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Shehu Transform for Solution of Some Integral Equations

Nitin D. Dhange

Assistant Professor
Department of Mathematics
Bhawabhuti Mahavidyalaya, Amgaon

Abstract

Shehu transform was introduced by Maitama as a generalization of Sumudu and Laplace transform. In this paper, convolution theorem and uniqueness theorem for Shehu transform is proved. Furthermore, both the theorems have been used to solve some integral equations.

Keywords: Integral transform, Shehu transform, Laplace transform, Integral equation.

1. Introduction

Abel's integral equation have found to be very applicable in science and engineering such as determination of potentials, stereology, seismic travel times, spectroscopy and optical fibers. Also, integro differential equations model many situations from science and engineering, such as circuit analysis. Shehu transform introduced by Maitama[1] is found to be applicable in solving differential equations, Heat and transport equations, Electric circuit problems, Particular Abel's equation, Volterra integral equation of first kind[2,3,4,5]. The main purpose of this paper is to solve convolution type Volterra integral equation of first kind including Abel's integral equation, Volterra integral equation of second kind and integro differential equation by Shehu transform.

2. Preliminaries

Definition [1]: The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions,

$$\mathcal{A} = \left\{ f(t)/\exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_i}}, if\ t \in (-1)^i \times [0, \infty) \right\}$$

by the following integral

$$\mathbb{S}[f(t)] = F(s,u) = \int_0^\infty e^{\frac{-st}{u}} f(t)dt$$

$$= \lim_{\alpha \to \infty} \int_0^\alpha e^{\frac{-st}{u}} f(t)dt \tag{1}$$

The inverse Shehu transform is given by

$$\mathbb{S}^{-1}[f(s,u)] = f(t), \text{for } t \geq 0 \tag{2}$$

Properties of Shehu transform [1]:

1) **Linearity Property:** Let the functions $\alpha v(t)$ and $\beta w(t)$ be in set $\mathcal{A}$, then $(\alpha v(t) + \beta w(t)) \in \mathcal{A}$ **,** where α and β are nonzero arbitrary constants and $\mathbb{S}[\alpha v(t) + \beta w(t)] = \alpha\mathbb{S}[v(t)] + \beta\mathbb{S}[w(t)]$ (3)
2) $\mathbb{S}[1] = \frac{u}{s}$
3) $\mathbb{S}[t] = \frac{u^2}{s^2}$
4) $\mathbb{S}\left[\frac{t^n}{n!}\right] = \left(\frac{u}{s}\right)^{n+1}, n = 0,1,2,---$
5) $\mathbb{S}[e^{at}] = \frac{u}{s-au}$
6) $\mathbb{S}[te^{at}] = \frac{u^2}{(s-au)^2}$
7) $\mathbb{S}\left[\frac{t^n e^{at}}{n!}\right] = \frac{u^{n+1}}{(s-au)^{n+1}}$
8) $\mathbb{S}[sin(at)] = \frac{au^2}{s^2+a^2u^2}$
9) $\mathbb{S}[cos(at)] = \frac{us}{s^2+a^2u^2}$
10) $\mathbb{S}\left[\frac{sinh(at)}{a}\right] = \frac{u^2}{s^2-a^2u^2}$
11) $\mathbb{S}[cosh(at)] = \frac{us}{s^2-a^2u^2}$
12) $\mathbb{S}[t^{-\alpha}] = \left(\frac{u}{s}\right)^{1-\alpha}\Gamma(1-\alpha), \quad 0 < \alpha < 1$

Proof: - By definition,

$$\mathbb{S}[f(t)] = \int_0^\infty e^{\frac{-st}{u}} t^{-\alpha} dt$$

Let $\frac{st}{u} = x$

$$\Rightarrow \quad \mathbb{S}[t^{-\alpha}] = \int_0^\infty e^{-x}\left(\frac{ux}{s}\right)^{-\alpha}\frac{u}{s}dx$$

$$= \left(\frac{u}{s}\right)^{1-\alpha}\int_0^\infty e^{-x}x^{(1-\alpha)-1}dx$$

$$= \left(\frac{u}{s}\right)^{1-\alpha}\Gamma(1-\alpha) \tag{4}$$

Replacing α by $(1-\alpha)$ in (4), we get

$$\mathbb{S}[t^{\alpha-1}] = \left(\frac{u}{s}\right)^{\alpha}\Gamma(\alpha) \tag{5}$$

Theorem [1]: - (Shehu transform of Derivative)

If the function $f^n(t)$ is the n[th] derivative of the function $f(t) \in \mathcal{A}$ with respect to t, then its Shehu transform is defined by

$$\mathbb{S}[f^n(t)] = \frac{s^n}{u^n}F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)} f^k(0) \tag{6}$$

When $n = 1, 2$ in equation (6) above, we obtain the following

$$\mathbb{S}[f'(t)] = \frac{s}{u}F(s,u) - f(0) \tag{7}$$

$$\mathbb{S}[f''(t)] = \frac{s^2}{u^2}F(s,u) - \frac{s}{u}f(0) - f'(0). \tag{8}$$

Where $F(s,u) = \mathbb{S}[f(t)]$.

Relation Between Shehu and Laplace transform

By definition of Laplace Transform, we have

$$L[f(t)] = \int_0^\infty e^{-st}f(t)dt = F(s) \tag{9}$$

And by definition of Shehu transform, we have

$$\mathbb{S}[f(t)] = \int_0^\infty e^{\frac{-st}{u}} f(t)dt = A(s,u) \tag{10}$$

From (9) and (10), we can establish the relation as

$$A(s,u) = F\left(\frac{s}{u}\right) \tag{11}$$

Convolution Theorem for Shehu Transform

Let $f(t)$ and $g(t)$ be defined in $\mathcal{A}$ having Laplace transform $F(s)$ and $G(s)$ and Shehu transforms $A(s,u)$ and $B(s,u)$. Then a Shehu transform of the convolution of f and g:

$(f * g)(t) = \int_0^t f(x)\, g(t-x)dx$ is given by

$$\mathbb{S}[(f * g)(t)] = A(s,u)B(s,u)$$

Proof: The Laplace transform of $(f * g)$ is given by

$$L[(f * g)(t)](s) = F(s)G(s)$$

From (11), we have

$$\mathbb{S}[(f * g)(t)](s,u) = L[(f * g)(t)]\left(\frac{S}{u}\right)$$

And since $A(s,u) = F\left(\frac{s}{u}\right)$, $B(s,u) = G\left(\frac{s}{u}\right)$

Then $\mathbb{S}[(f * g)(t)](s,u) = F\left(\frac{s}{u}\right) G\left(\frac{s}{u}\right)$

$$= A(s,u)B(s,u)$$

So, the theorem is proved.

Theorem: Uniqueness Theorem

If $A(s,u)$ and $B(s,u)$ are Shehu transforms of $f(t)$ and $g(t)$, then

$$A(s,u) = B(s,u) \Longrightarrow f(t) = g(t)$$

Proof: Consider $A(s,u) = B(s,u)$

$$\Longrightarrow \mathbb{S}[f(t);(s,u)] = \mathbb{S}[g(t);(s,u)]$$

$$\Longrightarrow \mathrm{L}\left[f(t), \frac{s}{u}\right] = \mathrm{L}\left[g(t), \frac{s}{u}\right]$$

By uniqueness of Laplace transform, we obtain

$$f(t) = g(t)$$

Some special types of Integral equations:

1) Integro-Differential equation:
   An integral equation in which various derivatives of the unknown function $y(t)$ can also be present is said to be an Integro-Differential equation.
   For example: $y'(t) = y(t) + f(t) + \int_0^t \sin(t-x)y(x)dx$
2) Integral Equation of Convolution Type:
   The integral equation

$$y(t) = f(t) + \int_0^t k(ct-x)y(x)dx$$

in which the kernel $k(t-x)$ is a function of the difference only.
Using the definition of convolution, the above integral equation can be written as

$$y(t) = f(t) + k(t) * y(t)$$

3. Main Result

Application of Shehu Transform to determine the solution of Volterra Integral Equation with Convolution Type Kernel:

Consider the Volterra integral equation of first kind

$$f(t) = \int_0^t k(t-x)u(x)dx \tag{12}$$

Where $k(t-x)$ depends only on the difference $(t-x)$

Applying Shehu transform to both sides of the equation (12), we have

$$\mathbb{S}[f(t)] = \mathbb{S}\left[\int_0^t k(t-x)u(x)dx\right]$$

$$= [k(t) * u(t)]$$

$$\Longrightarrow F(s,u) = K(s,u)U(s,u)$$

Where $F(s,u) = \mathbb{S}[f(t)]$, $K(s,u) = \mathbb{S}[k(t)]$ and $U(s,u) = \mathbb{S}[u(t)]$

$$\Longrightarrow U(s,u) = \frac{F(s,u)}{K(s,u)}$$

Applying inverse Shehu transform, we get

$$u(t) = \mathbb{S}^{-1}\left[\frac{F(s,u)}{K(s,u)}\right] \tag{13}$$

Similarly, consider the Volterra integral equation of second kind as

$$u(t) = f(t) + \int_0^t k\,(t-x)u(x)dx \tag{14}$$

Applying Shehu transform to both sides of the equation (14), we obtain

$$U(s,u) = F(s,u) + K(s,u)U(s,u)$$

$$\Longrightarrow U(s,u) = \frac{F(s,u)}{1-K(s,u)} \tag{15}$$

$$\Longrightarrow u(t) = \mathbb{S}^{-1}\left[\frac{F(s,u)}{1-K(s,u)}\right]$$

The resolvent kernel of equation (14) can be determined by the method of integral transform.

Let the kernel $k(t,x)$ be defined as a difference kernel, then so is the resolvent kernel. Since the resolvent kernel $R(t,x)$ is the sum of the iterated kernels and they all depend on the difference $(t-x)$, then

$R_1(t,x) = k(t,x) = k(t-x)$ and

$k_2(t,x) = \int_x^t k(t-z)\,k(z-x)dz$

Let $z - x = \mu \Longrightarrow z = x + \mu \Longrightarrow dz = d\mu$

Or $k_2(t,x) = \int_0^{t-x} k(t-x-\mu)\,k(\mu)d\mu$ (16)

Similarly, we can determine the other integration. Thus, the solution of integral equation (14) is given by

$$u(t) = f(t) + \int_0^t R(t-x)\,f(x)dx \tag{17}$$

Applying Shehu transform to both sides of the equation (17), we obtain

$$U(s,u) = F(s,u) + R(s,u)F(s,u) \tag{18}$$

Where $R(s,u) = \mathbb{S}[R(t-x)]$

Using (15) in (18), we have

$$\frac{F(s,u)}{1-K(s,u)} = F(s,u)[1 + R(s,u)]$$

$$\Longrightarrow R(s,u) = \frac{K(s,u)}{1-K(s,u)}$$

Applying inverse Shehu transform, we have

$$R(t-x) = \mathbb{S}^{-1}\left[\frac{K(s,u)}{1-K(s,u)}\right] \tag{19}$$

Substituting the values of $R(t-x)$ given by (19) in (17), we shall get the desired solution of (14).

Now we solve Abel's integral equation

$$f(t) = \int_0^t \frac{u(x)}{(t-x)^\alpha}dx, 0 < \alpha < 1 \tag{20}$$

by Shehu transform.

Applying Shehu transform on both sides of (20) and using convolution theorem, we obtain

$$\mathbb{S}[f(t)] = \mathbb{S}[u(t)]\mathbb{S}[t^{-\alpha}]$$

$$\Longrightarrow F(s,u) = U(s,u)\left(\frac{u}{s}\right)^{1-\alpha}\Gamma(1-\alpha)$$

$$\Longrightarrow U(s,u) = \frac{\left(\frac{u}{s}\right)^{\alpha-1}F(s,u)}{\Gamma(1-\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left\{\Gamma(\alpha)\left(\frac{u}{s}\right)^{\alpha-1}F(s,u)\right\}$$

$$= \frac{\sin\pi\alpha}{\pi}\frac{s}{u}\{\mathbb{S}[t^{\alpha-1} * f(t)]\}$$

$$\Longrightarrow \mathbb{S}[u(t)] = \frac{s}{u}\frac{\sin\pi\alpha}{\pi}\mathbb{S}\left[\int_0^t (t-x)^{\alpha-1}f(x)dx\right]$$

Or

$$\mathbb{S}[u(t)] = \frac{s}{u}\frac{\sin\pi\alpha}{\pi}\mathbb{S}[G(t)] \tag{21}$$

Where $G(t) = \int_0^t (t-x)^{\alpha-1}f(x)dx$, $G(0) = 0$

We know that

$$\mathbb{S}[G'(t)] = \frac{s}{u}\mathbb{S}[G(t)] - G(0) = \frac{s}{u}\mathbb{S}[G(t)]$$

From (21), we have

$$\mathbb{S}[u(t)] = \frac{\sin \pi\alpha}{\pi}\mathbb{S}[G'(t)]$$

Applying inverse Shehu transform, we get

$$u(t) = \frac{\sin \pi\alpha}{\pi}\frac{d}{dt}G(t)$$

Or $$u(t) = \frac{\sin \pi\alpha}{\pi}\frac{d}{dt}\left[\int_0^t (t-x)^{\alpha-1} f(x)dx\right] \quad (22)$$

Which is the solution of Abel's integral equation.

4. Examples:

In this section we solve some particular Abel's integral equations, convolution type Volterra integral equations of second kind and integro-differential equations.

1) Solve the Abel's equation $\int_0^t \frac{u(x)}{(t-x)^{\frac{1}{3}}} dx = t(1+t)$ (23)

Solution: - Applying Shehu transform on both sides of (23), we obtain

$$\mathbb{S}\left[u(t) * t^{\frac{-1}{3}}\right] = \mathbb{S}[t + t^2]$$

By using convolution theorem and linearity property of Shehu transform, we get

$$U(s,u)\left(\frac{u}{s}\right)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right) = \left(\frac{u}{s}\right)^2 + 2\left(\frac{u}{s}\right)^3$$

$$\Rightarrow U(s,u) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\left(\frac{u}{s}\right)^{\frac{4}{3}} + 2\left(\frac{u}{s}\right)^{\frac{7}{3}}\right]$$

On applying inverse Shehu transform, we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left\{\mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{4}{3}}\right] + 2\mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{7}{3}}\right]\right\} \\ &= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{t^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} + 2\frac{t^{\frac{4}{3}}}{\Gamma\left(\frac{7}{3}\right)}\right] \\ &= \frac{\sin\frac{\pi}{3}}{\frac{\pi}{3}} t^{\frac{1}{3}}\left(\frac{2+3t}{2}\right) = \frac{3\sqrt{3}}{4\pi} t^{\frac{1}{3}}(2+3t) \end{aligned}$$

2) Solve the inhomogeneous integral equation $u(t) = 1 + \int_0^t \sin(t-x)u(x)\,dx$ (24)

Solution: - Applying Shehu transform on both sides of (24), we get

$$\mathbb{S}[u(t)] = \mathbb{S}[1] + \mathbb{S}[\sin t * u(t)]$$

$$\Rightarrow \mathbb{S}[u(t)] = \frac{u}{s} + \mathbb{S}[\sin t]\mathbb{S}[u(t)]$$

$$\Rightarrow \mathbb{S}[u(t)]\left[1 - \frac{u^2}{s^2+u^2}\right] = \frac{u}{s}$$

$$\Rightarrow \mathbb{S}[u(t)] = \frac{u}{s} + \left(\frac{u}{s}\right)^3$$

Applying inverse Shehu transform, we get

$$u(t) = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}$$

3) Solve the inhomogeneous integral equation$u(t) = 1 - \int_0^t (t-x)\,u(x)dx$ (25)

Solution: - Applying Shehu transform on both sides of (25), we get

$$\mathbb{S}[u(t)] = \mathbb{S}[1] - \mathbb{S}[t]\mathbb{S}[u(t)]$$

$$\Rightarrow \left[1 + \left(\frac{u}{s}\right)^2\right]\mathbb{S}[u(t)] = \frac{u}{s}$$

$$\Rightarrow \mathbb{S}[u(t)] = \frac{us}{s^2+u^2}$$

Applying inverse Shehu transform, we get

$$u(t) = \mathbb{S}^{-1}\left[\frac{us}{s^2+u^2}\right] = \cos t$$

4) Find the resolvent kernel of the Volterra integral equation and hence its solution

$$u(t) = f(t) + \int_0^t (t-x)\,u(x)dx$$

Solution: - Given equation can be written as

$$u(t) = f(t) + u(t) * t \tag{26}$$

Applying Shehu transform on both sides o (26) and using the convolution theorem, we get

$$\mathbb{S}[u(t)] = \mathbb{S}[f(t)] + \mathbb{S}[u(t)]\left(\frac{u}{s}\right)^2$$

$$\Rightarrow \left[1-\left(\frac{u}{s}\right)^2\right]\mathbb{S}[u(t)] = \mathbb{S}[f(t)]$$

$$\Rightarrow \mathbb{S}[u(t)] = \frac{s^2}{s^2-u^2}\mathbb{S}[f(t)] \tag{27}$$

Let $R(t-x)$ be the resolvent kernel of the given integral equation. Then we know that the required solution is given by

$$u(t) = f(t) + \int_0^t R(t-x)\,f(x)dx \tag{28}$$

Or $$u(t) = f(t) + R(t) * f(t) \tag{29}$$

Applying Shehu transform on equation (29) and using convolution theorem, we get

$$\mathbb{S}[u(t)] = \mathbb{S}[f(t)] + \mathbb{S}[R(t)]\mathbb{S}[f(t)]$$

From (27), we get

$$\frac{s^2}{s^2-u^2} = 1 + \mathbb{S}[R(t)]$$

$$\Rightarrow \mathbb{S}[R(t)] = \frac{u^2}{s^2-u^2}$$

Applying inverse Shehu transform, we get

$$R(t) = \mathbb{S}^{-1}\left[\frac{u^2}{s^2-u^2}\right] = \sin ht$$

$$\Rightarrow R(t-x) = \sin h(t-x) \tag{30}$$

Using (30) in (28), the required solution is

$$u(t) = f(t) + \int_0^t \sin h\,(t-x) f(x)dx$$

5) Solve the integro-differential equation

$$u'(t) = \sin t + \int_0^t u\,(t-x)\cos x\,dx, where\ u(0) = 0$$

Solution: - Given equation can be written as

$$u'(t) = \sin t + u(t) * \cos t \tag{31}$$

Applying Shehu transform on both sides of (31) and using convolution theorem, we get

$$\mathbb{S}[u'(t)] = \mathbb{S}[\sin t] + \mathbb{S}[u(t)]\mathbb{S}[\cos t]$$

$$\Rightarrow \frac{s}{u}\mathbb{S}[u(t)] - u(0) = \frac{u^2}{s^2+u^2} + \mathbb{S}[u(t)].\frac{us}{s^2+u^2}$$

Using initial condition, we get

$$\mathbb{S}[u(t)]\left[\frac{s}{u} - \frac{us}{s^2+u^2}\right] = \frac{u^2}{s^2+u^2}$$

$$\Rightarrow \mathbb{S}[u(t)] = \left(\frac{u}{s}\right)^3$$

$$\Rightarrow u(t) = \mathbb{S}^{-1}\left(\frac{u}{s}\right)^3 = \frac{t^2}{2!} = \frac{t^2}{2}$$

5. Conclusion

Shehu transform is found to be very effective in solving Abel's integral equation, Volterra integral equation of first and second kind with resolvent kernel and integro-differential equation.

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