{T}_2^*(I - \mathcal{T}_1\mathcal{T}_2^{+}) & -\mathcal{T}_1^{-1}\mathcal{T}_2\Delta\mathcal{T}_3^* \\ \Delta\mathcal{T}_2^*(I - \mathcal{T}_1\mathcal{T}_1^{+}) & \Delta\mathcal{T}_3^* \end{bmatrix}$$

Remark

In Theorem 1, if R $(\mathcal{T}_2)$ is closed, we can show that $\mathcal{T}$ is G-invertible if and only if $R((I - \mathcal{T}_1\mathcal{T}_1^{+})\mathcal{T}_3(I - \mathcal{T}_2^{+}\mathcal{T}_2)$ is closed in a similar way. In this case, $\mathcal{T}^{+}$ has a very complicated representation. But we can show that $\mathcal{T}^{\{1\}}$ has the form as

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_3 \\ 0 & \mathcal{T}_2 \end{bmatrix}^{\{1\}} = \begin{bmatrix} \mathcal{T}_1^{\ +} - \mathcal{T}_1^{\ +}\mathcal{T}_3\mathcal{T}_{31}^{\ +} & -\mathcal{T}_1^{\ +}\mathcal{T}_3\mathcal{T}_2^{\ +} \\ \mathcal{T}_{31}^{\ +} & \mathcal{T}_2^{+} - \mathcal{T}_{31}^{\ +}\mathcal{T}_3\mathcal{T}_2^{\ +} \end{bmatrix},$$

Were $\mathcal{T}_{31} = \left((I - \mathcal{T}_1\mathcal{T}_1^{+})\mathcal{T}_3(I - \mathcal{T}_2^{+}\mathcal{T}_2)\right)$.

In addition, if $\mathcal{T}_1\mathcal{T}_1^{+}\mathcal{T}_3\left(I - \mathcal{T}_2^{+}\mathcal{T}_2 - \mathcal{T}_{31}^{\ +}\mathcal{T}_{31}\right) = 0$ and $\left(I - \mathcal{T}_2^{+}\mathcal{T}_2 - \mathcal{T}_{31}^{\ +}\mathcal{T}_{31}\right)\mathcal{T}_3\mathcal{T}_2\mathcal{T}_2^{+} = 0$, a directly calculation can show that,

$$\begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_3 \\ 0 & \mathcal{T}_2 \end{bmatrix}^{+} = \begin{bmatrix} \mathcal{T}_1^{\ +} - \mathcal{T}_1^{\ +}\mathcal{T}_3\mathcal{T}_{31}^{\ +} & -\mathcal{T}_1^{\ +}\mathcal{T}_3\mathcal{T}_2^{\ +} + \mathcal{T}_1^{\ +}\mathcal{T}_3\mathcal{T}_{31}^{\ +}\mathcal{T}_3\mathcal{T}_2^{+} \\ \mathcal{T}_{31}^{\ +} & \mathcal{T}_2^{+} - \mathcal{T}_{31}^{\ +}\mathcal{T}_3\mathcal{T}_2^{\ +} \end{bmatrix}.$$

(2) If we assume as well that $R(\mathcal{T}_3) \subset R(\mathcal{T}_1)$ and $R(\mathcal{T}_3^{\ +}) \subset R(\mathcal{T}_2^{+})$, then $\mathcal{T}$ satisfies remark (1), and $\mathcal{T}_{31} = (I - \mathcal{T}_1\mathcal{T}_1^{+})\mathcal{T}_3(I - \mathcal{T}_2^{+}\mathcal{T}_2) = 0$.Then we have

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