

# ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)

An International Scholarly Open Access, Peer-reviewed, Refereed Journal

# Some Properties of Stable Processes Characterized by Identically Stochastic Integrals

M . GEETHA HEAD AND ASSISTANT PROFESSOR OF MATHEMATICS PG DEPARTMENT OF MATHEMATICS SARADHA GANGADHARAN COLLEGE PONDICHERRY – 605004 , INDIA MAIL ID : m.geetha\_06@yahoo.co.in

ABSTRACT : The main object of this paper is to discuss some properties of stable processes when they are characterized by two identically distribute stochastic integrals formed by homogeneous and continuous stochastic process X (t) with independent increments . The main results based on Phragmen – Lindlelof

theory .

**KEYWORDS** : Wiener process, Levy Canonical Representation and Infinitely divisible distribution .

**INTRODUCTION** : Let X(t) be a homogeneous and continuous process with independent increments .  $f(u, \tau)$  denotes the characteristic functions (c.f) of the increment X(t+  $\tau$ ) - x(t) . It is infinitely divisible (i.d) and f (u,  $\tau$ ) = f (u,1)  $\tau$ . We can write f(u) for f(u,1) The process X(t) has a symmetric increments if there exist a real number b such that f(u)e\*ibu is a real characteristic function . A process X(t) is called stable if the d.f of its increments is a stable d.f and X(0) = 0 . If the d.f of the increments is normal and E X(t) = 0 for all t≥0,the stable process is called a Wiener process. A process X<sub>1</sub>(t) is said to be a Wiener process with linear mean value function m if X<sub>1</sub>(t) = X(t) +m(t) where X(t) is a Wiener process. where m is a linear (non-random ) function To define the integrals to be used and to indicate few of their properties ,

Assume that a, v are functions defined in [A, B] where v is non – negative. Let us form a sequence of subdivisions

$$A = t_{n,0} < t_{n,1} < \cdots \dots < t_{n,n} = B \quad (n = 1, 2, \dots \dots)$$

of the interval [A, B] such that

$$\lim_{n\to\infty}\max_{1\leq k\leq n}\quad \left(t_{n,k}-t_{n,k-1}\right)=0$$

and select a sequence of numbers  $t_{n,k}^*$  where  $t_{n,k-1} \leq t_{n,k}^* \leq t_{n,k}$  (k = 1,2, ..., n). Then for a given process X(t) let us construct a sequence of random variables in the following manner:

$$S_{n} = \sum_{k=1}^{n} a(t_{n,k}^{*}) \left[ X(\upsilon(t_{n,k})) - X(\upsilon(t_{n,k-1})) \right]$$

If the sequence  $S_n$  converges in probability to a random variable S, and if this limit is independent of the choice of the subdivision and the intermediate points  $t_{n,k}^*$ , then say that S is a stochastic integral and denote it by

$$\int_{A}^{B} \frac{1}{a(t)dX(v(t))}.$$

## Theorem: 1.1 (Representation Theorem)

Let the Levy canonical representation of the characteristic function of X(1) - X(0) be given by b,  $\sigma$ , M and N. Then the Levy canonical representation for the characteristic function of the stochastic integral

$$\int_{A}^{B} t dX(v(t))$$
(1.1)

is given by the following formulas:

$$b_{\upsilon} = \int_{A}^{B} \left( tb + t(1 - t^{2}) \right) \int_{0+}^{\infty} \frac{x^{3}}{(1 + (tx)^{2})(1 + x^{2})} d(M(-x) + N(x)) + d\upsilon(t);$$
(1.2)

$$\sigma_{\upsilon}^{2} = \sigma^{2} \int_{A}^{B} t^{2} d\upsilon(t)$$
(1.3)

$$M_{\upsilon}(x) = \int_{\min(A,0)}^{\min(B,0)} -N\left(\frac{x}{t}\right) d\upsilon(t) + \int_{\max(A,0)}^{\max(B,0)} M\left(\frac{x}{t}\right) d\upsilon(t) \qquad (x < 0) \quad (1.4)$$

$$N_{\upsilon}(x) = \int_{\min(A,0)}^{\min(B,0)} -M\left(\frac{x}{t}\right) d\upsilon(t) + \int_{\max(A,0)}^{\max(B,0)} N\left(\frac{x}{t}\right) d\upsilon(t) \qquad (x > 0) \quad (1.5)$$

## Lemma:1.1

The function g is an infinitely divisible characteristic function if, and only if, it can be written in the form

$$\log g(u) = iau + \frac{\sigma^2}{2} u^2 + \int_{-\infty}^{-0} r(u, x) dM(x) + \int_{+0}^{\infty} r(u, x) dN(x)$$

where a,  $\sigma$  are real constants; M and n are non – decreasing in the intervals ( $-\infty$ , 0) and ( $0,\infty$ ) respectively, with

$$M(-\infty) = N(\infty) = 0$$
$$\int_{-\epsilon}^{0-} x^2 dM(x) < \infty \text{ and } \int_{0+}^{\epsilon} x^2 dN(x) < \infty \text{ for every } \epsilon > 0$$

and

$$r(u, x) = e^{iux} - 1 - (iux/(1 + x^2))$$

### **Proof of theorem 1.1**

With out loss of generality let us assume that  $A \le 0 \le B$ .

1. First we assume that there exists a number  $t_0 > 0$  such that  $t_0$  is a point of continuity of v and

$$v(t_0) - v(-t_0) = 0 \tag{1.7}$$

The characteristic function of (1.1) is denoted by h. Then by theorem 1.1 we have

$$\log h(u) = \int_{A}^{B} \log f(u \cdot t) dv(t)$$
(1.8)

(1.6)

Now let us define a function s by

$$s(u, x, t) = r(ut, x) - r(u, tx)$$

where in view of (1.6)

$$s(u, x, t) = \frac{it(1 - t^2)x^3u}{(1 + (tx)^2)(1 + x^2)}$$

Since  $s(u, x, t) = o(x^2)$  as  $x \to 0$  and s(u, x, t) = o(1) as  $x \to \infty$  the function s is integrable with respect to M and N. By Lemma 1.1 and the definition of s we have,

$$\log f(ut) = iaut - \frac{\sigma^2}{2} (ut)^2 + \int_{-\infty}^{-0} (r(u, tx) + s(u, x, t)) dM(x) + \int_{+0}^{\infty} (r(u, tx) + s(u, x, t)) dN(x)$$
(1.9)

By virtue of (1.8) and (1.9) let us obtain

$$\log h(u) = iau \int_{A}^{B} t dv(t) - \frac{\sigma^2}{2} u^2 \int_{A}^{B} t^2 dv(t)$$

$$+ \int_{A}^{B} \left( \int_{-\infty}^{-0} s(u, x, t) dM(x) + \int_{+0}^{\infty} s(u, x, t) dN(x) \right) d\upsilon(t)$$
$$+ \int_{A}^{B} \left( \int_{-\infty}^{-0} r(u, tx) dM(x) + \int_{+0}^{\infty} r(u, tx) dN(x) \right) d\upsilon(t)$$

1

Using the definitions of  $a_\upsilon$  and  $\sigma_\upsilon$  we can write this relation in the form

$$\log h(u) = ia_{\upsilon}t - \frac{\sigma_{\upsilon}^{2}}{2} (ut)^{2} + \int_{A}^{B} \int_{-\infty}^{-0} r(u, tx) dM(x) d\upsilon(t) + \int_{A}^{B} \int_{0+}^{\infty} r(u, tx) dN(x) d\upsilon(t)$$
(1.10)

and in view of (1.7), Let us have

$$\log h(u) = ia_{v}t - \frac{\sigma_{v}^{2}}{2}(ut)^{2} + \int_{A}^{-t_{0}} \int_{-\infty}^{-0} r(u, tx) dM(x) dv(t) + \int_{t_{0}}^{B} \int_{-\infty}^{-0} r(u, tx) dM(x) dv(t) + \int_{A}^{B} \int_{0+}^{-0} r(u, tx) dN(x) dv(t) + \int_{A}^{b} \int_{0+}^{\infty} r(u, tx) dN(x) dv(t) (1.11)$$

Decomposing the third term on the right – hand side of (1.11) we get for every  $\varepsilon > 0$ .

$$I = \int_{A}^{-t_{0}} \int_{-\infty}^{-0} r(u, tx) dM(x) dv(t)$$
  
=  $\int_{A}^{-t_{0}} \int_{-\infty}^{-0} r(u, tx) dM(x) dv(t) + \int_{A}^{t_{0}} \int_{-\infty}^{-\epsilon} r(u, tx) dN(x) dv(t)$   
=  $I_{1} + I_{2}$ 

Applying L' Hospital's rule twice we find

$$\lim_{x\to\infty}\frac{r(u,tx)}{x^2}=-\frac{(ut)^2}{2}$$

Hence there is a constant  $C_1$  such that for fixed u and  $t \in [A, -t_0]$ 

$$|r(u,tx)| \le C_1(ut)^2 x^2$$

Therefore let us get for  $I_1$  the estimation ( $\varepsilon \rightarrow 0$  +)

$$|I_1| \le C_1 u^2 \int_A^{-t_0} t^2 dv(t) \int_{-\epsilon}^{0-} x^2 dM(x) = o(1)$$

Further we can transform  $I_2$  in the following way.

$$I_{2} = \int_{A}^{-t_{0}} \int_{\epsilon}^{\infty} r(u, x) d_{x} \left( -M\left(\frac{x}{t}\right) \right) dv(t)$$
$$= \int_{\epsilon}^{\infty} r(u, x) d_{x} \int_{A}^{-t_{0}} -M\left(\frac{x}{t}\right) dv(t)$$

so let us obtain as  $\varepsilon \to 0$ 

$$I = \int_{0+}^{\infty} r(u, x) d\left(\int_{A}^{-t_0} -M\left(\frac{x}{t}\right) dv(t)\right)$$

Transforming the fourth, fifth and sixth terms of (1.11) in a similar manner to the third one we get

$$\log h(u) = ia_{v}t - \frac{\sigma_{v}^{2}}{2}(ut)^{2} + \int_{0+}^{\infty} r(u,x)d\left(\int_{A}^{-t_{0}} -M\left(\frac{x}{t}\right)dv(t)\right) + \int_{-\infty}^{0-} r(u,x)d\int_{t_{0}}^{B} M\left(\frac{x}{t}\right)dv(t) + \int_{-\infty}^{0-} r(u,x)d\int_{A}^{-t_{0}} -N\left(\frac{x}{t}\right)dv(t) + \int_{-\infty}^{0-} r(u,x)d\int_{A}^{-t_{0}} -N\left(\frac{x}{t}\right)dv(t) + \int_{0+}^{\infty} r(u,x)d\int_{t_{0}}^{B} N\left(\frac{x}{t}\right)dv(t)$$

Finally, using the definition of  $M_v$  and  $N_v$ , we can rewrite this relation in the form,

$$\log h(u) = ia_{v}t - \frac{\sigma_{v}^{2}}{2}(ut)^{2} + \int_{-\infty}^{0-} r(u,x)dM_{v}(x) + \int_{0+}^{\infty} r(u,x)dN_{v}(x)$$

Let us complete the proof by showing that  $a_v, \sigma_v, M_v$  and  $N_v$  satisfy the condition of Lemma 1.1. Obviously  $a_v$  and  $\sigma_v^2$  are real constants and  $\sigma_v^2 \ge 0$ . By definition it is easily seen that  $M_v$  and  $N_v$  are non – decreasing in the intervals ( $-\infty$ , 0) and ( $0, \infty$ ), respectively, having the properties

$$M_v(-\infty) = N_v(\infty) = 0$$

For every  $\varepsilon > 0$  we obtain the inequality

$$\int_{0+}^{\epsilon} x^2 dN_{\nu}(x) = \int_{t_0}^{B} t^2 \int_{0+}^{\epsilon/t} x^2 dN(x) d\nu(t) + \int_{A}^{-t_0} t^2 \int_{\epsilon/t}^{0-} x^2 dM(x) d\nu(t)$$

$$\leq \int_{t_0}^{B} t^2 dv(t) \int_{0+}^{\epsilon/t_0} x^2 dN(x) + \int_{A}^{-t_0} t^2 dv(t) \int_{\epsilon/t_0}^{0-} x^2 dM(x) < \infty$$

Analogously, let us get

$$\int_{-\epsilon}^{0-} x^2 dM_v(x) < \infty$$

Then Lemma 1.1 shows he statement provided that (1.7) is valid.

2. Now let us turning to the general case *n* which (1.7) need not be true. Put for  $n \ge max(-1/A, 1/B)$ 

$$v_n(t) = \begin{cases} v(t) & \text{if } t \le -1/n \\ v(-1/n) & \text{if } -1/n < t \le 1/n \\ v(t) & \text{if } t > 1/n \end{cases}$$

Obviously, we have

$$\lim_{n \to \infty} v_n(t) = v(t) \tag{1.12}$$

and the functions  $v_n$  satisfying (1.7) are non – decreasing, non – negative and left – continuous. Hence let us can apply the first part of the proof to the stochastic integrals

$$\int_{A}^{B} t dX(v_n(t)) \tag{1.13}$$

and obtain representation of  $a_{v_n}, \sigma_{v_n}, M_{v_n}$  and  $N_{v_n}$  by formulas analogous to (1.2) – (1.5) using Helly's second theorem, we get

$$\lim_{n \to \infty} \int_{A}^{B} \log f(u \cdot t) dv(t) = \int_{A}^{B} \log f(u \cdot t) dv(t)$$
(1.14)

Because  $\log f(u \cdot t)$  – considered as function of t – is continuous and bounded and by (1.12) the sequence  $v_n$  converges weakly to v. Let us denote the characteristic function of (1.13) by  $h_v$ . In view of theorem 1.1. relation (1.14) is equivalent

$$\lim_{n\to\infty}h_n(u) = h(u)$$

Using the known fact that under this circumstance  $a_v \to a_{v_n}, \sigma_v \to \sigma_{v_n}, M_{v_n} \Rightarrow M_v$  and  $N_{v_n} \Rightarrow N_v$  ( $\Rightarrow$  stands for weak convergence) the statement follows.

#### 1. Proof of the main results

#### Lemma: 2.1

The Levy canonical representation of a stable characteristic function with characteristic exponent  $z_0$  is determined either by *b* is arbitrary,

$$\sigma = 0, \qquad M(x) = Q_1 |x|^{-z_0}, \qquad N(x) = -Q_2 x^{-z_0}$$
$$(Q_1 \ge 0, Q_2 \ge 0, Q_1 + Q_2 > 0) \text{ for } 0 < z_0 < 2 \qquad (2.1)$$

or by

b is arbitrary 
$$\sigma = 0$$
,  $M(x) \equiv 0$ ,  $N(x) \equiv 0$  for  $z_0 = 2$ .

#### Lemma: 2.2

Let  $0 \leq Re z_3 \leq 2$  and  $|t| \neq 0,1$ . Then

$$\int_{0^{+}}^{\infty} \frac{x^{2-z_{3}}}{(1+(tx)^{2})(1+x^{2})} dx = \begin{cases} \frac{-\pi}{2\cos\left(z_{3}\frac{\pi}{2}\right)} \frac{1-|t|^{z_{3}-1}}{1-t^{2}} & \text{if } z_{3} \neq 1\\ \frac{-\log|t|}{1-t^{2}} & \text{if } z_{3} = 1 \end{cases}$$
(2.2)

### **Proof:**

Let us compute the integral in (2.2) by contour integration. Assume at first  $z_3 \neq 1$  and  $Im z_3 \leq 0$ . Let us consider the integral

$$\int_{\Gamma} \frac{z^{2-z_3}}{\left(\left(\frac{1}{t}\right)^2 + z^2\right)(1+z^2)} dz$$

taken round the contour, consisting of the line segment  $(\varepsilon, R)(\Gamma_1)$  where  $0 < \varepsilon < R$ ; a semicircle  $\Gamma_2$  of radius R above the real axis; the line segment  $(-R, -\varepsilon)(\Gamma_3)$  and finally a semicircle  $\Gamma_4$  of radius  $\varepsilon$  above the real axis. Let us choose  $\varepsilon$  small and R large and denote the integrand by g(z). Then consider any branch of the many – valued function g. the function g has two poles inside  $\Gamma$ , at z = i/|t| and at z = i. From the definition of g obtain

$$\operatorname{Res}_{z=i} g(z) = \frac{i^{2-z_3}t^2}{2i(1-t^2)} \text{ and } \operatorname{Res}_{z=i/|t|} g(z) = \frac{-i^{2-z_3}|t|^{1+z_3}}{2i(1-t^2)}$$

Using the theorem of residues,

$$\int_{\Gamma} g(z)dz = \int_{\Gamma_1} g(z)dz + \int_{\Gamma_2} g(z)dz + \int_{\Gamma_3} g(z)dz + \int_{\Gamma_4} g(z)dz$$
$$= \pi i^{2-z_3} \frac{t^2 - t^{1+z_3}}{1 - t^2}$$
(2.3)

The two integrals together along the real axis give,

$$\int_{\Gamma_1} g(z)dz + \int_{\Gamma_3} g(z)dz = \int_{\varepsilon}^{R} g(x)dx + \int_{\varepsilon}^{R} g(-x)dx$$
$$= (1 + (-1)^{2-z_3}) \int_{\varepsilon}^{R} g(x)dx$$

The other two integrals tend to 0 as  $\varepsilon \to 0$  and  $R \to \infty$ . Namely, for g on  $\Gamma_2$ , let us have the estimate

$$|g(z)| \leq \frac{R^{2-Rez_3}}{\left(R^2 - \frac{1}{t^2}\right)(R^2 - 1)}$$

So that

$$\left| \int_{\Gamma_2} g(z) dz \right| \leq \frac{\pi R^{3 - \operatorname{Rez}_3}}{\left( R^2 - \frac{1}{t^2} \right) (R^2 - 1)} \to 0 \quad (R \to \infty)$$

Similarly,

$$\left| \int_{\Gamma_4} g(z) dz \right| \leq \frac{\pi \epsilon^{3 - \operatorname{Rez}_3}}{\left(\frac{1}{t^2} - \epsilon^2\right)(1 - \epsilon^2)} \to 0 \quad (\epsilon \to 0)$$

If we multiply the relation (2.3) by  $1/t^2(1 + (-1)^{2-z_3})$ , consider  $R \to \infty$  and  $\varepsilon \to 0$  and notice that

$$\frac{i^{2-z_3}}{1+(-1)^{2-z_3}} = -\frac{1}{2 \cos\left(z_3 \left(\frac{\pi}{2}\right)\right)}$$

then the statement of the theorem in this case.

$$\int_{0^{+}}^{\infty} \frac{x^{2-z_3}}{(1+(tx)^2)(1+x^2)} dx = \int_{0^{+}}^{\infty} \frac{x^{2-\overline{z}_3}}{(1+(tx)^2)(1+x^2)} dx$$

Since the formula obtained is true also for the case  $z_3 \neq 1$ ,  $\text{Im}z_3 > 0$ . The integral (2.2) depends continuously on  $z_3$ ; therefore get the other statement by tending  $z_3 \rightarrow 1$ .

## Lemma: 2.3

Let g be an analytic function of exponential type in Re  $z \ge 0$  which is continuous is Re  $z \ge 0$  and bounded on Re z = 0.If

$$\lim_{x \to \infty} \frac{\log|g(x)|}{x} \leq 0$$

then g is bounded in Re  $z \ge 0$ .

## Lemma: 2.4

Let g be an analytic in Re z < 0, continuous in Re  $z \le 0$  and bounded on Re z = 0. Moreover, let there exist a function h with the same properties which is bounded in Re  $z \le 0$ . If  $g \cdot h$  also is bounded in Re  $z \le 0$  then g is of exponential type in Re  $z \le 0$ .

## **References:**

- Riedal,M (1980) Characterization of stable processes by identically distributed stochastic integrals, Adv.Appl.Prob.12, 689 – 709.
- 2. Riedal, M. (1980) Representation of the characteristic function of a stochastic integral.J.Appl.Prob.17, 448 455.
- 3. Robbins, H. (1948) Mixture of distributions. Ann.Math.Sattist. 19, 360 369.
- Rossberg, H.J. (1974) On a problem of Kolmogorov concerning the normal distribution. Theor. Prob.Appl. 19, 824 – 828.